symmetry

MDPI

Article

Symbolic 4-Plithogenic Rings and 5-Plithogenic Rings

Ahmed Hatip

Department of Mathematics, Gaziantep University, Gaziantep 27310, Turkey; kollnaar5@gmail.com

Abstract: Symbolic n-plithogenic algebraic structures are considered as symmetric generalizations of classical algebraic structures because they have n + 1 symmetric components. This paper is dedicated to generalizing symbolic 3-plithogenic rings by defining symbolic 4-plithogenic rings and 5-plithogenic rings; these new classes of n-symbolic plithogenic algebraic structures will be defined for the first time, and their algebraic substructures will be studied. AH structures are considered to be a sign of the presence of symmetry within these types of ring, as they consist of several parts that are similar in structure and symmetrical, and when combined with each other, they have a broader structure resembling the classical consonant structure. Many related substructures will be presented such as 4-plithogenic/5-plithogenic AH-ideals, 4-plithogenic/5-plithogenic AH-homomorphisms, and 4-plithogenic AHS-isomorphisms will be discussed. We will show our results in terms of theorems, with many clear numerical examples that explain the novelty of this work.

Keywords: symbolic 4-plithogenic ring; 4-plithogenic ah-ideal; 4-plithogenic ah-homomorphism; symbolic 5-plithogenic ring; 5-plithogenic ah-ideal; 5-plithogenic ah-homomorphism

check for updates

Citation: Hatip, A. Symbolic 4-Plithogenic Rings and 5-Plithogenic Rings. *Symmetry* **2023**, *15*, 1588. https://doi.org/10.3390/sym15081588

Academic Editors: Emilio Acerbi, Jaume Giné and Sergei D. Odintsov

Received: 25 June 2023
Revised: 3 August 2023
Accepted: 8 August 2023
Published: 15 August 2023

1. Introduction

One of the most attractive concepts for mathematicians is algebraic structures due to their analog properties and close relationship with other branches of mathematics, such as geometry and matrix theory [1,2].

During the last two years, researchers have become interested in studying symbolic n-plithogenic algebraic structures. These structures were supposed by Smarandache in [3] as novel generalizations of classical algebraic structures that have symmetric logical elements combined with algebraic elements. These algebraic structures have been constructed in a manner similar to their analogues using neutrosophic logic, where it is possible to clearly see that the method that was used to construct the neutrosophic structures [4,5], the split-complex numbers [6,7], and the weak fuzzy numbers [8] was used in the extension of algebraic rings by plithogenic sets.

For the case of n = 2, we find many studies that deal with corresponding plithogenic structures. In [9], Merkeci et al. defined the symbolic 2-plithogenic ring and studied its elementary properties and substructures, such as AH-ideals, AH-homomorphisms, and kernels. Laterally, their results were used by Taffach and other authors to define and study symbolic 2-plithogenic vector spaces [10], symbolic 2-plithogenic modules [11], and symbolic 2-plithogenic number theory [12]. A wide review of symbolic 2-plithogenic algebraic structures is provided in [13,14].

This is what prompted other researchers to generalize the previous results to the symbolic 3-plithogenic case. In [15], symbolic 3-plithogenic rings were handled for the first time by Albasheer, et al.; then, symbolic 3-plithogenic vector spaces, modules, and number theoretical concepts were defined and studied (see [16–19]).

This is what prompted us to follow up the previous scientific efforts and to study 4-plithogenic rings for the first time by providing basic definitions and proofs that describe the algebraic behavior of the elements of these rings. It is noteworthy that these new rings will be very useful in more extensive classes of algebraic modules and vector spaces, and also cryptographic algorithms.

We study AH-substructures because they have a symmetric structure, where the components of an AH-ideal are classical ideals, and the components of the AH-homomorphism are classical homomorphisms. These symmetric structures of AH-substructures gives us many interesting results about the algebraic class of symbolic 4-plithogenic/5-plithogenic rings.

2. Main Discussion

Definition 1. *Let R be a ring; the symbolic 4-plithogenic ring is:*

$4-SP_R = \left\{a_0 + a_1P_1 + a_2P_2 + a_3P_3 + a_4P_4;\ a_i \in R,\ P_j^{\,2} = P_j,\ P_i \times P_j = P_{max(i,j)}\right\}$.

Operations on $4-SP_R$*:*

Addition:

$[a_0 + a_1P_1 + a_2P_2 + a_3P_3 + a_4P_4] + [b_0 + b_1P_1 + b_2P_2 + b_3P_3 + b_4P_4] =$
$(a_0 + b_0) + (a_1 + b_1)P_1 + (a_2 + b_2)P_2 + (a_3 + b_3)P_3 + (a_4 + b_4)P_4$.

Multiplication:

$[a_0 + a_1P_1 + a_2P_2 + a_3P_3 + a_4P_4]\cdot[b_0 + b_1P_1 + b_2P_2 + b_3P_3 + b_4P_4] =$
$a_0b_0 + (a_0b_1 + a_1b_0 + a_1b_1)P_1 + (a_0b_2 + a_1b_2 + a_2b_0 + a_2b_1 + a_2b_2)P_2+$
$(a_0b_3 + a_1b_3 + a_2b_3 + a_3b_3 + a_3b_0 + a_3b_1 + a_3b_2)P_3+$
$(a_0b_4 + a_1b_4 + a_2b_4 + a_3b_4 + a_4b_0 + a_4b_1 + a_4b_2 + a_4b_3 + a_4b_4)P_4$.

It is clear that ($4-SP_R$*) is a ring.*

If R is commutative, then $4-SP_R$ *is commutative, and if R has a unity, then* $4-SP_R$ *has the same unity.*

Example 1. *Consider the ring* $R = Z_3 = \{0,1,2\}$*, the corresponding* $4-SP_R$ *is:*

$4-SP_R = \{a + bP_1 + cP_2 + dP_3 + eP_4; a,b,c,d,e \in Z_3\}$.

If $X = 2 + 2P_1 + 1P_2 + P_3 + P_4,\ Y = P_2 + 2P_3$*, then:*

$X + Y = 2 + 2P_1 + 2P_2 + P_4$,
$X - Y = 2 + 2P_1 - P_3 + P_4$,
$X\cdot Y = 2P_2 + 4P_3 + 2P_2 + 4P_3 + P_2 + 2P_3 + P_3 + 2P_3 + P_4 + 2P_4 = 5P_2 + 13P_3 + 3P_4$.

Theorem 1. *Let* $4-SP_R$ *be a 4-plithogenic symbolic ring, with unity.*

Let $X = e_0 + e_1P_1 + e_2P_2 + e_3P_3 + e_4P_4$ *be an arbitrary element; then:*

1. *X is invertible if and only if* $e_0, e_0 + e_1, e_0 + e_1 + e_2, e_0 + e_1 + e_2 + e_3, e_0 + e_1 + e_2 + e_3 + e_4$ *are invertible.*
2. $X^{-1} = e_0^{-1} + [(e_0 + e_1)^{-1} - e_0^{-1}]P_1 + [(e_0 + e_1 + e_2)^{-1} - (e_0 + e_1)^{-1}]P_2 + [\ (e_0 + e_1 + e_2 + e_3)^{-1} - (e_0 + e_1 + e_2)^{-1}]P_3 + [(e_0 + e_1 + e_2 + e_3 + e_4)^{-1} - (e_0 + e_1 + e_2 + e_3)^{-1}]\ P_4$.

