SOME PROPERTIES OF THE PSEUDO-SMARANDACHE FUNCTION

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Abstract. Charles Ashbacher [1] has posed a number of questions relating to the pseudo-Smarandache function $Z(n)$. In this note we show that the ratio of consecutive values $Z(n+1)/Z(n)$ and $Z(n-1)/Z(n)$ are unbounded; that $Z(2n)/Z(n)$ is unbounded; that $n/Z(n)$ takes every integer value infinitely often; and that the series $\sum_n 1/Z(n)^\alpha$ is convergent for any $\alpha > 1$.

1. Introduction

We define the m-th triangular number $T(m) = \frac{m(m+1)}{2}$. Kashihara [2] has defined the *pseudo-Smarandache* function $Z(n)$ by

$$Z(n) = \min\{m : n \mid T(m)\}.$$

Charles Ashbacher [1] has posed a number of questions relating to the pseudo-Smarandache function $Z(n)$. In this note we show that the ratio of consecutive values $Z(n)/Z(n-1)$ and $Z(n)/Z(n+1)$ are unbounded; that $Z(2n)/Z(n)$ is unbounded; and that $n/Z(n)$ takes every integer value infinitely often. He notes that the series $\sum_n 1/Z(n)^\alpha$ is divergent for $\alpha = 1$ and asks whether it is convergent for $\alpha = 2$. He further suggests that the least value of α for which the series converges "may never be known" . We resolve this problem by showing that the series converges for all $\alpha > 1$.

2. Some properties of the pseudo-Smarandache function

We record some elementary properties of the function Z.

Lemma 1.
(1) *If $n \ge T(m)$ then $Z(n) \ge m$. $Z(T(m)) = m$.*
(2) *For all n we have $\sqrt{n} < Z(n)$.*
(3) *$Z(n) \le 2n - 1$, and if n is odd then $Z(n) \le n - 1$.*
(4) *If p is an odd prime dividing n then $Z(n) \ge p - 1$.*
(5) *$Z(2^k) = 2^{k+1} - 1$.*
(6) *If p is an odd prime then $Z(p^k) = p^k - 1$ and $Z(2p^k) = p^k - 1$ or p^k according as $p^k \equiv 1$ or $3 \bmod 4$.*

We shall make use of Dirichlet's Theorem on primes in arithmetic progression in the following form.

Lemma 2. *Let a, b be coprime integers. Then the arithmetic progression $a + bt$ is prime for infinitely many values of t.*

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3. Successive values of the pseudo-Smarandache function

Using properties (3) and (5), Ashbacher observed that $\left|Z(2^k) - Z(2^k - 1)\right| > 2^k$ and so the difference between the conecutive values of Z is unbounded. He asks about the ratio of consecutive values.

Theorem 1. *For any given $L > 0$ there are infinitely many values of n such that $Z(n+1)/Z(n) > L$, and there are infinitely many values of n such that $Z(n-1)/Z(n) > L$.*

Proof. Choose $k \equiv 3 \bmod 4$, so that $T(k)$ is even and k divides $T(k)$. We consider the conditions $k \mid m$ and $(k+1) \mid (m+1)$. These are satisfied if $m \equiv k \bmod k(k+1)$, that is, $m = k + k(k+1)t$ for some t. We have $m(m+1) = k(1+(k+1)t) \cdot (k+1)(1+kt)$, so that if $n = k(k+1)(1+kt)/2$ we have $n \mid T(m)$. Now consider $n+1 = T(k) + 1 + kT(k)t$. We have $k \mid T(k)$, so $T(k)+1$ is coprime to both k and $T(k)$. Thus the arithmetic progression $T(k) + 1 + kT(k)t$ has initial term coprime to its increment and by Dirichlet's Theorem contains infinitely many primes. We find that there are thus infinitely many values of t for which $n+1$ is prime and so $Z(n) \le m = k + k(k+1)t$ and $Z(n+1) = n = T(k)(1+kt)$. Hence

$$\frac{Z(n+1)}{Z(n)} \ge \frac{n}{m} = \frac{T(k) + kT(k)t}{k + 2T(k)t} > \frac{k}{3}.$$

A similar argument holds if we consider the arithmetic progression $T(k) - 1 + kT(k)t$. We then find infinitely many values of t for which $n-1$ is prime and

$$\frac{Z(n-1)}{Z(n)} \ge \frac{n-2}{m} = \frac{T(k) - 2 + kT(k)t}{k + 2T(k)t} > \frac{k}{4}.$$

The Theorem follows by taking $k > 4L$. □

We note that this Theorem, combined with Lemma 1(2), gives another proof of the result that the difference of consecutive values is unbounded.

