SPACES WITH M -STRUCTURES

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SPACES WITH $\mathscr{M}$-STRUCTURES

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Abstract. In this paper, we introduce the notion of $\mathscr{M}$-structures and study some properties of spaces endowed with $\mathscr{M}$-structures. We see that there is a $\mathscr{M}$-structure in every infinite set.

1. Introduction

Let X be a non-empty set. By a proper subset A of X we mean that A is a non-empty subset of X such that $A \neq X$ and in this case we write $A \subsetneq X$.

It is well known to us that $\{\emptyset\} \cup \{(a, b) : a, b \in \mathbb{R}, a \neq b\}$ forms a basis for the real number space $\mathbb{R}$. The collection $\mathscr{A} = \{(a, b) : a, b \in \mathbb{R}, a < b\}$ of proper subsets of $\mathbb{R}$ admits a special character: for any $A \in \mathscr{A}$ there exist $B, C \in \mathscr{A}$ such that $B \subsetneq A \subsetneq C$. Furthermore, if X is a T_1 connected topological space, then $\{\emptyset\} \cup \mathscr{T}_{mo}$ forms a basis (Theorem 2.4) satisfying the condition that for any $B \in \mathscr{M}$ there exist $A, C \in \mathscr{M}$ such that $A \subsetneq B \subsetneq C$, where $\mathscr{T}_{mo}$ is the collection of all mean open sets in X. Considering these facts, we develop a new kind of structure (resp., space) in nonempty sets namely $\mathscr{M}$-structures (resp., $\mathscr{M}$-space) (Definition 3.1). In recent years Smarandache multispace theory becomes a centre of attraction. Mao [3, 4, 5, 6] studied the Smarandache multispace theory significantly. Under the light of the Smarandache multispace theory, one can say that the study of $\mathscr{M}$-spaces is a particular case sudy of Smarandache multispaces.

2. Preliminaries

Firstly, we recall the following definitions and results:

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Definition 2.1 (Nakaoka and Oda [9, 10, 11])**.** A nonempty open set U of a topological space X is said to be a minimal open set if and only if any open set which is contained in U is $\emptyset$ or U.

Definition 2.2 (Mukharjee and Bagchi [7])**.** An open set M of a topological space X is said to be a mean open if there exist two distinct proper open sets U, V such that $U \subsetneq M \subsetneq V$.

Definition 2.3 (Benchalli et al. [2])**.** A topological space X is said to be a T_{min} space if every proper open set of X is minimal open.

Theorem 2.4 (Nakaoka and Oda [9])**.** *If U is a minimal open set and W is an open set of a topological space X, then either $U \cap W = \emptyset$ or $U \subset W$. If W is a minimal open set distinct from U, then $U \cap W = \emptyset$.*

Theorem 2.5 (Bagchi and Mukherjee [1])**.** *Let $(X, \mathscr{T})$ be a T_1 connected topological space and $\mathscr{T}_{mo}$ denotes the family of all mean open sets in X. Then $\mathscr{B} = \{\emptyset\} \cup \mathscr{T}_{mo}$ forms a basis of the topology $\mathscr{T}$ on X.*

3. $\mathscr{M}$-SPACES

Definition 3.1. Let X be a non-empty set. A collection $\mathscr{A}$ of proper subsets of X is said to be an $\mathscr{M}$-structure on X if for any $A \in \mathscr{A}$ there exist $B, C \in \mathscr{A}$ such that $B \subsetneq A \subsetneq C$. The ordered pair $(X, \mathscr{A})$ is said to be an $\mathscr{M}$-space.

Example 3.2. Let all the proper open sets of a topological space $(X, \mathscr{T}$ be mean open. We write $\mathscr{M} = \mathscr{T} - \{\emptyset, X\}$. Then $(X, \mathscr{M})$ is an $\mathscr{M}$-space.

