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# The ring of polyfunctions over $\mathbb{Z}/n\mathbb{Z}$

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## ABSTRACT

We study the ring of *polyfunctions* over  $\mathbb{Z}/n\mathbb{Z}$ . The ring of polyfunctions over a commutative ring  $R$  with unit element is the ring of functions  $f : R \rightarrow R$  which admit a polynomial representative  $p \in R[x]$  in the sense that  $f(x) = p(x)$  for all  $x \in R$ . This allows to define a ring invariant  $s$  which associates to a commutative ring  $R$  with unit element a value in  $\mathbb{N} \cup \{\infty\}$ . The function  $s$  generalizes the number theoretic Smarandache function. For the ring  $R = \mathbb{Z}/n\mathbb{Z}$  we provide a unique representation of polynomials which vanish as a function. This yields a new formula for the number  $\Psi(n)$  of polyfunctions over  $\mathbb{Z}/n\mathbb{Z}$ . We also investigate algebraic properties of the ring of polyfunctions over  $\mathbb{Z}/n\mathbb{Z}$ . In particular, we identify the additive subgroup of the ring and the ring structure itself. Moreover we derive formulas for the size of the ring of polyfunctions in several variables over  $\mathbb{Z}/n\mathbb{Z}$ , and we compute the number of polyfunctions which are units of the ring.

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## 1. Introduction

In a finite field  $F$ , every function  $f : F \rightarrow F$  can be represented by a polynomial, i.e., there exists a polynomial  $p \in F[x]$  such that  $f(x) = p(x)$  for all  $x \in F$ . Such a polynomial is, e.g., given by the Lagrange interpolation polynomial for  $f$ . Among the commutative rings with unit element, the finite fields are actually characterized by this representation property (see [18]):

**Theorem 1** (Rédei, Szele). *If  $R$  is a commutative ring with unit element then  $R$  is a finite field if and only if every function  $f : R \rightarrow R$  can be represented by a polynomial in  $R[x]$ .*

If a commutative ring  $R$  with unit element is *not* a field, it is natural to ask what can be said about the functions from  $R$  to  $R$  which *can* be represented by a polynomial in  $R[x]$ . These functions are called polynomial functions or *polyfunctions* for short. The set of polyfunctions

$$\{f : R \rightarrow R \mid \exists p \in R[x] \quad \forall x \in R : p(x) = f(x)\},$$

equipped with pointwise addition and multiplication, is a subring of  $R^R$ . This ring of polyfunctions over  $R$  will be denoted by  $G(R)$ . Of particular interest are the polynomials which correspond to the zero element in  $G(R)$ , they will be called *null-polynomials* (see, e.g., [19]). It is the objective of this article to investigate the algebraic structure and combinatorial properties of the ring of polyfunctions  $G(\mathbb{Z}/n\mathbb{Z})$ .

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Dedicated to the memory of the first author.

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More generally, one can study the ring of multivariate polyfunctions in  $d \in \mathbb{N}$  variables—this ring is defined as the set

$$\{f : R^d \rightarrow R \mid \exists p \in R[x_1, x_2, \dots, x_d] \quad \forall x = (x_1, \dots, x_d) \in R^d : p(x) = f(x)\},$$

equipped with pointwise addition and multiplication. We denote this ring by  $G_d(R)$  and write  $G(R) = G_1(R)$ , in accordance with the notation introduced above.

Polyfunctions in one variable over  $\mathbb{Z}/n\mathbb{Z}$  were already discussed by Kempner [12, 13], who gave a formula for the number  $\Psi(n)$  of polyfunctions over  $\mathbb{Z}/n\mathbb{Z}$ , which was subsequently simplified by Keller and Olson in [10] (see also the work of Carlitz [4] in the case where  $n$  is a power of a prime). Regarding polyfunctions in  $d$  variables we refer to Mullen [16] and more recently to [9]: In [9, Theorem 2, p. 5], a characterization theorem is proved which allows to tell whether a given function  $f : (\mathbb{Z}/n\mathbb{Z})^d \rightarrow \mathbb{Z}/n\mathbb{Z}$  is a polyfunction or not. Furthermore, a formula for the number of polyfunctions  $\Psi_d(n)$  in  $d$  variables over  $\mathbb{Z}/n\mathbb{Z}$  is obtained. In the present work, we provide an alternative formula for  $\Psi(n)$  and a new proof of the formula for  $\Psi_d(n)$  given in [9].