4. Divisibility of the pseudo-Smarandache function

Theorem 2. *For any integer $k \ge 2$, the equation $n/Z(n) = k$ has infinitely many solutions n.*

Proof. Fix an integer $k \ge 2$. Let p be a prime $\equiv -1 \bmod 2k$ and put $p+1 = 2kt$. Put $n = T(p)/t = p(p+1)/2t = pk$. Then $n \mid T(p)$ so that $Z(n) \le p$. We have $p \mid n$, so $Z(n) \ge p - 1$: that is, $Z(n)$ must be either p or $p-1$. Suppose, if possible, that it is the latter. In this case we have $2n \mid p(p+1)$ and $2n \mid (p-1)p$, so $2n$ divides $p(p+1) - (p-1)p = 2p$: but this is impossible since $k > 1$ and so $n > p$. We conclude that $Z(n) = p$ and $n/Z(n) = k$ as required. Further, for any given value of k there are infinitely many prime values of p satisfying the congruence condition and hence infinitely many values of $n = T(p)$ such $z/Z(n) = k$. □

5. Another divisibility question

Theorem 3. *The ratio $Z(2n)/Z(n)$ is not bounded above.*

Proof. Fix an integer k. Let $p \equiv -1 \bmod 2^k$ be prime and put $n = T(p)$. Then $Z(n) = p$. Consider $Z(2n) = m$. We have $2^k p \mid p(p+1) = 2n$ and this divides $m(m+1)/2$. We have $m \equiv \epsilon \bmod p$ and $m \equiv \delta \bmod 2^{k+1}$ where each of ϵ, δ can be either 0 or -1.

Let $m = pt + \epsilon$. Then $m \equiv \epsilon - t \equiv \delta \bmod 2^k$: that is, $t \equiv \epsilon - \delta \bmod 2^k$. This implies that either $t = 1$ or $t \geq 2^k - 1$. Now if $t = 1$ then $m \leq p$ and $T(m) \leq T(p) = n$, which is impossible since $2n \leq T(m)$. Hence $t \geq 2^k - 1$. Since $Z(2n)/Z(n) = m/p > t/2$, we see that the ratio $Z(2n)/Z(n)$ can be made as large as desired. ☐

6. Convergence of a series

Ashbacher observes that the series $\sum_n 1/Z(n)^\alpha$ diverges for $\alpha = 1$ and asks whether it converges for $\alpha = 2$.

In this section we prove convergence for all $\alpha > 1$.

Lemma 3.

$$\log n \leq \sum_{m=1}^{n} \frac{1}{m} \leq 1 + \log n;$$

$$\frac{1}{2}(\log n)^2 - 0.257 \leq \sum_{m=1}^{n} \frac{\log m}{m} \leq \frac{1}{2}(\log n)^2 + 0.110 \textit{ for } n \geq 4.$$

Proof. For the first part, we have $1/m \leq 1/t \leq 1/(m-1)$ for $t \in [m-1, m]$. Integrating,

$$\frac{1}{m} \leq \int_{m-1}^{m} \frac{1}{t}\mathrm{d}t \leq \frac{1}{m-1}.$$

Summing,

$$\sum_{2}^{n} \frac{1}{m} \leq \int_{1}^{n} \frac{1}{t}\mathrm{d}t \leq \sum_{2}^{n} \frac{1}{m-1},$$

that is,

$$\sum_{1}^{n} \frac{1}{m} \leq 1 + \log n \text{ and } \log n \leq \sum_{1}^{n-1} \frac{1}{m}.$$

The result follows.

