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Surfaces Family With Common Smarandache Asymptotic Curve According To Bishop Frame In Euclidean Space

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abstract: In this paper, we analyzed the problem of consructing a family of surfaces from a given some special Smarandache curves in Euclidean 3-space. Using the Bishop frame of the curve in Euclidean 3-space, we express the family of surfaces as a linear combination of the components of this frame, and derive the necessary and sufficient conditions for coefficents to satisfy both the asymptotic and isoparametric requirements. Finally, examples are given to show the family of surfaces with common Smarandache asymptotic curve.

Key Words: Smarandache Asymptotic Curve, Bishop Frame.

Contents

1. Introduction

In differential geometry, there are many important consequences and properties of curves [1], [2], [3]. Researches follow labours about the curves. In the light of the existing studies, authors always introduce new curves. Special Smarandache curves are one of them. Special Smarandache curves have been investigated by some differential geometers [4,5,6,7,8,9,10]. This curve is defined as, a regular curve in Minkowski space-time, whose position vector is composed by Frenet frame vectors on another regular curve, is Smarandache curve [4]. A.T. Ali has introduced some special Smarandache curves in the Euclidean space [5]. Special Smarandache curves according to Bishop Frame in Euclidean 3-space have been investigated by Çetin et al [6]. In addition, Special Smarandache curves according to Darboux Frame in Euclidean 3-space has introduced in [7]. They found some properties of these special curves and calculated normal curvature, geodesic curvature and geodesic torsion of these curves. Also, they investigate special Smarandache curves in Minkowski 3-space, [8]. Furthermore, they find some properties of these special curves and they calculate curvature and torsion of these curves. Special Smarandache curves

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such as -Smarandache curves according to Sabban frame in Euclidean unit sphere has introduced in [9]. Also, they give some characterization of Smarandache curves and illustrate examples of their results.On the Quaternionic Smarandache Curves in Euclidean 3-Space have been investigated in [10].

One of the most significant curve on a surface is asymptotic curve. Asymptotic curve on a surface has been a long-term research topic in Differential Geometry, [3,11,12]. A curve on a surface is called an asymptotic curve provided its velocity always points in an asymptotic direction, that is the direction in which the normal curvature is zero. Another criterion for a curve in a surface M to be asymptotic is that its acceleration always be tangent to M, [2].Asymptotic curves are also encountered in astronomy, astrophysics and CAD in architecture.

The concept of family of surfaces having a given characteristic curve was first introduced by Wang et.al. [12] in Euclidean 3-space. Kasap et.al. [13] generalized the work of Wang by introducing new types of marching-scale functions, coefficients of the Frenet frame appearing in the parametric representation of surfaces. With the inspiration of work of Wang, Li et.al. [14] changed the characteristic curve from geodesic to line of curvature and defined the surface pencil with a common line of curvature. Recently, in [15] Bayram et.al. defined the surface pencil with a common asymptotic curve. They introduced three types of marching-scale functions and derived the necessary and sufficient conditions on them to satisfy both parametric and asymptotic requirements.

Bishop frame, which is also called alternative or parallel frame of the curves, was introduced by L. R. Bishop in 1975 by means of parallel vector fields, [16]. Recently, many research papers related to this concept have been treated in the Euclidean space, see [17,18]. And, recently, this special frame is extended to study of canal and tubular surfaces, we refer to [19,20]. Bishop and Frenet–Serret frames have a common vector field, namely the tangent vector field of the Frenet–Serret frame. A practical application of Bishop frames is that they are used in the area of Biology and Computer Graphics. For example, it may be possible to compute information about the shope of sequences of DNA using a curve defined by the Bishop frame. The Bishop frame may also provide a new way to control virtual cameras in computer animatons, [21].

In this paper, we study the problem: given a curve (with Bishop frame), how to characterize those surfaces that posess this curve as a common isoasymptotic and Smarandache curve in Euclidean 3-space. In section 2, we give some preliminary information about Smarandache curves in Euclidean 3-space and define isoasymptotic curve. We express surfaces as a linear combination of the Bishop frame of the given curve and derive necessary and sufficient conditions on marching-scale functions to satisfy both isoasymptotic and Smarandache requirements in Section 3. We illustrate the method by giving some examples.

2. Preliminaries

The Bishop frame or parallel transport frame is an alternative approach to defining a moving frame that is well defined even when the curve has vanishing second derivative. One can express parallel transport of an orthonormal frame along a curve simply by parallel transporting each component of the frame [16]. The tangent vector and any convenient arbitrary basis for the remainder of the frame are used (for details, see [17]). The Bishop frame is expressed as [16,18].

$$\frac{d}{ds}\begin{pmatrix} T(s) \\ N_1(s) \\ N_2(s) \end{pmatrix} = \begin{pmatrix} 0 & k_1(s) & k_2(s) \\ -k_1(s) & 0 & 0 \\ -k_2(s) & 0 & 0 \end{pmatrix}\begin{pmatrix} T(s) \\ N_1(s) \\ N_2(s) \end{pmatrix} \tag{2.1}$$

Here, we shall call the set $\{T(s), N_1(s), N_2(s)\}$ as Bishop Frame and k_1 and k_2 as Bishop curvatures. The relation between Bishop Frame and Frenet Frame of curve $\alpha(s)$ is given as follows;

$$\begin{pmatrix} T(s) \\ N_1(s) \\ N_2(s) \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta\,(s) & -\sin\theta\,(s) \\ 0 & \sin\theta\,(s) & \cos\theta\,(s) \end{pmatrix}\begin{pmatrix} T(s) \\ N(s) \\ B(s) \end{pmatrix} \tag{2.2}$$

where

$$\begin{cases} \theta\,(s) = \arctan\left(\frac{k_2(s)}{k_1(s)}\right) \\ \tau\,(s) = -\frac{d\theta(s)}{ds} \\ \kappa\,(s) = \sqrt{k_1^2(s) + k_2^2(s)} \end{cases} \tag{2.3}$$