Polyfunctions from  $\mathbb{Z}/n\mathbb{Z}$  to  $\mathbb{Z}/m\mathbb{Z}$  have been discussed by Chen [5, 6] and Bhargava [3]. The focus there is to find conditions on the pair  $(m, n)$  such that all functions (or certain subclasses) from  $\mathbb{Z}/n\mathbb{Z}$  to  $\mathbb{Z}/m\mathbb{Z}$  are polyfunctions. These results have been generalized to polynomial functions in the residue class rings of Dedekind domains by Li and Sha in [14]. Dueball in [7] considered polynomials mod  $p^n$  with integer coefficients. He showed that the values of such a polynomial  $f(x)$  are already determined when  $x$  runs through a certain subset of residues. He also provided a formula to generate polynomials which vanish mod  $p^n$  for all integral values of  $x$ .

To each commutative ring  $R$  with unit element, we can associate a number  $s(R) \in \mathbb{N} \cup \{\infty\}$  which is defined to be the minimal degree  $m$  such that the function  $x \mapsto x^m$  can be represented by a polynomial in  $R[x]$  of degree strictly smaller than  $m$ , i.e.

$$s(R) := \min\{m \in \mathbb{N} \mid \exists p \in R[x], \deg(p) < m, \forall x \in R : p(x) = x^m\} \quad (1)$$

if such an  $m$  exists, and  $s(R) = \infty$  otherwise.

If  $s(R)$  is finite, the monomial  $x^{s(R)}$  can be represented by a polynomial  $p$  of degree less than  $s(R)$ . Therefore, the normed polynomial  $q(x) = x^{s(R)} - p(x)$  represents the zero-function. Vice versa, if  $r(x)$  is a normed null-polynomial of minimal degree  $m$ , then  $m = s(R)$ . Hence,  $s(R)$  can be interpreted as the minimal degree of a normed null-polynomial over  $R$ .

An alternative and, for reasons that will become clear later, preferable way to view the function defined by (1) is as follows: The building blocks of polynomials are the monomials  $x^0, x^1, x^2, \dots$ . We say, a monomial  $x^m$  is *reducible*, if the function  $x \mapsto x^m$  can be represented by a polynomial in  $R[x]$  of degree strictly smaller than  $m$ . Then,  $s(R)$  is the number of non-reducible monomials.

The function  $s$  is a ring invariant which generalizes the classical number theoretic Smarandache function  $s : \mathbb{N} \rightarrow \mathbb{N}$ ,

$$n \mapsto s(n) := \min\{k \in \mathbb{N} : n \mid k!\}, \quad (2)$$

which is named after the Romanian mathematician Florentin Smarandache, but which has been originally introduced by Lucas in [15] (for prime powers) and Kempner in [11] (for general  $n$ ). The function  $s$  defined in (1) will be called *Smarandache function* because  $n \mapsto s(\mathbb{Z}/n\mathbb{Z})$  coincides with the usual Smarandache function  $n \mapsto s(n)$  (see Theorem 2). In the context of general commutative rings with unit element, this function will be studied in a forthcoming paper [20]. We also refer to [17], where polyfunctions over general rings are discussed.

The article is organized as follows: Section 2 establishes a unique representation theorem for null-polynomials (Theorem 8). This provides a new formula for the number  $\Psi(n)$  of polyfunctions over  $\mathbb{Z}/n\mathbb{Z}$  (Corollary 9 and Proposition 11). In Section 3, we investigate algebraic properties of the ring of polyfunctions over  $\mathbb{Z}/n\mathbb{Z}$ . In particular, we identify the additive subgroup of the ring (Theorem 14) and the ring structure itself (Theorem 18). We also investigate the

multiplicative subgroup  $U_n$  of units in the ring (Propositions 22 and 27). Section 4 comprises a description of the ring of polyfunctions in several variables over  $\mathbb{Z}/n\mathbb{Z}$ . In particular, we give a new formula for the size of this ring (Proposition 26).