For the second part, we similarly have $\log m/m \leq \log t/t \leq \log(m-1)/(m-1)$ for $t \in [m-1, m]$ when $m \geq 4$, since $\log x/x$ is monotonic decreasing for $x > \mathrm{e}$. Integrating,

$$\frac{\log m}{m} \leq \int_{m-1}^{m} \frac{\log t}{t}\mathrm{d}t \leq \frac{\log(m-1)}{m-1}.$$

Summing,

$$\sum_{4}^{n} \frac{\log m}{m} \leq \int_{3}^{n} \frac{\log t}{t}\mathrm{d}t \leq \sum_{4}^{n} \frac{\log(m-1)}{m-1},$$

that is,

$$\begin{aligned} & \sum_1^n \frac{\log m}{m} - \frac{\log 2}{2} - \frac{\log 3}{3} \\ \le\ & \frac{1}{2}(\log n)^2 - \frac{1}{2}(\log 3)^2 \\ \le\ & \sum_1^n \frac{\log m}{m} - \frac{\log n}{n} - \frac{\log 2}{2}. \end{aligned}$$

We approximate the numerical values

$$\frac{\log 2}{2} + \frac{\log 3}{3} - \frac{1}{2}(\log 3)^2 < 0.110$$

and

$$\frac{\log 2}{2} - \frac{1}{2}(\log 3)^2 > -0.257.$$

to obtain the result. □

Lemma 4. *Let $d(m)$ be the function which counts the divisors of m. For $n \ge 2$ we have*

$$\sum_{m=1}^n d(m)/m < 7(\log n)^2.$$

Proof. We verify the assertion numerically for $n \le 6$. Now assume that $n \ge 8 > \mathrm{e}^2$. We have

$$\begin{aligned} \sum_{m=1}^n \frac{d(m)}{m} &= \sum_{m=1}^n \sum_{de=m} \frac{1}{m} = \sum_{d \le n} \sum_{de \le n} \frac{1}{de} \\ &= \sum_{d \le n} \frac{1}{d} \sum_{e < n/d} \frac{1}{e} \le \sum_{d \le n} \frac{1}{d}(1 + \log(n/d)) \\ &\le (1 + \log n)^2 - \frac{1}{2}(\log n)^2 + 0.257 \\ &= 1.257 + 2 \log n + \frac{1}{2}(\log n)^2 \\ &< \frac{4}{3}\left(\frac{\log n}{2}\right)^2 + 2 \log n \left(\frac{\log n}{2}\right) + \frac{1}{2}(\log n)^2 \\ &< 2(\log n)^2. \end{aligned}$$

□

Lemma 5. *Fix an integer $t \ge 5$. Let $e^t > Y > e^{(t-1)/2}$. The number of integers n with $e^{t-1} < n \le e^t$ such that $Z(n) \le Y$ is at most $196Yt^2$.*

Proof. Consider such an n with $m = Z(n) \le Y$. Now $n \mid m(m+1)$, say $k_1 n_1 = m$ and $k_2 n_2 = m + 1$, with $n = n_1 n_2$. Thus $k = k_1 k_2 = m(m+1)/n$ and $k_1 n_1 \le Y$. The value of k is bounded below by 2 and above by $m(m+1)/n \le 2Y^2/\mathrm{e}^{t-1} = K$, say. Given a pair (k_1, k_2), the possible values of n_1 are bounded above by Y/k_1 and must satisfy the congruence condition $k_1 n_1 + 1 \equiv 0$ modulo k_2: there are therefore at most $Y/k_1 k_2 + 1$ such values.

Since $Y/k \ge Y/K = e^{t-1}/2Y > 1/2e$, we have $Y/k + 1 < (2e+1)Y/k < 7Y/k$. Given values for k_1, k_2 and n_1, the value of n_2 is fixed as $n_2 = (k_1 n_1 + 1)/k_2$. There are thus at most $\sum_{k \le K} d(k)$ possible pairs (k_1, k_2) and hence at most $\sum_{k \le K} 7Y d(k)/k$ possible quadruples (k_1, k_2, n_1, n_2). We have $K > 2$ so that the previous Lemma applies and we can deduce that the number of values of n satisfying the given conditions is at most $49Y(\log K)^2$. Now $K = 2Y^2/e^{t-1} < 2e^{t+1}$ so $\log K < t + 1 + \log 2 < 2t$. This establishes the claimed upper bound of $196Yt^2$. □

Theorem 4. *Fix $\frac{1}{2} < \beta < 1$ and an integer $t \ge 5$. The number of integers n with $e^{t-1} < n \le e^t$ such that $Z(n) < n^\beta$ is at most $196t^2 e^{\beta t}$.*

Proof. We apply the previous result with $Y = e^{\beta t}$. The conditions of β ensure that the previous lemma is applicable and the upper bound on the number of such n is $196e^{\beta t}t^2$ as claimed. □

Theorem 5. *The series*

$$\sum_{n=1}^{\infty} \frac{1}{Z(n)^\alpha}$$

is convergent for any $\alpha > \sqrt{2}$.