Remark 3.3. $\mathscr{A} = \{(a, b) : a, b \in \mathbb{R}, a, b\}\}$ and $\mathscr{B} = \{[a, b] : a, b \in \mathbb{R}, a < b\}$ are $\mathscr{M}$-structures on $\mathbb{R}$. Here $(1, 2), (2, 3) \in \mathscr{R}$ but $(1, 2) \cup (2, 3) \notin \mathscr{R}$. On the other hand $[1, 2], [2, 3] \in \mathscr{B}$ but $\{2\} = [1, 2] \cap [2, 3] \notin \mathscr{B}$. Therefore $\mathscr{M}$-structures may not closed under unions as well as intersections.

Theorem 3.4. *Let $(X, \mathscr{A})$ be an $\mathscr{M}$-space. Then each member of the $\mathscr{M}$-structure $\mathscr{A}$ is infinite.*

Proof. Let $A \in \mathscr{A}$. If possible, let A be finite and $A = \{a_1, a_2, ..., a_n\}$, for some natural number $n \geq 1$. Then there is a $A_1 \in \mathscr{A}$ such that $A_1 \subsetneq A$. So $A_1 \subseteq A - \{a_{j_1}\}$, for some $j_1 \in \{1, 2, ..., n\}$. Again there is a $A_2 \in \mathscr{A}$ such that $A_2 \subsetneq A_1$. Thus $A_2 \subseteq A - \{a_{j_1}, a_{j_2}\}$, for some $j_2 \in \{1, 2, ..., n\}$ with $j_1 \neq j_2$. Continuing the process we can have $A_{n-1} \in \mathscr{A}$ such that $A_{n-1} \subseteq$

$A - \{a_{j_1}, a_{j_2}, ..., a_{j_{n-1}}\}$, where $j_k \in \{1, 2, ..., n\}$ with $j_1 \neq j_2 \neq ... \neq j_{n-1}$ and $k = 1, 2, ..., n-1$. Thus either A_{n-1} is a singleton set or $A_{n-1} = \emptyset$. Thus there is no $B \in \mathscr{A}$ such that $B \subsetneq A_{n-1}$, which contradicts $A_{n-1} \in \mathscr{A}$. So A is infinite. Since $A \in \mathscr{A}$ is arbitrary, each member of the $\mathscr{M}$-structure $\mathscr{A}$ is infinite. □

Theorem 3.5. *Let $(X, \mathscr{A})$ be an $\mathscr{M}$-space. Then $\mathscr{A}$ is an infinite collection of proper subsets of X.*

Proof. If possible, let $\mathscr{A} = \{A_1, A_2, ..., A_n\}$ for some natural number $n \geq 1$. Since $A_1 \in \mathscr{A}$, there is a $B \in \mathscr{A} - \{A_1\}$ such that $A_1 \subsetneq B$. Now after some finite steps we can have a $C \in \mathscr{A} - \{A_1, B\}$ such that there is no $D \in \mathscr{A}$ such that $C \subsetneq D$. Thus $\mathscr{A}$ is an infinite collection of proper subsets X. □

Let $(X, \mathscr{A})$ be an $\mathscr{M}$-space. Then X is infinite.

Proof. The proof follows from the fact that X has infinite subsets. □

Theorem 3.6. *Let $(X, \mathscr{A})$ be a $\mathscr{M}$-space. There exist $\mathscr{M}$-structures $\mathscr{B}$ and $\mathscr{C}$ such that $\mathscr{A} \neq \mathscr{B} \neq \mathscr{C}$. In other words, an $\mathscr{M}$-space contains at least three $\mathscr{M}$-structures.*

Proof. Let $\mathscr{B} = \{X - A : A \in \mathscr{A}\}$ and $B \in \mathscr{B}$. Then $B = X - A$ for some $A \in \mathscr{A}$. There exists $A_1, A_2 \in \mathscr{A}$ such that $A_1 \subsetneq A \subsetneq A_2$. So $X - A_2 \subsetneq X - A \subsetneq X - A_1$, i.e, $X - A_2 \subsetneq B \subsetneq X - A_1$. Furthermore $X - A_1, X - A_2 \in \mathscr{B}$. Thus $\mathscr{M}$ is an $\mathscr{M}$-structure on X different from $\mathscr{A}$. One can easily prove that $\mathscr{C} = \{A \subsetneq X : A \in \mathscr{A} or A \in \mathscr{B}\}$ is an $\mathscr{M}$-structure on X which is different from $\mathscr{A}$ as well as $\mathscr{B}$. □