Proof.

1. Assume that X is invertible, then there exists $Y = n_0 + n_1P_1 + n_2P_2 + n_3P_3 + n_4P_4$ such that $X\cdot Y = 1$; hence:

$$\begin{cases} e_0n_3 + e_1n_3 + e_2n_3 + e_3n_3 + e_3n_1 + e_3n_2 + e_3n_0 = 0 \text{ (1a)} \\ e_0n_0 = 1\ldots \text{ (1b)} \\ e_0n_1 + e_1n_0 + e_1n_1 = 0\ldots \text{ (1c)} \\ e_0n_2 + e_2n_0 + e_2n_2 + e_1n_2 + e_2n_1 = 0\ldots \text{ (1d)}, \\ e_0n_4 + e_1n_4 + e_2n_4 + e_3n_4 + e_4n_0 + e_4n_1 + e_4n_3 + e_4n_4 \text{(1e)} \end{cases} \quad (1)$$

From (1b), e_0 is invertible.
By adding (1c) to (1b), we obtain $(e_0 + e_1)(n_0 + n_1) = 1$; thus, $e_0 + e_1$ is invertible.
By adding (1d) to (1c) to (1b), $(e_0 + e_1 + e_2)(n_0 + n_1 + n_2) = 1$; hence, $e_0 + e_1 + e_2$ is invertible.
By adding (1a) to (1b) to (1c) to (1d),
$(e_0 + e_1 + e_2 + e_3)(n_0 + n_1 + n_2 + n_3) = 1$; hence, $e_0 + e_1 + e_2 + e_3$ is invertible.

Adding all equations gives:
$(e_0+e_1+e_2+e_3+e_4)(n_0+n_1+n_2+n_3+n_4) = 1$; hence, $e_0+e_1+e_2+e_3+e_4$ is invertible.

2. From the first part, we have:
$n_0 = {e_0}^{-1}, n_0+n_1 = (e_0+e_1)^{-1}, n_0+n_1+n_2 = (e_0+e_1+e_2)^{-1}$, $(e_0+e_1+e_2+e_3)^{-1}$ $= n_0+n_1+n_2+n_3, (e_0+e_1+e_2+e_3+e_4)^{-1} = n_0+n_1+n_2+n_3+n_4$; then:
$Y = {e_0}^{-1} + \left[(e_0+e_1)^{-1} - {e_0}^{-1}\right]P_1 + \left[(e_0+e_1+e_2)^{-1} - (e_0+e_1)^{-1}\right]P_2 +$
$\left[(e_0+e_1+e_2+e_3)^{-1} - (e_0+e_1+e_2)^{-1}\right]P_3 +$
$\left[(e_0+e_1+e_2+e_3+e_4)^{-1} - (e_0+e_1+e_2+e_3)^{-1}\right]P_4 = X^{-1}$. □

Example 2. *Take* $R = Z_3 = \{0,1,2\}$, $4-SP_{Z_3}$ *is the corresponding symbolic 4-plithogenic ring; consider* $X = 2+2P_2+P_4 \in 4-SP_{Z_3}$; *then:*
$e_0 = 2$ *is invertible with* ${e_0}^{-1} = 2$, $e_0+e_1 = 2$ *is invertible with* $(e_0+e_1)^{-1} = 2$,
$e_0+e_1+e_2 = 1$ *is invertible with* $(e_0+e_1+e_2)^{-1} = 1$,
$e_0+e_1+e_2+e_3 = 1$, $(e_0+e_1+e_2+e_3)^{-1} = 1$,
$e_0+e_1+e_2+e_3+e_4 = 2$, $(e_0+e_1+e_2+e_3+e_4)^{-1} = 2$ *hence:*
$X^{-1} = 2+(2-2)P_1+(1-2)P_2+(1-1)P_3+(2-1)P_4 = 2+2P_2+P_4$.

Definition 2. *If* $X = m+nP_1+cP_2+qP_3+lP_4 \in 4-SP_R$, *then X is idempotent if and only if* $X^2 = X$.

Theorem 2. *If* $X = m+nP_1+cP_2+qP_3+lP_4 \in 4-SP_R$, *then X is idempotent if and only if* $m, m+n, m+n+cm+n+c+qm+n+c+q+l$ *are idempotent.*

Proof.
$X^2 = X{\cdot}X = (m+nP_1+cP_2+qP_3+lP_4)(m+nP_1+cP_2+qP_3+lP_4) =$
$m^2 + (mn+nm+n{\cdot}n)P_1 + (mc+nc+cm+cn+c{\cdot}c)P_2 +$
$(mq+nq+cq+qm+qn+qc+q{\cdot}q)P_3 + (ml+nl+cl+ql+lm+ln+lc+lq+l{\cdot}l)$
P_4.

$$X^2 = X{\cdot}X \text{ equivalents } \begin{cases} mq+nq+cq+qm+qn+qc+q{\cdot}q = q \ (2a) \\ m^2 = m \ldots \ (2b) \\ mn+nm+n{\cdot}n = n \ldots \ (2c) \\ mc+nc+cm+cn+c{\cdot}c = c \ldots \ (2d) \\ ml+nl+cl+ql+lm+ln+lc+lq+l{\cdot}l = l \ (2e) \end{cases} \quad (2)$$

Equation (2b) implies that m is idempotent.

By adding (2c) to (2b), we obtain $(m+n)^2 = m+n$; hence, $m+n$ is idempotent.

By adding (2c) to (2b) to (2d), we obtain $(m+n+c)^2 = m+n+c$; hence, $m+n+c$ is idempotent.

By adding (2a) to (2b) to (2c) to (2d), we obtain $(m+n+c+q)^2 = m+n+c+q$; thus, $m+n+c+q$ is idempotent. By adding all equations to each other, we obtain:

$(m+n+c+q+l)^2 = m+n+c+q+l$, thus $m+n+c+l$ is idempotent. Thus the proof is complete. □

Example 3. *Take* $R = Z_4 = \{0,1,2,3\}$, $4-SP_{Z_4}$ *is the corresponding symbolic 4-plithogenic ring, and consider* $X = P_1+3P_4 \in 4-SP_{Z_5}$; *thus, we have:*
$X^2 = P_1+9P_4+6P_4 = P_1+3P_4 = X$.

Theorem 3. *Let* $4-SP_R$ *be a commutative symbolic 4-plithogenic ring; hence, if* $X = m+nP_1+cP_2+qP_3+lP_4$, *then*

$X^n = m^n + [(m+n)^n - m^n]P_1 + [(m+n+c)^n - (m+n)^n]P_2 + [(m+n+c+q)^n - (m+n+c)^n]P_3 + [(m+n+c+q+l)^n - (m+n+c+q)^n]P_4$ *for every* $n \in Z^+$.

Proof. For $n = 1$, it holds easily. Assume that it is true for $n = k$, and prove it for $n = k+1$.