Here Bishop curvatures are defined by

$$\begin{cases} k_1(s) = \kappa\,(s)\cos\theta\,(s) \\ k_2(s) = \kappa\,(s)\sin\theta\,(s) \end{cases} \tag{2.4}$$

Let r be a regular curve in a surface P passing through $s\,\epsilon P$, κ the curvature of r at s and $\cos\theta{=}n.N$,where N is the normal vector to r and is n normal vector to P at s and " ." denotes the standard inner product. The number $k_n = \kappa\cos\theta$ is then called the normal curvature of at s [2].

Let s be a point in a surface P. An asymptotic direction of P at s is a direction of the tangent plane of P for which the normal curvature is zero. An asymptotic curve of P is a regular curve $r \subset P$ such that for each $s\,\epsilon\, r$ the tangent line of r at s is an asymptotic direction [2].

An isoparametric curve $\alpha(s)$ is a curve on a surface $\Psi = \Psi(s,t)$ is that has a constant s or t-parameter value. In other words, there exist a parameter or such that $\alpha(s) = \Psi(s,t_0)$ or $\alpha(t) = \Psi(s_0,t)$.

Given a parametric curve $\alpha(s)$, we call $\alpha(s)$ an isoasymptotic of a surface Ψ if it is both a asymptotic and an isoparametric curve on Ψ.

Let $\alpha = \alpha(s)$ be a unit speed regular curve in E^3 and $\{T(s), N_1(s), N_2(s)\}$ be its moving Bishop frame . Smarandache TN_1 curves are defined by

$\beta = \beta(s^*) = \frac{1}{\sqrt{2}}\left(T(s) + N_1(s)\right),$

Smarandache TN_2 curves are defined by

$\beta = \beta(s^*) = \frac{1}{\sqrt{2}}\left(T(s) + N_2(s)\right),$

Smarandache N_1N_2 curves are defined by

$\beta = \beta(s^*) = \frac{1}{\sqrt{2}}\left(N_1(s) + N_2(s)\right),$

Smarandache TN_1N_2 curves are defined by

$\beta = \beta(s^*) = \frac{1}{\sqrt{3}}\left(T(s) + N_1(s) + N_2(s)\right),$ [5].

3. Surfaces with common Smarandache asymptotic curve

Let $\varphi = \varphi(s,v)$ be a parametric surface. The surface is defined by a given curve $\alpha = \alpha(s)$ as follows:

$$\varphi(s,v) = \alpha(s)+[x(s,v)T(s)+y(s,v)N_1(s)+z(s,v)N_2(s)], L_1 \leq s \leq L_2, T_1 \leq v \leq T_2 \tag{3.1}$$

where $x(s,v)$, $y(s,v)$ and $z(s,v)$ are C^1 functions. The values of the marching-scale functions $x(s,v)$, $y(s,v)$ and $z(s,v)$ indicate, respectively; the extension-like, flexion-like and retortion-like effects, by the point unit through the time v, starting from $\alpha(s)$ and $\{T(s), N_1(s), N_2(s)\}$ is the Bishop frame associated with the curve $\alpha(s)$.

Our goal is to find the necessary and sufficient conditions for which the some special Smarandache curves of the unit space curve $\alpha(s)$ is an parametric curve and an asymptotic curve on the surface $\varphi(s,v)$.

Firstly, since Smarandache curve of $\alpha(s)$ is an parametric curve on the surface $\varphi(s,v)$, there exists a parameter $v_0 \,\epsilon\, [T_1, T_2]$ such that

$$x(s,v_0) = y(s,v_0) = z(s,v_0) = 0, \; L_1 \leq s \leq L_2 \;, \; T_1 \leq v \leq T_2 \;. \tag{3.2}$$

Secondly, according to the above definitions, the curve $\alpha(s)$ is an asymptotic curve on the surface $\varphi(s,v)$ if and only if the normal curvature $k_n = \kappa \cos\theta \; = 0$, where θ is the angle between the surface normal $n(s,v_0)$ and the principal normal $N(s)$ of the curve $\alpha(s)$. Since $n(s,v_0).T(s) = 0$, $L_1 \leq s \leq L_2$, by derivating this equation with respect to the arc length parameter s, we have the equivalent constraint

$$\frac{dn}{ds}(s,v_0).T(s) = 0 \tag{3.3}$$

for the curve $\alpha(s)$ to be an asymptotic curve on the surface $\varphi(s,v)$, where " ." denotes the standard inner product.