**1.1. Notational conventions**

Unless stated otherwise,  $n$  will denote a natural number  $\geq 2$  and  $\mathbb{Z}_n = \mathbb{Z}/n\mathbb{Z}$  is the ring of integers modulo  $n$ . We adopt the notation  $(a, b)$  for the greatest common divisor of the integer numbers  $a$  and  $b$ , and we write  $a|b$  if  $b$  is an integer multiple of  $a$ . Furthermore, for  $f, g \in \mathbb{Z}_n[x]$  we will write  $f \equiv g \pmod n$  to mean the equality of polynomials and we will write  $f(x) \equiv g(x) \pmod n$  if the functions defined by  $f$  and  $g$  agree.

**2. Combinatorial aspects of polyfunctions over  $\mathbb{Z}_n$**

**2.1. The Smarandache function**

In this section, we want to determine the minimal degree of a normed null-polynomial in  $\mathbb{Z}_n[x]$ . We call a polynomial *normed*, if its leading coefficient is 1. The answer is given in the following theorem:

**Theorem 2.**  $s(\mathbb{Z}_n)$  equals the Smarandache function  $s(n)$  defined in (2).

**Remark 3.** According to our conventions,  $n \geq 2$  as the case  $n = 1$  should formally be excluded since  $\mathbb{Z}_1$  is not a ring with unit element. However, if  $n = 1$  we can still make sense of  $s(\mathbb{Z}_1)$  if we view  $\mathbb{Z}_1$  as  $\{0\}$  and it holds that  $s(\mathbb{Z}_1) = 0$  but  $s(1) = 1$ . Kempner originally defined  $s(1) = 1$  in [11] but changed it to  $s(1) = 0$  later on in [12, 13]. By defining

$$s(n) := \min\{k \in \mathbb{N}_0 : n|k!\},$$

this ambiguity can be avoided (see also [9, p. 7]) and the theorem might be stated for every  $1 \leq n \in \mathbb{N}$ . Another proof of Theorem 2 also appears in [8, Theorem 7, p. 126].

In order to prove Theorem 2 for  $n \geq 2$ , we first show that  $s(\mathbb{Z}_n) \leq s(n)$ . This is established by giving a normed null-polynomial of degree  $s(n)$ . In fact, we have

$$p(x) := \prod_{i=1}^{s(n)} (x + i) = \binom{x + s(n)}{s(n)} s(n)! \equiv 0 \pmod n$$

for all  $x \in \mathbb{Z}_n$ .

The second step consists in proving the reverse inequality  $s(\mathbb{Z}_n) \geq s(n)$ . This follows easily from the combinatorial identity which connects the binomial and the Stirling numbers of the second kind (see, e.g., [1, 3.39, p. 97] or [8, Lemma 3]): For all  $r, j \in \mathbb{N}_0$  there holds

$$\sum_{i=0}^r (-1)^{r-i} \binom{r}{i} i^j = r! \left\{ \begin{matrix} j \\ r \end{matrix} \right\}$$

(with the convention  $0^0 := 1$ ). In particular, it follows that

$$\sum_{i=0}^r (-1)^{i+r} \binom{r}{i} i^k = \delta_{kr} r! \tag{3}$$

for  $k \in \{0, 1, \dots, r\}$ . Now, we consider a null-polynomial  $p$  over  $\mathbb{Z}_n$ , i.e., we assume

$$p(i) = \sum_{k=0}^r a_k i^k \equiv 0 \pmod n$$

for all  $i \in \mathbb{Z}_n$ . Then, it follows from (3) that modulo  $n$

$$\begin{aligned} 0 &\equiv \sum_{i=0}^r \sum_{k=0}^r (-1)^{i+r} \binom{r}{i} a_k i^k \\ &= \sum_{k=0}^r a_k \sum_{i=0}^r (-1)^{i+r} \binom{r}{i} i^k \\ &= \sum_{k=0}^r a_k \delta_{kr} r! = a_r r! \end{aligned}$$

This establishes the desired inequality  $s(\mathbb{Z}_n) \geq s(n)$  and the proof of [Theorem 2](#) is complete.  $\square$

