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On the pseudo Smarandache square-free function

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Abstract For any positive integer n, the famous Pseudo Smarandache Square-free function $Z_w(n)$ is defined as the smallest positive integer m such that m^n is divisible by n. That is, $Z_w(n) = \min\{m : n|m^n, m \in N\}$, where N denotes the set of all positive integers. The main purpose of this paper is using the elementary method to study the properties of $Z_w(n)$, and give an inequality for it. At the same time, we also study the solvability of an equation involving the Pseudo Smarandache Square-free function, and prove that it has infinity positive integer solutions.

Keywords The Pseudo Smarandache Square-free function, Vinogradov's three-primes theorem, inequality, equation, positive integer solution.

§1. Introduction and results

For any positive integer n, the famous Pseudo Smarandache Square-free function $Z_w(n)$ is defined as the smallest positive integer m such that m^n is divisible by n. That is,

$$Z_w(n) = \min\{m : \ n|m^n, \ m \in N\},$$

where N denotes the set of all positive integers. This function was proposed by Professor F. Smarandache in reference [1], where he asked us to study the properties of $Z_w(n)$. From the definition of $Z_w(n)$ we can easily get the following conclusions: If $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_r^{\alpha_r}$ denotes the factorization of n into prime powers, then $Z_w(n) = p_1 p_2 \cdots p_r$. From this we can get the first few values of $Z_w(n)$ are: $Z_w(1) = 1$, $Z_w(2) = 2$, $Z_w(3) = 3$, $Z_w(4) = 2$, $Z_w(5) = 5$, $Z_w(6) = 6$, $Z_w(7) = 7$, $Z_w(8) = 2$, $Z_w(9) = 3$, $Z_w(10) = 10$, $\cdots$. About the elementary properties of $Z_w(n)$, some authors had studied it, and obtained some interesting results, see references [2], [3] and [4]. For example, Maohua Le [3] proved that

$$\sum_{n=1}^{\infty} \frac{1}{(Z_w(n))^{\alpha}}, \ \alpha \epsilon R, \ \alpha > 0$$

is divergence. Huaning Liu [4] proved that for any real numbers $\alpha > 0$ and $x \geq 1$, we have the asymptotic formula

$$\sum_{n \leq x} (Z_w(n))^{\alpha} = \frac{\zeta(\alpha+1) x^{\alpha+1}}{\zeta(2)(\alpha+1)} \prod_p \left[1 - \frac{1}{p^{\alpha}(p+1)}\right] + O\left(x^{\alpha+\frac{1}{2}+\epsilon}\right),$$

where $\zeta(s)$ is the Riemann zeta-function.

Now, for any positive integer $k > 1$, we consider the relationship between $Z_w\left(\prod_{i=1}^{k} m_i\right)$ and $\sum_{i=1}^{k} Z_w(m_i)$. In reference [2], Felice Russo suggested us to study the relationship between them. For this problem, it seems that none had studied it yet, at least we have not seen such a paper before. The main purpose of this paper is using the elementary method to study this problem, and obtained some progress on it. That is, we shall prove the following:

Theorem 1. Let $k > 1$ be an integer, then for any positive integers $m_1, m_2, \cdots, m_k$, we have the inequality

$$\sqrt[k]{Z_w\left(\prod_{i=1}^{k} m_i\right)} < \frac{\sum_{i=1}^{k} Z_w(m_i)}{k} \leq Z_w\left(\prod_{i=1}^{k} m_i\right),$$

and the equality holds if and only if all $m_1,\ m_2,\ \cdots,\ m_k$ have the same prime divisors.

Theorem 2. For any positive integer $k \geq 1$, the equation

$$\sum_{i=1}^{k} Z_w(m_i) = Z_w\left(\sum_{i=1}^{k} m_i\right)$$

has infinity positive integer solutions $(m_1,\ m_2,\ \cdots,\ m_k)$.

§2. Proof of the theorems

In this section, we shall prove our Theorems directly. First we prove Theorem 1. For any positive integer $k > 1$, we consider the problem in two cases:

(a). If $(m_i,\ m_j) = 1,\ i,\ j = 1,\ 2,\ \cdots,\ k$, and $i \neq j$, then from the multiplicative properties of $Z_w(n)$, we have

$$Z_w\left(\prod_{i=1}^{k} m_i\right) = \prod_{i=1}^{k} Z_w(m_i).$$

Therefore, we have

$$\sqrt[k]{Z_w\left(\prod_{i=1}^{k} m_i\right)} = \sqrt[k]{\prod_{i=1}^{k} Z_w(m_i)} < \frac{\sum_{i=1}^{k} Z_w(m_i)}{k} < \prod_{i=1}^{k} Z_w(m_i) = Z_w\left(\prod_{i=1}^{k} m_i\right).$$