Theorem 3.1. *: Smarandache TN_1 curve of the curve $\alpha(s)$ is isoasymptotic on a surface $\varphi(s,v)$ if and only if the following conditions are satisfied:*

$$\begin{cases} x(s,v_0) = y(s,v_0) = z(s,v_0) = 0, \\ \frac{\partial z}{\partial v}(s,v_0) = \tan\theta\,(s)\,\frac{\partial y}{\partial v}(s,v_0). \end{cases}$$

Proof: Let $\alpha(s)$ be a Smarandache TN_1 curve on surface $\varphi(s,v)$.From (3.1) , $\varphi(s,v)$, parametric surface is defined by a given Smarandache TN_1 curve of curve $\alpha(s)$ as follows:

$$\varphi(s,v) = \frac{1}{\sqrt{2}}\,(T(s)+N_1(s)) + [x(s,v)T(s)+y(s,v)N_1(s)+z(s,v)N_2(s)].$$

If Smarandache TN_1 curve is an parametric curve on this surface , then there exist a parameter $v = v_0$ such that , $\frac{1}{\sqrt{2}}\,(T(s)+N_1(s)) = \varphi(s,v_0)$, that is,

$$x(s,v_0) = y(s,v_0) = z(s,v_0) = 0 \tag{3.4}$$

The normal vector of $\varphi(s,v)$ can be written as

$$n(s,v)=\frac{\partial\varphi(s,v)}{ds}\times\frac{\partial\varphi(s,v)}{\partial v}$$

Since

The normal vector can be expressed as

$$\begin{aligned} n(s,v) \quad = \quad & [\frac{\partial z(s,v)}{dv}(\frac{k_1(s)}{\sqrt{2}}+\frac{\partial y(s,v)}{ds}+k_1(s)x(s,v)) \\ & -\frac{\partial y(s,v)}{dv}(\frac{k_2(s)}{\sqrt{2}}+\frac{\partial z(s,v)}{ds}+k_2(s)x(s,v))]T(s) \\ & +[\frac{\partial x(s,v)}{dv}(\frac{k_2(s)}{\sqrt{2}}+\frac{\partial z(s,v)}{ds}+k_2(s)x(s,v)) \\ & -\frac{\partial z(s,v)}{dv}(-\frac{k_1(s)}{\sqrt{2}}+\frac{\partial x(s,v)}{ds}-k_1(s)y(s,v)-k_2(s)z(s,v))]N_1(s) \\ & +[\frac{\partial y(s,v)}{dv}(-\frac{k_1(s)}{\sqrt{2}}+\frac{\partial x(s,v)}{ds}-k_1(s)y(s,v)-k_2(s)z(s,v)) \\ & -\frac{\partial x(s,v)}{dv}(\frac{k_1(s)}{\sqrt{2}}+\frac{\partial y(s,v)}{ds}+k_1(s)x(s,v))]N_2(s) \end{aligned} \tag{3.5}$$

Using (3.4), if we let

$$\begin{cases} \Phi_1(s,v_0)=\frac{k_1(s)}{\sqrt{2}}\frac{\partial z}{dv}(s,v_0)-\frac{k_2(s)}{\sqrt{2}}\frac{\partial y}{dv}(s,v_0), \\ \Phi_2(s,v_0)=\frac{k_1(s)}{\sqrt{2}}\frac{\partial z}{dv}(s,v_0)+\frac{k_2(s)}{\sqrt{2}}\frac{\partial x}{dv}(s,v_0), \\ \Phi_3(s,v_0)=-\frac{k_1(s)}{\sqrt{2}}\frac{\partial y}{dv}(s,v_0)-\frac{k_1(s)}{\sqrt{2}}\frac{\partial x}{dv}(s,v_0). \end{cases} \tag{3.6}$$

we obtain

$$n(s,v_0)=\Phi_1(s,v_0)T(s)+\Phi_2(s,v_0)N_1(s)+\Phi_3(s,v_0)N_2(s).$$

From Eqn. (2.2), we get

$$\begin{aligned} n(s,v_0) \quad = \quad & \Phi_1(s,v_0)T(s)+\cos\theta\,(s)\,\Phi_2(s,v_0)+\sin\theta\,(s)\,\Phi_3(s,v_0))N(s) \\ & +(\cos\theta\,(s)\,\Phi_3(s,v_0)-\sin\theta\,(s)\,\Phi_2(s,v_0))B(s). \end{aligned}$$

From the eqn. (3.2), we should have

$$\begin{aligned} & \frac{\partial n}{\partial s}(s,v_0).T(s)=0 \\ \Leftrightarrow\ & \tfrac{\partial(\cos\theta(s)\Phi_2(s,v_0)+\sin\theta(s)\Phi_3(s,v_0))N(s)+(\cos\theta(s)\Phi_3(s,v_0)-\sin\theta(s)\Phi_2(s,v_0))B(s))}{\partial s}.T(s)=0 \\ \Leftrightarrow\ & \frac{\partial\Phi_1}{\partial s}(s,v_0)-\kappa(s)\,(\cos\theta\,(s)\,\Phi_2(s,v_0)+\sin\theta\,(s)\,\Phi_3(s,v_0))=0. \end{aligned} \tag{3.7}$$

From (3.4), we have $\dfrac{\partial \Phi_1}{\partial s}(s,v_0)=0$.We using Eqn. (3.6) and Eqn.(2.4) in Eqn. (3.7), since $\kappa(s)\neq 0$, we get

$$\frac{\partial z(s,v_0)}{dv}=\tan\theta\,(s)\,\frac{\partial y(s,v_0)}{dv} \tag{3.8}$$

which completes the proof. □

Combining the conditions (3.4) and (3.8), we have found the necessary and sufficient conditions for the $\varphi(s,v)$ to have the Smarandache TN_1 curve of the curve is an isoasymptotic.