$X^{k+1} = X \cdot X^k =$

$(m + nP_1 + cP_2 + qP_3 + lP_4)(m^k + \left[(m+n)^k - m^k\right]P_1 + \left[(m+n+c)^k - (m+n)^k\right]P_2 + \left[(m+n+c+q)^k - (m+n+c)^k\right]P_3 + \left[(m+n+c+q+l)^k - (m+n+c+q)^k\right]P_4) =$

$m^{k+1} + \left[(m+n)^{k+1} - m^{k+1}\right]P_1 + \left[(m+n+c)^{k+1} - (m+n)^{k+1}\right]P_2 +$

$\left[(m+n+c+q)^{k+1} - (m+n+c)^{k+1}\right]P_3 +$

$\left[(m+n+c+q+l)^{k+1} - (m+n+c+q)^{k+1}\right]P_4.$

So, this proof is complete by induction. □

Definition 3. *Let* G_0, G_1, G_2, G_3, *and* G_4 *be ideals of the ring R and define the symbolic 4-plithogenic AH-ideal:*

$G = G_0 + G_1P_1 + G_2P_2 + G_3P_3 + G_4P_4 = \{w_0 + w_1P_1 + w_2P_2 + w_3P_3 + w_4P_4;\ w_i \in G_i\}$.

If $G_0 = G_1 = G_2 = G_3 = G_4$, *then G is called an AHS-ideal.*

Example 4. *Let* $R = Z$ *be the ring of integers; then,* $G_0 = 7Z$ *and* $G_1 = 11Z$ *are ideals of* R.

$G = \{7m + 7nP_1 + 11tP_2 + 11sP_3 + 11lP_4;\ m \cdot n \cdot t, s, l \in Z\}$ *is an AHS-ideal of* $4 - SP_Z$.

$M = \{11m + 11nP_1 + 11tP_2 + 11sP_3 + 11lP_4;\ m \cdot n \cdot t,\ s,\ l \in Z\}$ *is an AHS-ideal of* $4 - SP_Z$.

Theorem 4. *Let G be an AHS- ideal of* $4 - SP_R$*; thus,G is an ideal with an ordinary meaning.*

Proof. G can be written as $G = G_0 + G_0P_1 + G_0P_2 + G_0P_3 + G_0P_4$, where G_0 is an ideal of R.

It is clear that $(G, +)$ is a subgroup of $(4 - SP_R, +)$.

Let $F = f_0 + f_1P_1 + f_2P_2 + f_3P_3 + f_4P_4 \in 4 - SP_R$,

Then if $X = m + nP_1 + cP_2 + qP_3 + lP_4 \in G$, we have:

$F \cdot X = f_0m + (f_0n + f_1m + f_1n)P_1 + (f_0c + f_1c + f_2m + f_2n + f_2c)P_2 + (f_0q + f_1q + f_2q + f_3q + f_3m + f_3n + f_3c)P_3 + (f_0l + f_1l + f_2l + f_3l + f_4m + f_4n + f_4c + f_4q + f_4l)P_4 \in G$; thus, G is an ideal. □

Definition 4. *Let R and T be two rings;* $4 - SP_R$ *and* $4 - SP_T$ *are the corresponding symbolic 4-plithogenic rings. Let* $f_0, f_1, f_2, f_3, f_4 : R \to T$ *be ring homomorphisms; thus, we define the AH-homomorphism:*

$f : 4 - SP_R \to 4 - SP_T$ *such that:*

$f(m + nP_1 + cP_2 + qP_3 + lP_4) = f_0(m) + f_1(n)P_1 + f_2(c)P_2 + f_3(q)P_3 + f_4(l)P_4$

If $f_0 = f_1 = f_2 = f_3 = f_4$, *then f is called an AHS-homomorphism.*

Remark 1. *If* f_0, f_1, f_2, f_3, *and* f_4 *are isomorphisms, then f is called an AH-isomorphism.*

Example 5. *Take* $R = Z$, $T = Z_4$, $f_0, f_1 : R \to T$ *such that:*

$f_0(x) = x(mod\ 4), f_1(2) = 2x(mod\ 4)$. *It is clear that* f_0 *and* f_1 *are homomorphisms.*

We define $f : 4 - SP_R \to 4 - SP_T$, *where:*

$f(m + nP_1 + cP_2 + qP_3 + lP_4) = f_0(m) + f_1(n)P_1 + f_0(c)P_2 + f_1(q)P_3 + f_1(l)P_4 =$

$m(mod\ 4) + 2n(mod\ 4)P_1 + (c\ mod\ 4)P_2 + (2q\ mod\ 4)P_3 + (2l\ mod\ 4)P_4$,

Which is an AH-homomorphism.

Theorem 5. *Let* $f = f_0 + f_1P_1 + f_2P_2 + f_3P_3 + f_4P_4 : 4 - SP_R \to 4 - SP_T$ *be a mapping, then:*

1. *If f is an AHS-homomorphism, then f is a ring homomorphism.*
2. *If f is an AHS-homomorphism, then it is an isomorphism.*

Proof.

1. Assume that f is an AHS-homomorphism, then, $f_0 = f_1 = f_2 = f_3 = f_4$ are homomorphisms.

Let $X = d_0 + d_1P_1 + d_2P_2 + d_3P_3 + d_4P_4, Y = c_0 + c_1P_1 + c_2P_2 + c_3P_3 + c_4P_4 \in 4 - SP_R$, and we have:
$f(X+Y) =$
$f_0(d_0+c_0) + f_0(d_1+c_1)P_1 + f_0(d_2+c_2)P_2 + f_0(d_3+c_3)P_3 + f_0(d_4+c_4)P_4 = f(X) + f(Y)$
$f(X \cdot Y) = f_0(d_0c_0) + f_0(d_0c_1 + d_1c_0 + d_1c_1)P_1 + f_0(d_0c_2 + d_2c_0 + d_2c_2 + d_2c_1 + d_1c_2)P_2$
$+f_0(d_0c_3 + d_1c_3 + d_2c_3 + d_3c_3 + d_3c_1 + d_3c_0 + d_3c_2)P_3+$
$(d_0c_4 + d_1c_4 + d_2c_4 + d_3c_4 + d_4c_0 + d_4c_1 + d_4c_2 + d_4c_3 + d_4c_4)P_4$
$= f_0(d_0)f_0(c_0) + (f_0(d_0)f_0(c_1) + f_0(d_1)f_0(c_0) + f_0(d_1)f_0(c_1))P_1$
$+(f_0(d_0)f_0(c_2) + f_0(d_2)f_0(y_0) + f_0(d_2)f_0(c_2) + f_0(d_2)f_0(c_1)$
$+f_0(d_1)f_0(c_2))P_2$
$+[(f_0(d_0)f_0(c_3) + f_0(d_1)f_0(c_3) + f_0(d_2)f_0(c_3) + f_0(d_3)f_0(c_3)$
$+f_0(d_3)f_0(c_1) + f_0(d_3)f_0(c_2) + f_0(d_3)f_0(c_0))]P_3$
$+[(f_0(d_0)f_0(c_4) + f_0(d_1)f_0(c_4) + f_0(d_2)f_0(c_4) + f_0(d_3)f_0(c_4)$
$+f_0(d_4)f_0(c_1) + f_0(d_4)f_0(c_2) + f_0(d_4)f_0(c_0) + f_0(d_4)f_0(c_4)$
$+f_0(d_4)f_0(c_3))]P_4 =$
$[f_0(d_0) + f_0(d_1)P_1 + f_0(d_2)P_2 + f_0(d_3)P_3 + f_0(d_4)P_4][f_0(c_0) + f_0(c_1)P_1 + f_0(c_2)P_2+$
$f_0(c_3)P_3 + f_0(c_4)P_4] = f(X) \cdot f(Y)$.
This implies the proof.