(b). If $(m_i,\ m_j) > 1,\ i,\ j = 1,\ 2,\ \cdots,\ k$, and $i \neq j$, then let $m_i = p_1^{\alpha_{i1}} p_2^{\alpha_{i2}} \cdots p_r^{\alpha_{ir}}$, $\alpha_{is} \geq 0,\ i = 1,\ 2,\ \cdots,\ k;\ s = 1,\ 2,\ \cdots,\ r$. we have $Z_w(m_i) = p_1^{\beta_{i1}} p_2^{\beta_{i2}} \cdots p_r^{\beta_{ir}}$, where

$$\beta_{is} = \begin{cases} 0, & \text{if } \alpha_{is} = 0\ ; \\ 1, & \text{if } \alpha_{is} \geq 1. \end{cases}$$

Thus

$$\frac{\sum_{i=1}^{k} Z_w(m_i)}{k} = \frac{p_1^{\beta_{11}} p_2^{\beta_{12}} \cdots p_r^{\beta_{1r}} + p_1^{\beta_{21}} p_2^{\beta_{22}} \cdots p_r^{\beta_{2r}} + \cdots + p_1^{\beta_{k1}} p_2^{\beta_{k2}} \cdots p_r^{\beta_{kr}}}{k}$$

$$\leq \frac{p_1 p_2 \cdots p_r + p_1 p_2 \cdots p_r + \cdots + p_1 p_2 \cdots p_r}{k} = p_1 p_2 \cdots p_r = Z_w\left(\prod_{i=1}^{k} m_i\right),$$

and equality holds if and only if $\alpha_{is} \geq 1$, $i = 1,\ 2,\ \cdots,\ k$, $s = 1,\ 2,\ \cdots,\ r$.

$$\sqrt[k]{Z_w\left(\prod_{i=1}^{k} m_i\right)} = \sqrt[k]{p_1 p_2 \cdots p_r} \leq \sqrt[k]{p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_r^{\alpha_r}}$$

$$\leq \frac{p_1^{\beta_{11}} p_2^{\beta_{12}} \cdots p_r^{\beta_{1r}} + p_1^{\beta_{21}} p_2^{\beta_{22}} \cdots p_r^{\beta_{2r}} + \cdots + p_1^{\beta_{k1}} p_2^{\beta_{k2}} \cdots p_r^{\beta_{kr}}}{k} = \frac{\sum_{i=1}^{k} Z_w(m_i)}{k},$$

where $\alpha_s = \sum_{i=1}^{k} \beta_{is}$, $s = 1,\ 2,\ \cdots,\ r$, but in this case, two equal sign in the above can't be hold in the same time.

So, we obtain

$$\sqrt[k]{Z_w\left(\prod_{i=1}^{k} m_i\right)} < \frac{\sum_{i=1}^{k} Z_w(m_i)}{k}.$$

From (a) and (b) we have

$$\sqrt[k]{Z_w\left(\prod_{i=1}^{k} m_i\right)} < \frac{\sum_{i=1}^{k} Z_w(m_i)}{k} \leq Z_w\left(\prod_{i=1}^{k} m_i\right),$$

and the equality holds if and only if all $m_1,\ m_2,\ \cdots,\ m_k$ have the same prime divisors. This proves Theorem 1.

To complete the proof of Theorem 2, we need the famous Vinogradov's three-primes theorem, which was stated as follows:

Lemma 1. Every odd integer bigger than c can be expressed as a sum of three odd primes, where c is a constant large enough.

Proof. (See reference [5]).

Lemma 2. Let $k \geq 3$ be an odd integer, then any sufficiently large odd integer n can be expressed as a sum of k odd primes

$$n = p_1 + p_2 + \cdots + p_k.$$

Proof. (See reference [6]).

Now we use these two Lemmas to prove Theorem 2. From Lemma 2 we know that for any odd integer $k \geq 3$, every sufficient large prime p can be expressed as

$$p = p_1 + p_2 + \cdots + p_k.$$

By the definition of $Z_w(n)$ we know that $Z_w(p) = p$. Thus,

$$\begin{aligned} & Z_w(p_1) + Z_w(p_2) + \cdots + Z_w(p_k) = p_1 + p_2 + \cdots + p_k = p = Z_w(p) \\ = \quad & Z_w(p_1 + p_2 + \cdots + p_k). \end{aligned}$$

This means that Theorem 2 is true for odd integer $k \geq 3$.

If $k \geq 4$ is an even number, then for every sufficient large prime p, $p - 2$ is an odd number, and by Lemma 2 we have

$$p - 2 = p_1 + p_2 + \cdots + p_{k-1} \quad \text{or} \quad p = 2 + p_1 + p_2 + \cdots + p_{k-1}.$$

Therefore,

$$\begin{aligned} & Z_w(2) + Z_w(p_1) + Z_w(p_2) + \cdots + Z_w(p_{k-1}) = 2 + p_1 + p_2 + \cdots + p_{k-1} = p \\ = \quad & Z_w(p) = Z_w(2 + p_1 + p_2 + \cdots + p_{k-1}). \end{aligned}$$

This means that Theorem 2 is true for even integer $k \geq 4$.

At last, for any prime $p \geq 3$, we have

$$Z_w(p) + Z_w(p) = p + p = 2p = Z_w(2p),$$

so Theorem 2 is also true for $k = 2$. This completes the proof of Theorem 2.

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