Theorem 3.2. *: Smarandache TN_2 curve of the curve $\alpha(s)$ is isoasymptotic on a surface $\varphi(s,v)$ if and only if the following conditions are satisfied:*

$$\begin{cases} x(s,v_0)=y(s,v_0)=z(s,v_0)=0, \\ \frac{\partial z}{\partial v}(s,v_0)=\tan\theta\,(s)\,\frac{\partial y}{\partial v}(s,v_0). \end{cases}$$

Proof: Let $\alpha(s)$ be a Smarandache TN_2 curve on surface $\varphi(s,v)$.From (3.1) , $\varphi(s,v)$ parametric surface is defined by a given Smarandache TN_2 curve of curve $\alpha(s)$ as follows:

$$\varphi(s,v)=\frac{1}{\sqrt{2}}\,(T(s)+N_2(s))+[x(s,v)T(s)+y(s,v)N_1(s)+z(s,v)N_2(s)].$$

If Smarandache TN_2 curve is an parametric curve on this surface , then there exist a parameter $v=v_0$ such that , $\frac{1}{\sqrt{2}}\,(T(s)+N_2(s))=\varphi(s,v_0)$, that is,

$$x(s,v_0)=y(s,v_0)=z(s,v_0)=0 \tag{3.9}$$

The normal vector can be expressed as

$$\begin{aligned} n(s,v) \;\;=\;\; & [\frac{\partial z(s,v)}{dv}(\frac{k_1(s)}{\sqrt{2}}+\frac{\partial y(s,v)}{ds}+k_1(s)x(s,v)) \\ & -\frac{\partial y(s,v)}{dv}(\frac{k_2(s)}{\sqrt{2}}+\frac{\partial z(s,v)}{ds}+k_2(s)x(s,v))]T(s) \\ & +[\frac{\partial x(s,v)}{dv}(\frac{k_2(s)}{\sqrt{2}}+\frac{\partial z(s,v)}{ds}+k_2(s)x(s,v)) \\ & -\frac{\partial z(s,v)}{dv}(-\frac{k_2(s)}{\sqrt{2}}+\frac{\partial x(s,v)}{ds}-k_1(s)y(s,v)-k_2(s)z(s,v))]N_1(s) \\ & +[\frac{\partial y(s,v)}{dv}(-\frac{k_2(s)}{\sqrt{2}}+\frac{\partial x(s,v)}{ds}-k_1(s)y(s,v)-k_2(s)z(s,v)) \\ & -\frac{\partial x(s,v)}{dv}(\frac{k_1(s)}{\sqrt{2}}+\frac{\partial y(s,v)}{ds}+k_1(s)x(s,v))]N_2(s) \end{aligned} \tag{3.10}$$

Using (3.9), if we let

$$\begin{cases} \Phi_1(s,v_0) = \frac{k_1(s)}{\sqrt{2}}\frac{\partial z}{dv}(s,v_0) - \frac{k_2(s)}{\sqrt{2}}\frac{\partial y}{dv}(s,v_0), \\ \Phi_2(s,v_0) = \frac{k_2(s)}{\sqrt{2}}\frac{\partial z}{dv}(s,v_0) + \frac{k_2(s)}{\sqrt{2}}\frac{\partial x}{dv}(s,v_0), \\ \Phi_3(s,v_0) = -\frac{k_2(s)}{\sqrt{2}}\frac{\partial y}{dv}(s,v_0) - \frac{k_1(s)}{\sqrt{2}}\frac{\partial x}{dv}(s,v_0). \end{cases} \tag{3.11}$$

we obtain

$$n(s,v_0) = \Phi_1(s,v_0)T(s) + \Phi_2(s,v_0)N_1(s) + \Phi_3(s,v_0)N_2(s).$$

From Eqn. (2.2), we get

$$\begin{aligned} n(s,v_0) &= \Phi_1(s,v_0)T(s) + \cos\theta\,(s)\,\Phi_2(s,v_0) + \sin\theta\,(s)\,\Phi_3(s,v_0))N(s) \\ &\quad +(\cos\theta\,(s)\,\Phi_3(s,v_0) - \sin\theta\,(s)\,\Phi_2(s,v_0))B(s). \end{aligned}$$

From the eqn. (3.2), we should have

$$\begin{aligned} &\frac{\partial n}{\partial s}(s,v_0).T(s) = 0 \\ &\Leftrightarrow \tfrac{\partial(\cos\theta(s)\Phi_2(s,v_0)+\sin\theta(s)\Phi_3(s,v_0))N(s)+(\cos\theta(s)\Phi_3(s,v_0)-\sin\theta(s)\Phi_2(s,v_0))B(s))}{\partial s}.T(s) = 0 \\ &\Leftrightarrow \frac{\partial \Phi_1}{\partial s}(s,v_0) - \kappa(s)\left(\cos\theta\,(s)\,\Phi_2(s,v_0) + \sin\theta\,(s)\,\Phi_3(s,v_0)\right) = 0. \end{aligned} \tag{3.12}$$

From (3.9), we have $\dfrac{\partial \Phi_1}{\partial s}(s,v_0) = 0$.We using Eqn. (3.11) and Eqn.(2.4) in Eqn. (3.12), since $\kappa(s) \neq 0$, we get

$$\frac{\partial z(s,v_0)}{dv} = \tan\theta\,(s)\,\frac{\partial y(s,v_0)}{dv} \tag{3.13}$$

which completes the proof. □

Theorem 3.3. *: Smarandache N_1N_2 curve of the curve $\alpha(s)$ is isoasymptotic on a surface $\varphi(s,v)$ if and only if the following conditions are satisfied:*

$$\begin{cases} x(s,v_0) = y(s,v_0) = z(s,v_0) = 0, \\ k_1\,(s) + k_2\,(s) \neq 0, \\ \frac{\partial z}{\partial v}(s,v_0) = \tan\theta\,(s)\,\frac{\partial y}{\partial v}(s,v_0). \end{cases}$$

Proof: Let $\alpha(s)$ be a Smarandache N_1N_2 curve on surface $\varphi(s,v)$.From (3.1) , $\varphi(s,v)$ parametric surface is defined by a given Smarandache N_1N_2 curve of curve $\alpha(s)$ as follows:

$$\varphi(s,v) = \frac{1}{\sqrt{2}}\,(N_1(s) + N_2(s)) + [x(s,v)T(s) + y(s,v)N_1(s) + z(s,v)N_2(s)].$$