Proof. We note that if $\alpha > 2$ then $1/Z(n)^\alpha < 1/n^{\alpha/2}$ and the series is convergent. So we may assume $\sqrt{2} < \alpha \le 2$. Fix β with $1/\alpha < \beta < \alpha/2$. We have $\frac{1}{2} < \beta < \sqrt{\frac{1}{2}} < \alpha/2$.

We split the positive integers $n > e^4$ into two classes A and B. We let class A be the union of the A_t where, for positive integer $t \ge 5$ we put into class A_t those integers n such that $e^{t-1} < n \le e^t$ for integer t and $Z(n) \le n^\beta$. All values of n with $Z(n) > n^\beta$ we put into class B. We consider the sum of $1/Z(n)^\alpha$ over each of the two classes. Since all terms are positive, it is sufficient to prove that each series separately is convergent.

Firstly we observe that for $n \in B$, we have $1/Z(n)^\alpha < 1/n^{\alpha\beta}$ and since $\alpha\beta > 1$ the series summed over the class B is convergent.

Consider the elements n of A_t: so for such n we have $e^{t-1} < n \le e^t$ and $Z(n) < n^\beta$. By the previous result, the number of values of n satisfying these conditions is at most $196t^2e^{\beta t}$. For $n \in A_t$, we have $Z(n) \ge \sqrt{n}$, so $1/Z(n)^\alpha \le 1/n^{\alpha/2} < 1/e^{\alpha(t-1)/2}$. Hence the sum of the subseries $\sum_{n \in A_t} 1/Z(n)^\alpha$ is at most $196e^{\alpha/2}t^2e^{(\beta-\alpha/2)t}$. Since $\beta < \alpha/2$ for $\alpha > \sqrt{2}$, the sum over all t of these terms is finite.

We conclude that $\sum_{n=1}^{\infty} 1/Z(n)^\alpha$ is convergent for $\alpha > \sqrt{2}$ □

Theorem 6. *The series*

$$\sum_{n=1}^{\infty} \frac{1}{Z(n)^\alpha}$$

is convergent for any $\alpha > 1$.

Proof. We fix $\beta_0 = 1 > \beta_1 > \cdots > \beta_r = \frac{1}{2}$ with $\beta_j < \alpha\beta_{j+1}$ for $0 \le j \le r-1$. We define a partition of the integers $e^{t-1} < n < e^t$ into classes B_t and $C_t(j)$, $1 \le j \le r-1$. Into B_t place those n with $Z(n) > n^{\beta_1}$. Into $C_t(j)$ place those n

with $n^{\beta_{j+1}} < Z(n) < n^{\beta_j}$. Since $\beta_r = \frac{1}{2}$ we see that every n with $e^{t-1} < n < e^t$ is placed into one of the classes.

The number of elements in $C_t(j)$ is at most $196t^2 e^{\beta_j t}$ and so

$$\sum_{n \in C_t(j)} \frac{1}{Z(n)^\alpha} < 196t^2 e^{\beta_j t} e^{-\beta_{j+1}\alpha(t-1)} = 196 e^{\beta_{j+1}\alpha} t^2 e^{(\beta_j - \alpha\beta_{j+1})t}.$$

For each j we have $\beta_j < \alpha\beta_{j+1}$ so each sum over t converges.

The sum over the union of the B_t is bounded above by

$$\sum_n \frac{1}{n^{\alpha\beta_1}},$$

which is convergent since $\alpha\beta_1 > \beta_0 = 1$.

We conclude that $\sum_{n=1}^\infty 1/Z(n)^\alpha$ is convergent. □

References

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