2. Using a similar discussion, we obtain the desired proof. □

The Characterization Of Symbolic 4-Plithogenic Ideals

Theorem 6. *Let $Q_i; 0 \leq i \leq 4$ be ideals of the ring R; then:*
$Q = \{q_j^{(0)} + \left(q_k^{(1)} - q_j^{(0)}\right)P_1 + \left(q_s^{(2)} - q_k^{(1)}\right)P_2 + \left(q_l^{(3)} - q_s^{(2)}\right)P_3 + \left(q_t^{(4)} - q_l^{(3)}\right)P_4;$
$q^{(i)} \in Q_i, j, k, l, s, t \in I\}$ *is an ideal of* $4 - SP_R$.

Proof.
It is clear that Q is non-empty set.
Let $X = q_{j_1}^{(0)} + \left(q_{k_1}^{(1)} - q_{j_1}^{(0)}\right)P_1 + \left(q_{s_1}^{(2)} - q_{k_1}^{(1)}\right)P_2 + \left(q_{l_1}^{(3)} - q_{s_1}^{(2)}\right)P_3 + \left(q_{t_1}^{(4)} - q_{l_1}^{(3)}\right)P_4$
$Y = q_{j_2}^{(0)} + \left(q_{k_2}^{(1)} - q_{j_2}^{(0)}\right)P_1 + \left(q_{s_2}^{(2)} - q_{k_2}^{(1)}\right)P_2 + \left(q_{l_2}^{(3)} - q_{s_2}^{(2)}\right)P_3 + \left(q_{t_2}^{(4)} - q_{l_2}^{(3)}\right)P_4$
For two arbitrary elements of Q, then:
$X - Y = \left(q_{j_1}^{(0)} - q_{j_2}^{(0)}\right) + \left[\left(q_{k_1}^{(1)} - q_{k_2}^{(1)}\right) - \left(q_{j_1}^{(0)} - q_{j_2}^{(0)}\right)\right]P_1 + [(q_{s_1}^{(2)} - q_{s_2}^{(2)}) - (q_{k_1}^{(1)} - q_{k_2}^{(1)})]P_2 + \left[\left(q_{l_1}^{(3)} - q_{l_2}^{(3)}\right) - \left(q_{s_1}^{(2)} - q_{s_2}^{(2)}\right)\right]P_3 + \left[\left(q_{t_1}^{(4)} - q_{t_2}^{(4)}\right) - \left(q_{l_1}^{(3)} - q_{l_2}^{(3)}\right)\right]P_4 \in Q$, which is because:

$$\begin{cases} q_{j_1}^{(0)} - q_{j_2}^{(0)} \in Q_0 \\ q_{k_1}^{(1)} - q_{k_2}^{(1)} \in Q_1 \\ q_{s_1}^{(2)} - q_{s_2}^{(2)} \in Q_2 \\ q_{l_1}^{(3)} - q_{l_2}^{(3)} \in Q_3 \\ q_{t_1}^{(4)} - q_{t_2}^{(4)} \in Q_4 \end{cases}$$

Let $r = r_0 + r_1P_1 + r_2P_2 + r_3P_3 + r_4P_4 \in 4 - SP_R$; then:
$rX = r_0q_{j_1}^{(0)} + [r_0q_{k_1}^{(1)} - r_0q_{j_1}^{(0)} + r_1q_{k_1}^{(1)} - r_1q_{j_1}^{(0)} + r_1q_{j_1}^{(0)}]P_1 + [r_0q_{s_1}^{(2)} - r_0q_{k_1}^{(1)} + r_1q_{s_1}^{(2)} - r_1q_{k_1}^{(1)} + r_2q_{s_1}^{(2)} - r_2q_{k_1}^{(1)} + r_2q_{j_1}^{(0)} + r_2q_{j_1}^{(1)} - r_2q_{j_1}^{(0)}]P_2 + [r_0q_{l_1}^{(3)} - r_0q_{s_1}^{(2)} + r_1q_{l_1}^{(3)} - r_1q_{s_1}^{(2)} + r_2q_{l_1}^{(3)} - r_2q_{s_1}^{(2)} + r_3q_{l_1}^{(3)} - r_3q_{s_1}^{(2)} + r_3q_{j_1}^{(0)} + r_3q_{k_1}^{(1)} - r_3q_{j_1}^{(0)} + r_3q_{s_1}^{(2)} - r_3q_{k_1}^{(3)}]P_3 + [r_0q_{t_1}^{(4)} - r_0q_{l_1}^{(3)} + r_1q_{t_1}^{(4)} - r_1q_{l_1}^{(3)} + r_2q_{t_1}^{(4)} - r_2q_{l_1}^{(3)} + r_3q_{t_1}^{(4)} - r_3q_{l_1}^{(3)} + r_4q_{t_1}^{(4)} - r_4q_{l_1}^{(3)} + r_4q_{j_1}^{(0)} + r_4q_{k_1}^{(1)} - r_4q_{j_1}^{(0)} + r_4q_{s_1}^{(2)} - r_4q_{k_1}^{(3)} + r_4q_{l_1}^{(3)} - r_4q_{s_1}^{(2)}]P_4 = r_0q_{j_1}^{(0)} + [(r_0 + r_1)q_{k_1}^{(1)} - r_0q_{j_1}^{(0)}]P_1 + [(r_0 + r_1 + r_2)q_{s_1}^{(2)} - (r_0 + r_1)q_{k_1}^{(1)}]P_2 + [(r_0 + r_1 + r_2 + r_3)q_{l_1}^{(3)} - (r_0 + r_1 + r_2)q_{s_1}^{(2)}]P_3 + [(r_0 + r_1 + r_2 + r_3 + r_4)q_{t_1}^{(4)} - (r_0 + r_1 + r_2 + r_3)q_{l_1}^{(3)}]P_4 \in Q$; thus, Q is ideal of $4 - SP_R$.
□

Example 6. *Take* $R = Z$, *the ring of integers, and consider the ideals* $Q_0 = 2Z, Q_1 = 3Z, Q_2 = 4Z, Q_3 = 5Z, Q_4 = 6Z$; *then:*
$Q = \{2x + (3y - 2x)P_1 + (4z - 3y)P_2 + (5t - 4z)P_3 + (6s - 5t)P_4, x, y, z, t, s \in Z\}$
Also,

$M = \{3x + (2y - 3x)P_1 + (5t - 2y)P_2 + (4s - 5t)P_3 + (6k - 4s)P_4, x, y, k, t, s \in Z\}$ *is another ideal of* $4 - SP_R$.

Symbolic 5-plithogenic rings

Definition 5. *Let R be a ring; the symbolic 5-plithogenic ring is:*

$5 - SP_R = \{a_0 + a_1P_1 + a_2P_2 + a_3P_3 + a_4P_4 + a_5P_5;\ a_i \in R,\ P_j^2 = P_j,\ P_i \times P_j = P_{max(i,j)}\}$.

Operations on $5 - SP_R$*:*

Addition:

$[a_0 + a_1P_1 + a_2P_2 + a_3P_3 + a_4P_4 + a_5P_5] + [b_0 + b_1P_1 + b_2P_2 + b_3P_3 + b_4P_4 + b_5P_5] = (a_0 + b_0) + (a_1 + b_1)P_1 + (a_2 + b_2)P_2 + (a_3 + b_3)P_3 + (a_4 + b_4)P_4 + (a_5 + b_5)P_5$.