In order to gain more insight in the ideal of null-polynomials in  $\mathbb{Z}_n[x]$ , we need a stronger version of [Theorem 2](#). First we consider the following simple lemma:

**Lemma 4.** *Let  $A$  and  $C$  denote matrices with integer coefficients,  $y$  a vector with integer components and  $\mathbb{I}$  the identity matrix. If  $A^t C \equiv m \mathbb{I} \pmod{n}$ , then  $Ay \equiv 0 \pmod{n}$  implies  $my \equiv 0 \pmod{n}$ .*

*Proof.* Modulo  $n$  we have

$$0 \equiv C^t Ay = (y^t A^t C)^t \equiv (y^t m \mathbb{I})^t = my. \quad \square$$

[Lemma 4](#) allows to prove the following stronger form of [Theorem 2](#). This will be the technical key to the understanding of the null-polynomials in [Section 2.2](#), the structure of the additive group of the polyfunctions in [Section 3.1](#), and of their ring structure in [Section 3.2](#).

**Theorem 5.** *If  $p(x) = a_0 + a_1x + \dots + a_r x^r$  vanishes in  $\mathbb{Z}_n$  on the set  $x \in \{\alpha, \alpha + 1, \dots, \alpha + r\}$  (in particular, if  $p$  is a null-polynomial over  $\mathbb{Z}_n$ ), then  $a_k r! \equiv 0 \pmod{n}$  holds for all  $k \in \{0, 1, \dots, r\}$ .*

*Proof.* For  $\alpha \in \{0, 1, \dots, n - 1\}$  and  $j \in \{\alpha, \alpha + 1, \dots, \alpha + r\}$ , we consider the polynomials

$$g_{j,\alpha}(x) := \prod_{\substack{k=\alpha \\ k \neq j}}^{\alpha+r} (x - k) = \sum_{k=0}^r g_{jzk} x^k.$$

Obviously, we have  $g_{j,\alpha}(i) = 0$  whenever  $i \in \{\alpha, \alpha + 1, \dots, \alpha + r\}$  is different from  $j$ , and  $g_{j,\alpha}(j) = (j - \alpha)! (-1)^{\alpha+r-j} (\alpha + r - j)!$ . Hence, we obtain for  $i, j \in \{\alpha, \alpha + 1, \dots, \alpha + r\}$

$$(-1)^{\alpha+r-j} \binom{r}{j - \alpha} g_{j,\alpha}(i) = \delta_{ij} r!$$

This identity can be read as  $AD = r! \mathbb{I}$  for the matrix  $(A)_{ik} = i^k$ ,  $i \in \{\alpha, \alpha + 1, \dots, \alpha + r\}$ ,  $k \in \{0, 1, \dots, r\}$ , and the matrix

$$(D)_{kj} = (-1)^{\alpha+r-j} \binom{r}{j - \alpha} g_{jzk},$$

$k \in \{0, 1, \dots, r\}$ ,  $j \in \{\alpha, \alpha + 1, \dots, \alpha + r\}$ . Finally, from it follows  $A^t C = r! \mathbb{I}$  for  $C = D^t$ . Thus, the hypotheses of [Lemma 4](#) are fulfilled with  $m = r!$ .

From the hypothesis of [Theorem 5](#) it follows moreover, that  $Ay \equiv 0 \pmod{n}$  for the vector  $y = (a_0, a_1, \dots, a_r)^t$  and hence, the conclusion of [Lemma 4](#) gives the desired result.  $\square$

### 2.2. Decomposition of null-polynomials

In this section we analyze the null-polynomials in  $\mathbb{Z}_n[x]$ , i.e. the polynomials which vanish as a function from  $\mathbb{Z}_n$  to  $\mathbb{Z}_n$ . In particular we will determine the number of null-polynomials which then allows to compute the number of polyfunctions over  $\mathbb{Z}_n$ .