If Smarandache N_1N_2 curve is an parametric curve on this surface , then there exist a parameter $v = v_0$ such that , $\frac{1}{\sqrt{2}}(N_1(s) + N_2(s)) = \varphi(s, v_0)$, that is,

$$x(s, v_0) = y(s, v_0) = z(s, v_0) = 0 \tag{3.14}$$

The normal vector can be expressed as

$$\begin{aligned} n(s,v) &= [\frac{\partial z(s,v)}{dv}(\frac{\partial y(s,v)}{ds} + k_1(s)x(s,v)) - \frac{\partial y(s,v)}{dv}(\frac{\partial z(s,v)}{ds} \\ &\quad + k_2(s)x(s,v))]T(s) + [\frac{\partial x(s,v)}{dv}(\frac{\partial z(s,v)}{ds} + k_2(s)x(s,v)) \\ &\quad - \frac{\partial z(s,v)}{dv}(-\frac{k_1(s)+k_2(s)}{\sqrt{2}} + \frac{\partial x(s,v)}{ds} - k_1(s)y(s,v) \\ &\quad - k_2(s)z(s,v))]N_1(s) + [\frac{\partial y(s,v)}{dv}(-\frac{k_1(s)+k_2(s)}{\sqrt{2}} + \frac{\partial x(s,v)}{ds} \\ &\quad - k_1(s)y(s,v) - k_2(s)z(s,v)) \\ &\quad - \frac{\partial x(s,v)}{dv}(\frac{\partial y(s,v)}{ds} + k_1(s)x(s,v))]N_2(s) \end{aligned} \tag{3.15}$$

Using (3.14), if we let

$$\begin{cases} \Phi_1(s, v_0) = 0, \\ \Phi_2(s, v_0) = \left(\frac{k_1(s)+k_2(s)}{\sqrt{2}}\right)\frac{\partial z}{dv}(s, v_0), \\ \Phi_3(s, v_0) = -\left(\frac{k_1(s)+k_2(s)}{\sqrt{2}}\right)\frac{\partial y}{dv}(s, v_0). \end{cases} \tag{3.16}$$

we obtain

$$n(s, v_0) = \Phi_2(s, v_0)N_1(s) + \Phi_3(s, v_0)N_2(s).$$

From Eqn. (2.2), we get

$$\begin{aligned} n(s, v_0) &= \cos\theta(s)\,\Phi_2(s, v_0) + \sin\theta(s)\,\Phi_3(s, v_0))N(s) \\ &\quad + (\cos\theta(s)\,\Phi_3(s, v_0) - \sin\theta(s)\,\Phi_2(s, v_0))B(s). \end{aligned}$$

From the eqn. (3.2), we should have

$$\begin{aligned} &\frac{\partial n}{\partial s}(s, v_0).T(s) = 0 \\ &\Leftrightarrow \frac{\partial(\cos\theta(s)\Phi_2(s,v_0)+\sin\theta(s)\Phi_3(s,v_0))N(s)+(\cos\theta(s)\Phi_3(s,v_0)-\sin\theta(s)\Phi_2(s,v_0))B(s))}{\partial s}.T(s) = 0 \\ &\Leftrightarrow -\kappa(s)\left(\cos\theta(s)\,\Phi_2(s, v_0) + \sin\theta(s)\,\Phi_3(s, v_0)\right) = 0. \end{aligned} \tag{3.17}$$

We using Eqn. (3.16) and Eqn.(2.4) in Eqn. (3.17), since $\kappa(s) \neq 0$, we get

$$\cos\theta(s)\left(\frac{k_1(s)+k_2(s)}{\sqrt{2}}\right)\frac{\partial z(s, v_0)}{dv} = \sin\theta(s)\left(\frac{k_1(s)+k_2(s)}{\sqrt{2}}\right)\frac{\partial y(s, v_0)}{dv}$$

For , $k_1(s)+k_2(s)\neq 0$ we obtain

$$\frac{\partial z(s,v_0)}{dv}=\tan\theta(s)\frac{\partial y(s,v_0)}{dv} \tag{3.18}$$

which completes the proof. □

Theorem 3.4. *Smarandache TN_1N_2 curve of the curve $\alpha(s)$ is isoasymptotic on a surface $\varphi(s,v)$ if and only if the following conditions are satisfied:*

$$\begin{cases} x(s,v_0)=y(s,v_0)=z(s,v_0)=0, \\ k_1(s)+k_2(s)\neq 0, \\ \frac{\partial z}{\partial v}(s,v_0)=\tan\theta(s)\frac{\partial y}{\partial v}(s,v_0). \end{cases}$$

Proof: Let $\alpha(s)$ be a Smarandache TN_1N_2 curve on surface $\varphi(s,v)$.From (3.1) , $\varphi(s,v)$ parametric surface is defined by a given Smarandache TN_1N_2 curve of curve $\alpha(s)$ as follows:

$$\varphi(s,v)=\frac{1}{\sqrt{2}}(T(s)+N_1(s)+N_2(s))+[x(s,v)T(s)+y(s,v)N_1(s)+z(s,v)N_2(s)].$$