Multiplication:

$[a_0 + a_1P_1 + a_2P_2 + a_3P_3 + a_4P_4 + a_5P_5]\cdot[b_0 + b_1P_1 + b_2P_2 + b_3P_3 + b_4P_4 + b_5P_5] = a_0b_0 + (a_0b_1 + a_1b_0 + a_1b_1)P_1 + (a_0b_2 + a_1b_2 + a_2b_0 + a_2b_1 + a_2b_2)P_2 + (a_0b_3 + a_1b_3 + a_2b_3 + a_3b_3 + a_3b_0 + a_3b_1 + a_3b_2)P_3 + (a_0b_4 + a_1b_4 + a_2b_4 + a_3b_4 + a_4b_0 + a_4b_1 + a_4b_2 + a_4b_3 + a_4b_4)P_4 + (a_0b_5 + a_1b_5 + a_2b_5 + a_3b_5 + a_4b_5 + a_5b_0 + a_5b_1 + a_5b_2 + a_5b_3 + a_5b_4 + a_5b_5)P_5$.

It is clear that $(5 - SP_R)$ *is a ring.*

If R is commutative, then $5 - SP_R$ *is commutative, and if R has a unity, then* $5 - SP_R$ *has the same unity.*

Example 7. *Consider the ring* $R = Z_3 = \{0, 1, 2\}$*; the corresponding* $5 - SP_R$ *is:*

$4 - SP_R = \{a + bP_1 + cP_2 + dP_3 + eP_4 + wP_5; a, b, c, d, e, w \in Z_3\}$.

If $X = 2 + 2P_1 + P_4,\ Y = P_2 + 2P_5$*, then:*

$X + Y = 2 + 2P_1 + P_2 + P_4 + 2P_5$,

$X - Y = 2 + 2P_1 - P_2 + P_4 - 2P_5$,

$X\cdot Y = 2P_2 + 4P_3 + 2P_2 + 4P_5 + P_4 + 2P_5 = 4P_2 + 4P_3 + P_4 + 2P_5$.

Theorem 7. *Let* $5 - SP_R$ *be a 5-plithogenic symbolic ring, with unity.*

Let $X = e_0 + e_1P_1 + e_2P_2 + e_3P_3 + e_4P_4 + e_5P_5$ *be an arbitrary element; then:*

1. *X is invertible if and only if* $e_0, e_0 + e_1, e_0 + e_1 + e_2,\ e_0 + e_1 + e_2 + e_3,\ e_0 + e_1 + e_2 + e_3 + e_4, e_0 + e_1 + e_2 + e_3 + e_4 + e_5$ *are invertible.*
2. $X^{-1} = e_0^{-1} + \left[(e_0 + e_1)^{-1} - e_0^{-1}\right]P_1 + \left[(e_0 + e_1 + e_2)^{-1} - (e_0 + e_1)^{-1}\right]P_2 + [(e_0 + e_1 + e_2 + e_3)^{-1} - (e_0 + e_1 + e_2)^{-1}]P_3 + [(e_0 + e_1 + e_2 + e_3 + e_4)^{-1} - (e_0 + e_1 + e_2 + e_3)^{-1}]P_4 + \left[(e_0 + e_1 + e_2 + e_3 + e_4 + e_5)^{-1} - (e_0 + e_1 + e_2 + e_3 + e_4)^{-1}\right]P_4$.

Proof.

1. Assume that X is invertible, then there exists $Y = n_0 + n_1P_1 + n_2P_2 + n_3P_3 + n_4P_4 + n_5P_5$ such that $X\cdot Y = 1$; hence:

$$\begin{cases} e_0n_3 + e_1n_3 + e_2n_3 + e_3n_3 + e_3n_1 + e_3n_2 + e_3n_0 = 0 \text{ (3a)} \\ e_0n_0 = 1\ldots \text{ (3b)} \\ e_0n_1 + e_1n_0 + e_1n_1 = 0\ldots \text{ (3c)} \\ e_0n_2 + e_2n_0 + e_2n_2 + e_1n_2 + e_2n_1 = 0\ldots \text{ (3d)}, \\ e_0n_4 + e_1n_4 + e_2n_4 + e_3n_4 + e_4n_0 + e_4n_1 + e_4n_3 + e_4n_4 \text{(3e)} \\ e_0n_5 + e_1n_5 + e_2n_5 + e_3n_5 + e_4n_5 + e_5n_0 + e_5n_1 + e_5n_2 + e_5n_3 + e_5n_4 + e_5n_5 = 0 \text{ (3f)} \end{cases} \quad (3)$$

From (3b), e_0 is invertible.

By adding (3c) to (3b), we obtain $(e_0 + e_1)(n_0 + n_1) = 1$; thus, $e_0 + e_1$ is invertible.

By adding (3d) to (3c) to (3b), $(e_0 + e_1 + e_2)(n_0 + n_1 + n_2) = 1$; hence, $e_0 + e_1 + e_2$ is invertible.

By adding (3a) to (3b) to (3c) to (3d), $(e_0 + e_1 + e_2 + e_3)(n_0 + n_1 + n_2 + n_3) = 1$; hence, $e_0 + e_1 + e_2 + e_3$ is invertible.

Adding all Equations (3a) to (3e) gives:

$(e_0 + e_1 + e_2 + e_3 + e_4)(n_0 + n_1 + n_2 + n_3 + n_4) = 1$; hence, $e_0 + e_1 + e_2 + e_3 + e_4$ is invertible.

Adding all Equations (3a) to (3f) gives:

$(e_0 + e_1 + e_2 + e_3 + e_4 + e_5)(n_0 + n_1 + n_2 + n_3 + n_4 + n_5) = 1$;

Hence, $e_0 + e_1 + e_2 + e_3 + e_4 + e_5$ is invertible.

2. From the first part, we have:
$n_0 = {e_0}^{-1}, n_0 + n_1 = (e_0 + e_1)^{-1}, n_0 + n_1 + n_2 = (e_0 + e_1 + e_2)^{-1}$,
$(e_0 + e_1 + e_2 + e_3)^{-1} = n_0 + n_1 + n_2 + n_3, (e_0 + e_1 + e_2 + e_3 + e_4)^{-1} = n_0 + n_1 + n_2 + n_3 + n_4$, $(e_0 + e_1 + e_2 + e_3 + e_4 + e_5)^{-1} = n_0 + n_1 + n_2 + n_3 + n_4 + n_5$; then:
$Y = {e_0}^{-1} + [(e_0 + e_1)^{-1} - {e_0}^{-1}]P_1 + [(e_0 + e_1 + e_2)^{-1} - (e_0 + e_1)^{-1}]P_2 +$
$\left[(e_0 + e_1 + e_2 + e_3)^{-1} - (e_0 + e_1 + e_2)^{-1}\right]P_3 +$
$\left[(e_0 + e_1 + e_2 + e_3 + e_4)^{-1} - (e_0 + e_1 + e_2 + e_3)^{-1}\right]P_4 +$
$\left[(e_0 + e_1 + e_2 + e_3 + e_4 + e_5)^{-1} - (e_0 + e_1 + e_2 + e_3 + e_4)^{-1}\right]P_5 = X^{-1}$. □