Remark 3.7. Let $(X, \mathscr{A})$ be an $\mathscr{M}$-space. An $\mathscr{M}$-structure $\mathscr{B}$ on X is said to be conjugate to $\mathscr{A}$ iff $\mathscr{B} = \{X - A : A \in \mathscr{A}\}$. In this case, we write $\mathscr{B} = \mathscr{A}^c$. Furthermore, the $\mathscr{M}$-structures $\mathscr{A}$ and $\mathscr{B} = \mathscr{A}^c$ are said to be conjugate to each other.

Theorem 3.8. *There exists $\mathscr{M}$-spaces.*

Proof. Let X be an infinite set. We consider the collection $\mathscr{A} = \{A \subsetneq X : A$ and $X - A$ both are infinite$\}$. Now let $A \in \mathscr{A}$. Then both A and $X - A$ are infinite proper subsets of X. There are points $x \in A$ and $y \in X - A$ such that $A - \{x\} \subsetneq A \subsetneq A \cup \{y\}$. By the definition of $\mathscr{A}$, $A - \{x\}$ and $A \cup \{y\}$ are members of $\mathscr{A}$. So $\mathscr{A}$ is an $\mathscr{M}$-structure on X, i.e., $(X, \mathscr{A})$ is a $\mathscr{M}$-space. □

The $\mathscr{M}$-structure $\mathscr{A}$ defined on an infinite set X discussed on the previous theorem is said to be the trivial $\mathscr{M}$-structure and the $\mathscr{M}$-space $(X, \mathscr{A})$ is said to be the trivial $\mathscr{M}$-space.

Definition 3.9. Let $(X, \mathscr{A})$ be a $\mathscr{M}$-space and $M \subseteq X$. M is said to be an $\mathscr{M}$-set Of X if there exist $A, B \in \mathscr{A}$ such that $A \subsetneq M \subsetneq B$.

If M is an $\mathscr{M}$-set then $M \neq \emptyset, X$. Clearly if $A \in \mathscr{A}$ then A is an $\mathscr{M}$-set.

We denote the collection of all $\mathscr{M}$-sets of X by $\mathscr{M}^*$. One can easily verify that $\mathscr{M}^*$ is an $\mathscr{M}$ structure on X. If $\bigcup_{A \in \mathscr{A}} A = X$, then $\mathscr{M}^*$ is an s-refinement ([8]) of $\mathscr{A}$.

Example 3.10. Let us consider the $\mathscr{M}$-space $(\mathbb{R}, \mathscr{A})$, where $\mathscr{A} = \{(a, b) : a < b \text{ and } a, b \in \mathbb{R}\}$. If M is a countable subset of $\mathbb{R}$, then M is not a $\mathscr{M}$-set. Again for any $a, b \in \mathbb{R}$ with $a < b$, $(a, b]$ and $[a, b)$ are $\mathscr{M}$-sets.

Now let $(X, \mathscr{A})$ be a $\mathscr{M}-space$ and M be a $\mathscr{M}$-set. Then $\{P \in \mathscr{A} : P \subsetneq A\}$ and $\{P \in \mathscr{A} : A \subsetneq P\}$ are nonempty collection of nonempty subsets of X. We write $M_L = \bigcup\{P \in \mathscr{A} : P \subsetneq A\}$ and $M_R = \bigcap\{P \in \mathscr{A} : A \subsetneq P\}$. Clearly $M_L \subseteq M \subseteq M_R$. We call M_L and M_R are the left variation and right variation of the $\mathscr{M}$-set M respectively and $M_R - M_L$ is said to be the variation of the $\mathscr{M}$-set M. We denote the variation of an $\mathscr{M}$-set M by $v(M)$.

Let ρ be the relation on $\mathscr{M}^*$ defined by: "$M \rho N$ if and only if $v(A) = v(B)$, for any $M, N \in \mathscr{M}^*$". We can prove that ρ is an equivalence relation on $\mathscr{M}^*$.