If Smarandache TN_1N_2 curve is an parametric curve on this surface , then there exist a parameter $v=v_0$ such that , $\frac{1}{\sqrt{2}}(T(s)+N_1(s)+N_2(s))=\varphi(s,v_0)$, that is,

$$x(s,v_0)=y(s,v_0)=z(s,v_0)=0 \tag{3.19}$$

The normal vector can be expressed as

$$\begin{aligned} n(s,v) \;=\; & [\frac{\partial z(s,v)}{dv}(\frac{k_1(s)}{\sqrt{3}}+\frac{\partial y(s,v)}{ds}+k_1(s)x(s,v)) \\ & -\frac{\partial y(s,v)}{dv}(\frac{k_2(s)}{\sqrt{3}}+\frac{\partial z(s,v)}{ds}+k_2(s)x(s,v))]T(s) \\ & +[\frac{\partial x(s,v)}{dv}(\frac{\partial z(s,v)}{ds}+k_2(s)x(s,v)+\frac{k_2(s)}{\sqrt{3}}) \\ & -\frac{\partial z(s,v)}{dv}(-\frac{k_1(s)+k_2(s)}{\sqrt{3}}+\frac{\partial x(s,v)}{ds}-k_1(s)y(s,v) \\ & -k_2(s)z(s,v))]N_1(s) \\ & +[\frac{\partial y(s,v)}{dv}(-\frac{k_1(s)+k_2(s)}{\sqrt{3}}+\frac{\partial x(s,v)}{ds}-k_1(s)y(s,v)-k_2(s)z(s,v)) \\ & -\frac{\partial x(s,v)}{dv}(\frac{k_1(s)}{\sqrt{3}}+\frac{\partial y(s,v)}{ds}+k_1(s)x(s,v))]N_2(s) \end{aligned} \tag{3.20}$$

Using (3.19), if we let

$$\begin{cases} \Phi_1(s,v_0)=\frac{k_1(s)}{\sqrt{3}}\frac{\partial z}{dv}(s,v_0)-\frac{k_2(s)}{\sqrt{3}}\frac{\partial y}{dv}(s,v_0), \\ \Phi_2(s,v_0)=\frac{k_2(s)}{\sqrt{3}}\frac{\partial x}{dv}(s,v_0)+\left(\frac{k_1(s)+k_2(s)}{\sqrt{3}}\right)\frac{\partial z}{dv}(s,v_0), \\ \Phi_3(s,v_0)=-\frac{k_1(s)}{\sqrt{3}}\frac{\partial x}{dv}(s,v_0)-\left(\frac{k_1(s)+k_2(s)}{\sqrt{3}}\right)\frac{\partial y}{dv}(s,v_0). \end{cases} \tag{3.21}$$

we obtain

$$n(s,v_0) = \Phi_1(s,v_0)T(s) + \Phi_2(s,v_0)N_1(s) + \Phi_3(s,v_0)N_2(s).$$

From Eqn. (2.2), we get

$$\begin{aligned} n(s,v_0) &= \Phi_1(s,v_0)T(s) + \cos\theta\,(s)\,\Phi_2(s,v_0) + \sin\theta\,(s)\,\Phi_3(s,v_0))N(s) \\ &\quad +(\cos\theta\,(s)\,\Phi_3(s,v_0) - \sin\theta\,(s)\,\Phi_2(s,v_0))B(s). \end{aligned}$$

From the Eqn. (3.2), we should have

$$\begin{aligned} &\frac{\partial n}{\partial s}(s,v_0).T(s) = 0 \\ &\Leftrightarrow \tfrac{\partial(\cos\theta(s)\Phi_2(s,v_0)+\sin\theta(s)\Phi_3(s,v_0))N(s)+(\cos\theta(s)\Phi_3(s,v_0)-\sin\theta(s)\Phi_2(s,v_0))B(s))}{\partial s}.T(s) = 0 \\ &\Leftrightarrow -\kappa(s)\,(\cos\theta\,(s)\,\Phi_2(s,v_0) + \sin\theta\,(s)\,\Phi_3(s,v_0)) = 0. \end{aligned} \tag{3.22}$$

From (3.19) , we have $\frac{\partial\Phi_1}{\partial s}(s,v_0) = 0$.We using Eqn. (3.21) and Eqn.(2.4) in Eqn. (3.22), since $\kappa(s) \neq 0$, we get

$$\cos\theta\,(s)\left(\frac{k_1(s)+k_2(s)}{\sqrt{3}}\right)\frac{\partial z(s,v_0)}{dv} = \sin\theta\,(s)\left(\frac{k_1(s)+k_2(s)}{\sqrt{3}}\right)\frac{\partial y(s,v_0)}{dv}$$

For , $k_1(s) + k_2(s) \neq 0$ we obtain

$$\frac{\partial z(s,v_0)}{dv} = \tan\theta\,(s)\,\frac{\partial y(s,v_0)}{dv} \tag{3.23}$$

which completes the proof. □

Now let us consider other types of the marching-scale functions. In the Eqn. (3.1) marching-scale functions $x(s,v)$, $y(s,v)$ and $z(s,v)$ can be choosen in two different forms:

1) If we choose

$$\left\{\begin{array}{c} x(s,v) = \sum\limits_{k=1}^{p} a_{1k}l(s)^k x(v)^k, \\ y(s,v) = \sum\limits_{k=1}^{p} a_{2k}m(s)^k y(v)^k, \\ z(s,v) = \sum\limits_{k=1}^{p} a_{3k}n(s)^k z(v)^k. \end{array}\right.$$

then we can simply express the sufficient condition for which the curve $\alpha(s)$ is an Smarandache asymptotic curve on the surface $\varphi(s,v)$ as

$$\left\{\begin{array}{l} x(v_0) = y(v_0) = z(v_0) = 0, \\ a_{31}n(s)\frac{dz(v_0)}{dv} = \tan\theta\,(s)\,a_{21}m(s)\frac{dy(v_0)}{dv}. \end{array}\right. \tag{3.24}$$

where $l(s)$, $m(s)$, $n(s)$, $x(v)$, $y(v)$ and $z(v)$ are C^1 functions, $a_{ij}\epsilon\ IR$, $i = 1,2,3,\ j = 1,2,...,p$.