Example 8. *Take* $R = Z_3 = \{0, 1, 2\}$, $5 - SP_{Z_3}$ *is the corresponding symbolic 5-plithogenic ring, and consider* $X = 2 + 2P_2 + P_5 \in 5 - SP_{Z_3}$; *then:*
$e_0 = 2$ *is invertible with* ${e_0}^{-1} = 2$, $e_0 + e_1 = 2$ *is invertible with* $(e_0 + e_1)^{-1} = 2$,
$e_0 + e_1 + e_2 = 1$ *is invertible with* $(e_0 + e_1 + e_2)^{-1} = 1$,
$e_0 + e_1 + e_2 + e_3 = 1$, $(e_0 + e_1 + e_2 + e_3)^{-1} = 1$,
$e_0 + e_1 + e_2 + e_3 + e_4 = 1$, $(e_0 + e_1 + e_2 + e_3 + e_4)^{-1} = 1$, $e_0 + e_1 + e_2 + e_3 + e_4 + e_5 = 2$,
$(e_0 + e_1 + e_2 + e_3 + e_4 + e_5)^{-1} = 2$; *hence,*
$X^{-1} = 2 + (2 - 2)P_1 + (1 - 2)P_2 + (1 - 1)P_3 + (1 - 1)P_4 + (2 - 1)P_5 = 2 + 2P_2 + P_5$.

Definition 6. *Let* $X = m + nP_1 + cP_2 + qP_3 + lP_4 + kP_5 \in 5 - SP_R$, *then, X is idempotent if and only if* $X^2 = X$.

Theorem 8. *Let* $X = m + nP_1 + cP_2 + qP_3 + lP_4 + kP_5 \in 5 - SP_R$; *then, X is idempotent if and only if* $m, m + n, m + n + c, m + n + c + q, m + n + c + q + l, m + n + c + q + l + k$ *are idempotent.*

Proof.
$X^2 = X{\cdot}X = (m + nP_1 + cP_2 + qP_3 + lP_4 + kP_5)(m + nP_1 + cP_2 + qP_3 + lP_4 + kP_5) =$
$m^2 + (mn + nm + n{\cdot}n)P_1 + (mc + nc + cm + cn + c{\cdot}c)P_2 +$
$(mq + nq + cq + qm + qn + qc + q{\cdot}q)P_3 + (ml + nl + cl + ql + lm + ln + lc + lq + l{\cdot}l)P_4 +$
$(mk + nk + ck + qk + lk + km + kn + kc + kq + kl + k{\cdot}k)P_5$.

$$X^2 = X{\cdot}X \text{ equivalents } \begin{cases} mq + nq + cq + qm + qn + qc + q{\cdot}q = q \text{ (4a)} \\ m^2 = m \ldots \text{ (4b)} \\ mn + nm + n{\cdot}n = n \ldots \text{ (4c)} \\ mc + nc + cm + cn + c{\cdot}c = c \ldots \text{ (4d)} \\ ml + nl + cl + ql + lm + ln + lc + lq + l{\cdot}l = l \text{ (4e)} \\ mk + nk + ck + qk + lk + km + kn + kc + kq + kl + k{\cdot}k = k \text{ (4f)} \end{cases} \quad (4)$$

Equation (4b) implies that m is idempotent.
By adding (4c) to (4b), we obtain $(m + n)^2 = m + n$; hence, $m + n$ is idempotent.
By adding (4c) to (4b) to (4d), we obtain $(m + n + c)^2 = m + n + c$; hence, $m + n + c$ is idempotent.
By adding (4a) to (4b) to (4c) to (4d), we obtain $(m + n + c + q)^2 = m + n + c + q$; thus, $m + n + c + q$ is idempotent.
By adding all equations from (4a) to (4e) to each other, we obtain:
$(m + n + c + q + l)^2 = m + n + c + q + l$, thus $m + n + c + l$ is idempotent.
By adding all equations from (4a) to (4f), we obtain:
$(m + n + c + q + l + k)^2 = m + n + c + q + l + k$, thus $m + n + c + l + k$ is idempotent.
Thus, the proof is complete. □

Example 9. *Take* $R = Z_4 = \{0, 1, 2, 3\}$, $5 - SP_{Z_4}$ *is the corresponding symbolic 5-plithogenic ring, and consider* $X = P_1 + 3P_4 \in 5 - SP_{Z_5}$; *thus, we have:*
$X^2 = P_1 + 9P_4 + 6P_4 = P_1 + +3P_4 = X$.

Theorem 9. *Let* $5 - SP_R$ *be a commutative symbolic 5-plithogenic ring; hence, if* $X = m + nP_1 + cP_2 + qP_3 + lP_4 + kP_5$, *then*

$X^n = m^n + [(m+n)^n - m^n]P_1 + [(m+n+c)^n - (m+n)^n]P_2 + [(m+n+c+q)^n - (m+n+c)^n]P_3 + [(m+n+c+q+l)^n - (m+n+c+q)^n]P_4 + [(m+n+c+q+l+k)^n - (m+n+c+q+l)^n] P_5$ *for every* $n \in Z^+$.

Proof.

For $n = 1$, it holds easily. Assume that it is true for $n = k$ and prove it for $n = k + 1$.

$X^{k+1} = X \cdot X^k =$

$(m + nP_1 + cP_2 + qP_3 + lP_4 + kP_5)(m^k + [(m+n)^k - m^k]P_1 + [(m+n+c)^k - (m+n)^k]P_2 + [(m+n+c+q)^k - (m+n+c)^k]P_3 + [(m+n+c+q+l)^k - (m+n+c+q)^k]P_4 + [(m+n+c+q+l+k)^k - (m+n+c+q+l)^k]P_5) = m^{k+1} + [(m+n)^{k+1} - m^{k+1}]P_1 + [(m+n+c)^{k+1} - (m+n)^{k+1}]P_2 + [(m+n+c+q)^{k+1} - (m+n+c)^{k+1}]P_3 +$

$[(m+n+c+q+l)^{k+1} - (m+n+c+q)^{k+1}]P_4 + [(m+n+c+q+l+k)^{k+1} - (m+n+c+q+l)^{k+1}]P_5$.

So, this proof is complete by induction. □

Definition 7. *Let* G_0, G_1, G_2, G_3, G_4, *and* G_5 *be ideals of the ring* R; *define the symbolic 5-plithogenic AH-ideal:*

$G = G_0 + G_1P_1 + G_2P_2 + G_3P_3 + G_4P_4 + G_5P_5 = \{w_0 + w_1P_1 + w_2P_2 + w_3P_3 + w_4P_4 + w_5P_5;\ w_i \in G_i\}$.

If $G_0 = G_1 = G_2 = G_3 = G_4 = G_5$, *then* G *is called an AHS-ideal.*

Example 10. *Let* $R = Z$ *be the ring of integers; then,* $G_0 = 7Z$, $G_1 = 11Z$, *are ideals of* R.

$G = \{7m + 7nP_1 + 11tP_2 + 11sP_3 + 11lP_4 + 11kP_5;\ m \cdot n \cdot t, s, l, k \in Z\}$ *is an AHS-ideal of* $5 - SP_Z$.

$M = \{11m + 11nP_1 + 11tP_2 + 11sP_3 + 11lP_4 + 11kP_5;\ m \cdot n \cdot t, s, l, k \in Z\}$ *is an AHS-ideal of* $5 - SP_Z$.

Theorem 10. *Let* G *be an AHS- ideal of* $5 - SP_R$; *then,* G *is an ideal with an ordinary meaning.*

Proof.