Example 3.11. Let $X = \mathbb{R}^2$ and $\mathscr{A} = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq n^2, n \in \mathbb{N}\} \bigcup \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1/n^2, n \in \mathbb{N} - \{1\}\}$. Then $\mathscr{A}$ is an $\mathscr{M}$-structure on $\mathscr{R}^2$. Let $M = \{(x, y) \in \mathbb{R} : x^2 + y^2 = 1\}$. Then $M_L = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1/4\}$ and $M_R = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq 4\}$.

Theorem 3.12. *Let $(X, \mathscr{A})$ be an $\mathscr{M}$-space and $M, N \in \mathscr{A}$ be such that $M \subseteq N$. Then*

(i) $M_L \subseteq N_L$*; and*
(ii) $M_R \subseteq N_R$.

Proof. (i) Let $x \in M_L$. Then $x \in P$ for some $P \in \mathscr{A}$ with $P \subsetneq M$. Since $M \subseteq N$, it follows that $P \subsetneq B$ and so $x \in B_L$. Thus $A_L \subseteq B_L$.

(ii) If $P \in \{P \in \mathscr{A} : N \subsetneq P\}$ then $P \in \{P \in \mathscr{A} : M \subsetneq P\}$, since $M \subseteq N$. Therefore $\{P \in \mathscr{A} : N \subsetneq P\}$ is a subcollection of $\{P \in \mathscr{A} : M \subsetneq P\}$ and thus $\bigcap\{P \in \mathscr{A} : M \subsetneq P\} \subseteq \bigcap\{P \in \mathscr{A} : N \subsetneq P\}$, i.e., $M_R \subseteq N_R$. □

Theorem 3.13. *Let $(X, \mathscr{A})$ be a $\mathscr{M}$-space and M be a $\mathscr{M}$. Then*

(i) $M \not\subseteq v(M)$;

(ii) $v(M) \subsetneq M$ iff $M = M_R$.

Proof. (i) $M \subseteq v(M) \Rightarrow M \subseteq M_R - M_L \Rightarrow M \subseteq X - M_L \Rightarrow M_L \subseteq X - M_L$, which is a contradiction.

(ii) $M = M_R \Rightarrow v(M) = M - M_L \Rightarrow V(M) \subseteq A$. Using (i) we have $v(M) \subsetneq M$.

Now $v(M) \subsetneq M \Rightarrow M_R \cap (X - M_L) \subsetneq M \Rightarrow M_L \cup [M_R \cap (X - M_L)] \subseteq M_L \cup M = M \Rightarrow (M_L \cup M_R) \cap [M_L \cup (X - M_L)] \subsetneq M \Rightarrow M_R \cap X \subsetneq M \Rightarrow M_R \subsetneq M \subseteq M_R \Rightarrow M = M_R$. □

Theorem 3.14. *Let $(X, \mathscr{A})$ be the trivial $\mathscr{M}$-space. Then:*

(i) $\mathscr{A} = \mathscr{M}^*$; and

(ii) $v(M) = \emptyset$, for each $M \in \mathscr{A}$.

Proof. (i) It is sufficient to prove that if $M \in \mathscr{M}^*$, then $M \in \mathscr{A}$ for each $M \in \mathscr{M}^*$. Let $M \in \mathscr{M}^*$. Then there exist $A, B \in \mathscr{A}$ such that $A \subsetneq M \subsetneq B$. As $A \subsetneq M$ and A is infinite, so M is also infinite. Now $M \subsetneq B \Rightarrow X - B \subsetneq X - M$. Since $X - B$ is infinite, it follows that $X - M$ is infinite. Thus $M \in \mathscr{A}$. Therefore $\mathscr{A} = \mathscr{M}^*$.