2) If we choose

$$\begin{cases} x(s,v) = f\left(\sum\limits_{k=1}^{p} a_{1k} l(s)^k x(v)^k\right), \\ y(s,v) = g\left(\sum\limits_{k=1}^{p} a_{2k} m(s)^k y(v)^k\right), \\ z(s,v) = h\left(\sum\limits_{k=1}^{p} a_{3k} n(s)^k z(v)^k\right). \end{cases}$$

then we can write the sufficient condition for which the curve $\alpha(s)$ is an Smarandache asymptotic curve on the surface $\varphi(s,v)$ as

$$\begin{cases} x(v_0) = y(v_0) = z(v_0) = f(0) = g(0) = h(0) = 0, \\ a_{31} n(s) \frac{dz(v_0)}{dv} h'(0) = \tan\theta\,(s)\, a_{21} m(s) \frac{dy(v_0)}{dv} g'(0). \end{cases} \tag{3.25}$$

where $l(s)$, $m(s)$, $n(s)$, $x(v)$, $y(v)$, $z(v)$, f, g and h are C^1 functions.

Also conditions for different types of marching-scale functions can be obtained by using the Eqn. (3.4) and (3.9).

4. Examples of generating simple surfaces with common Smarandache asymptotic curve

Example 4.1. *Let* $\alpha(s) = \left(\frac{4}{5}\cos(s), -\frac{3}{5}\cos(s), 1-\sin(s)\right)$ *be a unit speed curve. Then it is easy to show that*

$T(s) = \left(-\frac{4}{5}\sin(s), \frac{3}{5}\sin(s), -\cos(s)\right),\quad \kappa = 1\ ,\ \tau = 0.$

From Eqn.(2.3), $\tau\,(s) = -\frac{d\theta(s)}{ds} \Rightarrow \theta\,(s) = c, c = cons\tan t$. Here $c = \frac{\pi}{4}$ can be taken.

From Eqn. (2.4),$k_1 = \frac{1}{2}, k_2 = \frac{\sqrt{3}}{2}$.

From Eqn. (2.1),$N_1 = -\int k_1 T$, $N_2 = -\int k_2 T$.

$N_1(s) = \left(\frac{2}{5}\cos(s), -\frac{3}{5}\cos(s), \frac{1}{2}\sin(s)\right)$,

$N_2(s) = \left(\frac{2\sqrt{3}}{5}\cos(s), \frac{3\sqrt{3}}{10}\cos(s), \frac{\sqrt{3}}{2}\sin(s)\right)$.

If we take $x(s,v) = 0,\ y(s,v) = e^v - 1,\ z(s,v) = \sqrt{3}\,(e^v - 1)$,we obtain a member of the surface with common curve $\alpha(s)$ as

$\varphi_1(s,v) = \left(\frac{4}{5}\cos(s)\,(1 + 2(e^v - 1)), -\frac{3}{5}\cos(s)\left(1 - \frac{1}{2}\,(e^v - 1)\right), 1 - \sin(s)\,(1 + 2\,(e^v - 1))\right)$

where $0 \le s \le 2\pi$, $-1 \le v \le 1$ (Fig. 1).

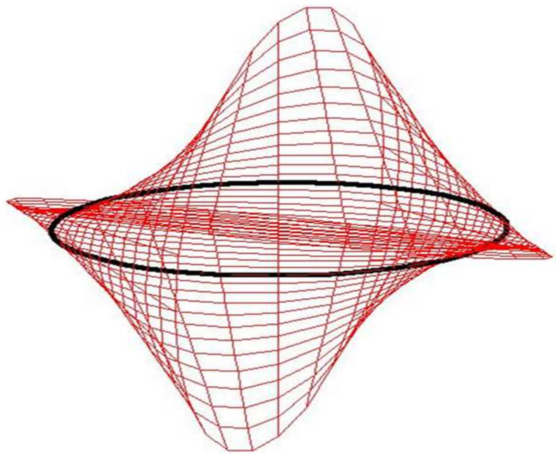

Figure 1: $\varphi_1(s,v)$ as a member of surfaces and curve $\alpha(s)$.

If we take $x(s,v)=0,\ y(s,v)=e^v-1,\ z(s,v)=\sqrt{3}\,(e^v-1)$and $v_0=0$ then the Eqns.(3.4) and (3.7) are satisfied. Thus, we obtain a member of the surface with common Smarandache TN_1 asymptotic curve as

$$\varphi_2(s,v)=\begin{pmatrix} \frac{2}{5\sqrt{2}}(-2\sin(s)+\cos(s))+\frac{8}{5}\cos(s)(e^v-1),\ \frac{3}{5\sqrt{2}}(\sin(s)-\cos(s))+\frac{3}{10}\cos(s)(e^v-1), \\ \frac{1}{\sqrt{2}}(\frac{1}{2}\sin(s)-\cos(s))+2\sin(s)(e^v-1) \end{pmatrix}$$

where $0\le s\le 2\pi$, $-1\le v\le 1$ (Fig. 2).