G can be written as $G = G_0 + G_0P_1 + G_0P_2 + G_0P_3 + G_0P_4 + G_0P_5$, where G_0 is an ideal of R.

It is clear that $(G, +)$ is a subgroup of $(5 - SP_R, +)$.

Let $F = f_0 + f_1P_1 + f_2P_2 + f_3P_3 + f_4P_4 + f_5P_5 \in 5 - SP_R$,

Then if $X = m + nP_1 + cP_2 + qP_3 + lP_4 + kP_5 \in G$, we have:

$F \cdot X = f_0m + (f_0n + f_1m + f_1n)P_1 + (f_0c + f_1c + f_2m + f_2n + f_2c)P_2 + (f_0q + f_1q + f_2q + f_3q + f_3m + f_3n + f_3c)P_3 + (f_0l + f_1l + f_2l + f_3l + f_4m + f_4n + f_4c + f_4q + f_4l)P_4 + (f_0k + f_1k + f_2k + f_3k + f_4k + f_5m + f_5n + f_5c + f_5q + f_5l)P_5 \in G$; thus, G is an ideal. □

Definition 8. *Let* R, T *be two rings,* $5 - SP_R, 5 - SP_T$ *are the corresponding symbolic 5-plithogenic rings, and let* $f_0, f_1, f_2, f_3, f_4, f_5 : R \to T$ *be ring homomorphisms; we define the AH-homomorphism:*

$f : 5 - SP_R \to 5 - SP_T$ *such that:*

$f(m + nP_1 + cP_2 + qP_3 + lP_4 + kP_5) = f_0(m) + f_1(n)P_1 + f_2(c)P_2 + f_3(q)P_3 + f_4(l)P_4 + f_5(k)P_5$

If $f_0 = f_1 = f_2 = f_3 = f_4 = f_5$, *then* f *is called an AHS-homomorphism.*

Remark 2. *If* f_0, f_1, f_2, f_3, f_4, *and* f_5 *are isomorphisms, then* f *is called an AH-isomorphism.*

Example 11. *Take* $R = Z$, $T = Z_4$, $f_0, f_1 : R \to T$ *such that:*

$f_0(x) = x(mod\ 4)$, $f_1(2) = 2x(mod\ 4)$. *It is clear that* f_0 *and* f_1 *are homomorphisms.*

We define $f : 5 - SP_R \to 5 - SP_T$, *where:*

$f(m + nP_1 + cP_2 + qP_3 + lP_4 + kP_5) = f_0(m) + f_1(n)P_1 + f_0(c)P_2 + f_1(q)P_3 + f_1(l)P_4 + f_1(k)P_5 = m(mod\ 4) + 2n(mod\ 4)P_1 + (c\ mod\ 4)P_2 + (2q\ mod\ 4)P_3 + (2l\ mod\ 4)P_4 + (2k\ mod\ 4)P_5$,

Which is an AH-homomorphism.

Theorem 11. *Let* $f = f_0 + f_1P_1 + f_2P_2 + f_3P_3 + f_4P_4 + f_5P_5 : 5 - SP_R \to 5 - SP_T$ *be a mapping; then:*

1. *If* f *is an AHS-homomorphism, then* f *is a ring homomorphism.*
2. *If* f *is an AHS-homomorphism, then it is an isomorphism.*

Proof.

1. Assume that f is an AHS-homomorphism; then, $f_0 = f_1 = f_2 = f_3 = f_4 = f_5$ are homomorphisms. Let
$X = d_0 + d_1P_1 + d_2P_2 + d_3P_3 + d_4P_4 + d_5P_5, Y = c_0 + c_1P_1 + c_2P_2 + c_3P_3 + c_4P_4 + c_5P_5 \in 5 - SP_R$; we have:
$f(X+Y) =$
$f_0(d_0+c_0) + f_0(d_1+c_1)P_1 + f_0(d_2+c_2)P_2 + f_0(d_3+c_3)P_3 + f_0(d_4+c_4)P_4 + f_0(d_5+c_5)P_5 = f(X) + f(Y)$
$f(X \cdot Y) = f_0(d_0c_0) + f_0(d_0c_1 + d_1c_0 + d_1c_1)P_1 + f_0(d_0c_2 + d_2c_0 + d_2c_2 + d_2c_1 + d_1c_2)P_2$
$+ f_0(d_0c_3 + d_1c_3 + d_2c_3 + d_3c_3 + d_3c_1 + d_3c_0 + d_3c_2)P_3 +$
$(d_0c_4 + d_1c_4 + d_2c_4 + d_3c_4 + d_4c_0 + d_4c_1 + d_4c_2 + d_4c_3 + d_4c_4)P_4$
$+ (d_0c_5 + d_1c_5 + d_2c_5 + d_3c_5 + d_4c_5 + d_5c_0 + d_5c_1 + d_5c_2 + d_5c_3 + d_5c_4 + d_5c_5)P_5$
$= f_0(d_0)f_0(c_0) + (f_0(d_0)f_0(c_1) + f_0(d_1)f_0(c_0) + f_0(d_1)f_0(c_1))P_1$
$+ (f_0(d_0)f_0(c_2) + f_0(d_2)f_0(y_0) + f_0(d_2)f_0(c_2) + f_0(d_2)f_0(c_1)$
$+ f_0(d_1)f_0(c_2))P_2$
$+ [(f_0(d_0)f_0(c_3) + f_0(d_1)f_0(c_3) + f_0(d_2)f_0(c_3) + f_0(d_3)f_0(c_3)$
$+ f_0(d_3)f_0(c_1) + f_0(d_3)f_0(c_2) + f_0(d_3)f_0(c_0))]P_3$
$+ [(f_0(d_0)f_0(c_4) + f_0(d_1)f_0(c_4) + f_0(d_2)f_0(c_4) + f_0(d_3)f_0(c_4)$
$+ f_0(d_4)f_0(c_1) + f_0(d_4)f_0(c_2) + f_0(d_4)f_0(c_0) + f_0(d_4)f_0(c_4)$
$+ f_0(d_4)f_0(c_3))]P_4 + [f_0(d_0)f_0(c_5) + f_0(d_1)f_0(c_5) + f_0(d_2)f_0(c_5)$
$+ f_0(d_3)f_0(c_5) + f_0(d_4)f_0(c_5) + f_0(d_5)f_0(c_0) + f_0(d_5)f_0(c_1) + f_0(d_5)f_0(c_2)$
$+ f_0(d_5)f_0(c_3) + f_0(d_5)f_0(c_4) + f_0(d_5)f_0(c_5)]P_5 =$
$[f_0(d_0) + f_0(d_1)P_1 + f_0(d_2)P_2 + f_0(d_3)P_3 + f_0(d_4)P_4 + f_0(d_5)P_5][f_0(c_0) + f_0(c_1)P_1 +$
$f_0(c_2)P_2 + f_0(c_3)P_3 + f_0(c_4)P_4 + f_0(c_5)P_5] = f(X) \cdot f(Y)$. This implies the proof.
2. Using a similar discussion, we obtain the desired proof. □

The following table shows the number of units in the ring R, symbolic 2-plithogenic ring $2 - SP_R$, symbolic 3-plithogenic ring $3 - SP_R$, and symbolic 4-plithogenic ring $4 - SP_R$, and symbolic 5-plithogenic ring $5 - SP_R$