(ii) Let $M \in \mathscr{A} = \mathscr{M}^*$. So M and $X - M$ are infinite proper subsets of X. We can choose distinct points $m, n \in M$ such that $M - \{m\} \subseteq M_L$ and $M - \{n\} \subseteq M_L$. Now $(M - \{m\}) \cup (M - \{n\}) = M$ and so $M = M_L$. On the other hand we can choose two distinct points $p, q \in X - M$ such that $M_R \subseteq M \cup \{p\}$ as well as $M_R \subseteq M \cup \{q\}$. So $M = (M \cup \{p\}) \cap (M \cup \{q\})$ and thus $M = M_R$. Hence $v(M) = M_R - M_L = M - M = \emptyset$. Since $M \in \mathscr{A}$ is arbitrary, $v(M) = \emptyset$, for each $M \in \mathscr{A}$. □

Definition 3.15. Let $M \subsetneq X$. M is said to be a common $\mathscr{M}$-set of X if M is an $\mathscr{M}$-set of X with respect to $\mathscr{A}$ as well as an $\mathscr{M}$-set of X with respect to $\mathscr{A}^c$.

Theorem 3.16. *Let $(X, \mathscr{A})$ be a $\mathscr{M}$-space. Then followings are equivalent:*

(i) M is a common $\mathscr{M}$-set of X.
(ii) M and $X - M$ are $\mathscr{M}$-sets with respect to $\mathscr{A}$.
(iii) M and $X - M$ are $\mathscr{M}$-sets with respect to $\mathscr{A}^c$.

Proof. $(i) \Rightarrow (ii)$:
There exist $A, B \in \mathscr{A}$ and $C, D \in \mathscr{A}^c$ such that $A \subsetneq M \subsetneq B$ and $C \subsetneq M \subsetneq D$. By the definition of $\mathscr{A}^c$, $C = X - A_1$ and $D = B_1$, for some $A_1, B_1 \in \mathscr{A}$. Then $B_1 \subsetneq X - M \subsetneq A_1$. Thus M and $X - M$ are $\mathscr{M}$-sets with respect to $\mathscr{A}$.

$(ii) \Rightarrow (iii)$:
There exist $A, B, C, D \in \mathscr{A}$ such that $A \subsetneq M \subsetneq B$ and $C \subsetneq X - M \subsetneq D$. Now $X - A, X - B \in \mathscr{A}^c$ and $X - B \subsetneq X - M \subsetneq X - A$. Also $X - C, X - D \in \mathscr{A}^c$ such that $X - D \subsetneq M \subsetneq X - C$. Thus M and $X - M$ are $\mathscr{M}$-sets with respect to $\mathscr{A}^c$.

$(iii) \Rightarrow (i)$:
There exist $A, B, C, D \in \mathscr{A}$ such that $X - A \subsetneq M \subsetneq X - B$ and $X - C \subsetneq X - M \subsetneq X - D$. Then $D \subsetneq M \subsetneq C$ and so M is a common $\mathscr{M}$-set of X. □

Theorem 3.17. *Let $(X, \mathscr{A})$ be a $\mathscr{M}$-space and M be a common $\mathscr{M}$-set of X. Then followings are true:*

(i) There exist $A, B \in \mathscr{A}$ such that $A \cap B = \emptyset$ and $A \cup B = X$.
(ii) There exist $C, D \in \mathscr{A}^c$ such that $C \cap D = \emptyset$ and $C \cup D = X$.

Proof. $(i) \Rightarrow (ii)$:
By the previous Theorem, we choose $A = M$ and $B = X - M$.

$(ii) \Rightarrow (iii)$:
By the previous Theorem, we choose $C = M$ and $D = X - M$. □

4. Remark on T_{min} spaces

Let X be a T_{min} space. Then all the proper open sets of X are minimal open sets. By Theorem 2.4, all the proper open sets of X mutually disjoint. We claim that X can have atmost two proper open sets. In fact, if X has more than two proper open sets then union of any two proper open sets must be a proper open set containing two proper open sets (since all the proper open sets are mutually disjoint). Consequently, X has a proper open set which is not a minimal open set, but this contradicts the fact that X

is a T_{min} space. On the other hand, if a topological space X has only one proper open set then X must be a T_{min} space. Further more, if a topological space X has only two disjonit proper open sets then the proper open sets must be minimal, i.e., X must be a T_{min} space.

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