A member of the surface with common Smarandache TN_2 asymptotic curve as

$$\varphi_3(s,v)=\begin{pmatrix} -\frac{2}{5\sqrt{2}}(2\sin(s)-\sqrt{3}\cos(s))+\frac{8}{5}\cos(s)(e^v-1), \\ \frac{3}{5\sqrt{2}}(\sin(s)+\frac{\sqrt{3}}{2}\cos(s))+\frac{3}{10}\cos(s)(e^v-1), \\ \frac{1}{\sqrt{2}}(\frac{\sqrt{3}}{2}\sin(s)-\cos(s))+2\sin(s)(e^v-1) \end{pmatrix}$$

where $0\le s\le 2\pi$, $-1\le v\le 1$ (Fig. 3).

A member of the surface with common Smarandache N_1N_2 asymptotic curve as

$$\varphi_4(s,v)=\begin{pmatrix} \frac{2}{5\sqrt{2}}\cos(s)(1+\sqrt{3})+\frac{8}{5}\cos(s)(e^v-1),-\frac{3}{5\sqrt{2}}\cos(s)(1+\frac{\sqrt{3}}{2})+\frac{3}{10}\cos(s)(e^v-1), \\ \frac{1}{2\sqrt{2}}\sin(s)(1+\sqrt{3})+2\sin(s)(e^v-1) \end{pmatrix}$$

where $0\le s\le 2\pi$, $-1\le v\le 1$ (Fig. 4).

A member of the surface and its Smarandache TN_1N_2 asymptotic curve as

$$\varphi_5(s,v)=\begin{pmatrix} \frac{2}{5\sqrt{3}}(-2\sin(s)+\cos(s)+\sqrt{3}\cos(s))+\frac{8}{5}\cos(s)(e^v-1), \\ -\frac{3}{5\sqrt{3}}(\sin(s)-\cos(s)+\frac{\sqrt{3}}{2})+\frac{3}{10}\cos(s)(e^v-1), \\ \frac{1}{\sqrt{3}}(-\cos(s)+\frac{1}{2}\sin(s)(1+\sqrt{3}))+2\sin(s)(e^v-1) \end{pmatrix}$$

where $0\le s\le 2\pi$, $-1\le v\le 1$ (Fig. 5).

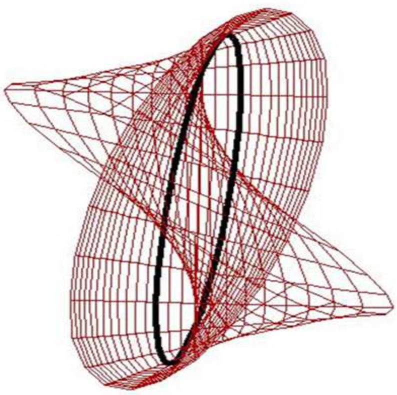

Figure 2: $\varphi_2(s, v)$ as a member of surfaces and its Smarandache TN_1 asymptotic curve of $\alpha(s)$.

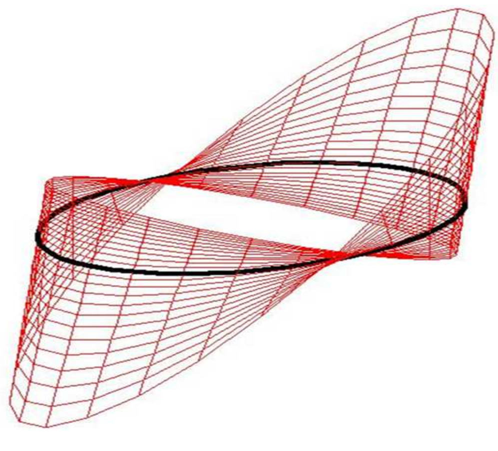

Figure 3: $\varphi_3(s, v)$ as a member of surfaces and its Smarandache TN_2 asymptotic curve of $\alpha(s)$.

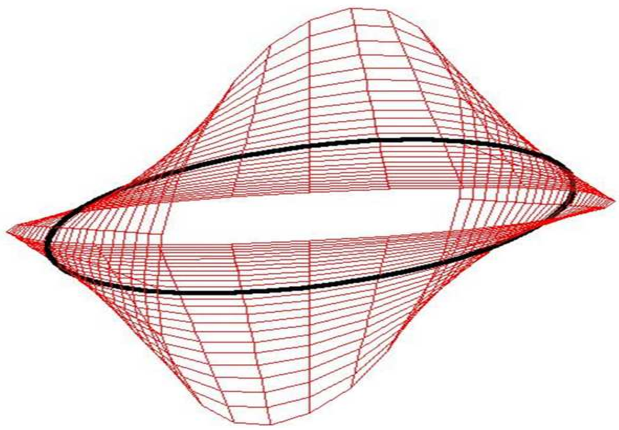

Figure 4: $\varphi_4(s,v)$ as a member of surfaces and its Smarandache N_1N_2 asymptotic curve of $\alpha(s)$.

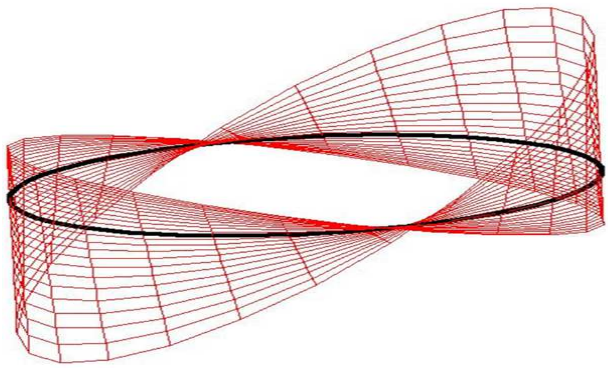

Figure 5: $\varphi_5(s,v)$ as a member of surfaces and its Smarandache TN_1N_2 asymptotic curve of $\alpha(s)$.

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