Classical Ring	Symbolic 5-Plithogenic Ring	Symbolic 2-Plithogenic Ring	Symbolic 3-Plithogenic Ring	Symbolic 4-Plithogenic Ring
Z (2 units)	*Z(I) (64 units)*	$2 - SP_Z$ *(8 units)*	$3 - SP_Z$ *(16 units)*	$4 - SP_Z$ *(32 units)*
Z_2 *(1 unit)*	Z_2*(I) (1 unit)*	$2 - SP_{Z_2}$ *(1 unit)*	$3 - SP_{Z_2}$ *(1 unit)*	$4 - SP_{Z_2}$ *(1 unit)*
Z_3 *(2 units)*	Z_3*(I) (64 units)*	$2 - SP_{Z_3}$ *(8 units)*	$3 - SP_{Z_3}$ *(16 units)*	$4 - SP_{Z_3}$ *(32 units)*
Z_4 *(2 units)*	Z_4*(I) (64 units)*	$2 - SP_{Z_4}$ *(8 units)*	$3 - SP_{Z_4}$ *(16 units)*	$4 - SP_{Z_4}$ *(32 units)*
Z_5 *(4 units)*	Z_5*(I) (4096 units)*	$2 - SP_{Z_5}$ *(64 units)*	$3 - SP_{Z_5}$ *(256 units)*	$4 - SP_{Z_5}$ *(1024 units)*
Z_6 *(2 units)*	Z_6*(I) (64 units)*	$2 - SP_{Z_6}$ *(8 units)*	$3 - SP_{Z_6}$ *(16 units)*	$4 - SP_{Z_6}$ *(32 units)*
Z_7 *(6 units)*	Z_7*(I)* $(6^6$ *units)*	$2 - SP_{Z_7}$ *(216 units)*	$3 - SP_{Z_7}$ *(1296 units)*	$4 - SP_{Z_7}$ *(7776 units)*
Z_8 *(4 units)*	Z_8*(I) (4096 units)*	$2 - SP_{Z_8}$ *(64 units)*	$3 - SP_{Z_8}$ *(256 units)*	$4 - SP_{Z_8}$ *(1024 units)*
Z_9 *(6 units)*	Z_9*(I)* $(6^6$ *units)*	$2 - SP_{Z_9}$ *(216 units)*	$3 - SP_{Z_9}$ *(1296 units)*	$4 - SP_{Z_9}$ *(7776 units)*
Z_{10} *(4 units)*	Z_{10}*(I) (4096 units)*	$2 - SP_{Z_{10}}$ *(64 units)*	$3 - SP_{Z_{10}}$ *(256 units)*	$4 - SP_{Z_{10}}$ *(1024 units)*

3. Conclusions

In this paper, we have defined the 4-plithogenic rings and 5-plithogenic rings, the elements of which have many algebraic properties such as invertibility, nilpotency, and idempotency.

Also, we have shown some related substructures, such as 4-plithogenic AH-ideals, 4-plithogenic AH-kernels and homomorphisms, 5-plithogenic AH-ideals, 5-plithogenic AH-kernels, and homomorphisms.

As a future direction, we aim to use 4-plithogenic numbers and rings to generalize symbolic 3-plithogenic algebraic modules and equations.

Funding: This research received no external funding.

Data Availability Statement: Not applicable.

Conflicts of Interest: The author declares no conflict of interest.

References

1. Olgun, N.; Hatip, A.; Bal, M.; Abobala, M. A Novel Approach to Necessary and Sufficient Conditions for the Diagonalization of Refined Neutrosophic Matrices. *Int. J. Neutrosophic Sci.* **2021**, *16*, 72–79. [CrossRef]
2. Abobala, M. On Some Algebraic Properties of n-Refined Neutrosophic Elements and n-Refined Neutrosophic Linear Equations. *Math. Probl. Eng. Hindawi* **2021**, *2021*, 5573072. [CrossRef]
3. Smarandache, F. Introduction to the Symbolic Plithogenic Algebraic Structures (revisited). *Neutrosophic Sets Syst.* **2023**, *53*, 39.
4. Abobala, M. On Refined Neutrosophic Matrices and Their Applications in Refined Neutrosophic Algebraic Equations. *J. Math. Hindawi* **2021**, *2021*, 5531093. [CrossRef]
5. Abobala, M. A Study of Nil Ideals and Kothe's Conjecture in Neutrosophic Rings. *Int. J. Math. Math. Sci. Hindawi* **2021**, *2021*, 9999707. [CrossRef]
6. Khaldi, A. A Study on Split-Complex Vector Spaces. *Neoma J. Math. Comput. Sci.* **2023**, *1*. [CrossRef]
7. Ahmad, K. On Some Split-Complex Diophantine Equations. *Neoma J. Math. Comput. Sci.* **2023**, *6*, 32–35. [CrossRef]
8. Ali, R. On The Weak Fuzzy Complex Inner Products on Weak Fuzzy Complex Vector Spaces. *Neoma J. Math. Comput. Sci.* **2023**, *1*. [CrossRef]
9. Merkepci, H.; Abobala, M. On The Symbolic 2-Plithogenic Rings. *Int. J. Neutrosophic Sci.* **2023**, *54*, 3. [CrossRef]
10. Taffach, N. An Introduction to Symbolic 2-Plithogenic Vector Spaces Generated from the Fusion of Symbolic Plithogenic Sets and Vector Spaces. *Neutrosophic Sets Syst.* **2023**, *54*, 4. [CrossRef]
11. Taffach, N.; Ben Othman, K. An Introduction to Symbolic 2-Plithogenic Modules Over Symbolic 2-Plithogenic Rings. *Neutrosophic Sets Syst.* **2023**, *54*, 3. [CrossRef]
12. Merkepci, H.; Rawashdeh, A. On The Symbolic 2-Plithogenic Number Theory and Integers. *Neutrosophic Sets Syst.* **2023**, *54*, 10. [CrossRef]
13. Taffach, N.M.; Hatip, A. A Review on Symbolic 2-Plithogenic Algebraic Structures. *Galoitica J. Math. Struct. Appl.* **2023**, *5*, 8–16. [CrossRef]
14. Taffach, N.M.; Hatip, A. A Brief Review on The Symbolic 2-Plithogenic Number Theory and Algebraic Equations. *Galoitica J. Math. Struct. Appl.* **2023**, *5*, 36–44. [CrossRef]
15. Albasheer, O.; Hajjari, A.; Dalla, R. On The Symbolic 3-Plithogenic Rings and Their Algebraic Properties. *Neutrosophic Sets Syst.* **2023**, *54*, 5. [CrossRef]
16. Ali, R.; Hasan, Z. An Introduction to The Symbolic 3-Plithogenic Modules. *Galoitica J. Math. Struct. Appl.* **2023**, *6*, 13–17. [CrossRef]
17. Ali, R.; Hasan, Z. An Introduction to The Symbolic 3-Plithogenic Vector Spaces. *Galoitica J. Math. Struct. Appl.* **2023**, *6*, 8–12. [CrossRef]
18. Rawashdeh, A. An Introduction to The Symbolic 3-plithogenic Number Theory. *Neoma J. Math. Comput. Sci.* **2023**, *1*. [CrossRef]
19. Ben Othman, K. On Some Algorithms for Solving Symbolic 3-Plithogenic Equations. *Neoma J. Math. Comput. Sci.* **2023**, *1*. [CrossRef]