We introduce the following notation for  $2 \leq n \in \mathbb{N}$  :  $q(n)$  denotes the smallest prime divisor of  $n$ ,  $t(n) := \text{card}\{s((n, \alpha!)) | s((n, \alpha!)) \geq q(n), \alpha \in \mathbb{N}\}$  and

$$\{s((n, \alpha!)) | s((n, \alpha!)) \geq q(n), \alpha \in \mathbb{N}\} =: \{\beta_1, \beta_2, \dots, \beta_{t(n)}\},$$

where the numbers  $\beta_k$  are numbered in descending order, i.e.

$$s(n) = \beta_1 > \beta_2 > \dots > \beta_{t(n)} = q(n). \tag{4}$$

Here,  $s$  continues to denote the number-theoretic Smarandache function. We have

$$\beta_{l+1} = s((n, (\beta_l - 1)!))$$

for  $l = 1, \dots, t(n) - 1$  : To see this, let  $\alpha \in \{q(n), q(n) + 1, \dots, s(n)\}$  be such that  $\beta_l = s((n, \alpha!))$ . If  $k = (n, \alpha!)$ , then  $s(k)$  is the smallest number such that  $k | s(k)!$ . If  $\alpha > s(k)$  we might replace  $\alpha$  by  $s(k)$  and obtain  $(n, \alpha!) = (n, s(k)!) = (n, \beta_l!)$ . Therefore  $\beta_l = s((n, \beta_l!))$  and  $\beta_{l+1} = s((n, (\beta_l - 1)!)) < \beta_l$ , as claimed.

Furthermore, we define

$$\alpha_k := \frac{n}{(n, \beta_k!)} \tag{5}$$

and consider the *basic null-polynomials* in  $\mathbb{Z}_n[x]$  :

$$b_k(x) := \alpha_k \prod_{i=1}^{\beta_k} (x + i) \tag{6}$$

Why the null-polynomials are important becomes clear in [Theorem 8](#) below. But first we consider an example and give some computational remarks.

**Example 6.** The smallest prime divisor of  $n = 90$  is  $q(90) = 2$ , and  $s(90) = 6$ . In order to compute the degrees  $\beta_k$  according to (4), notice that we only need to consider values  $\alpha \in \{q(n), q(n) + 1, \dots, s(n)\}$ . For these values, we have

$\alpha$	$(90, \alpha!)$	$s((90, \alpha!))$
2	2	2
3	6	3
4	6	3
5	30	5
6	90	6

From this table, we read off  $t(90) = 4$  and

$$\beta_1 = 6, \quad \beta_2 = 5, \quad \beta_3 = 3, \quad \beta_4 = 2.$$

The coefficients  $\alpha_k$  are now computed by (5):

$$\alpha_1 = 1, \quad \alpha_2 = 3, \quad \alpha_3 = 15, \quad \alpha_4 = 45.$$

The basic null-polynomials for  $n = 90$  are therefore

$$\begin{aligned} b_1(x) &= (1 + x)(2 + x)(3 + x)(4 + x)(5 + x)(6 + x) \\ b_2(x) &= 3(1 + x)(2 + x)(3 + x)(4 + x)(5 + x) \\ b_3(x) &= 15(1 + x)(2 + x)(3 + x) \\ b_4(x) &= 45(1 + x)(2 + x) \end{aligned}$$

□

**Remark 7.** It is useful to note, that by construction we have

$$(n, (k + 1)!) = (n, \beta_j!)$$

for all  $k + 1 \in \{\beta_j, \beta_j + 1, \dots, \beta_{j-1} - 1\}$ .

Note that Kempner [12, 13] also introduces basic null-polynomials of the form

$$\tilde{b}(x) = \frac{n}{d} \prod_{i=0}^{s(d)-1} (x - i)$$

where  $d > 1$  is a divisor of  $n$ . If  $d > 1$  runs through all divisors of  $n$  in decreasing order, we only list polynomials which are not multiples of polynomials that already appeared. In the present case, when  $n = 90$ , one obtains in this way the basic null-polynomials

$$\begin{aligned} \tilde{b}_1(x) &= x(x - 1)(x - 2)(x - 3)(x - 4)(x - 5) \\ \tilde{b}_2(x) &= 3x(x - 1)(x - 2)(x - 3)(x - 4) \\ \tilde{b}_3(x) &= 15x(x - 1)(x - 2) \\ \tilde{b}_4(x) &= 45x(x - 1) \end{aligned}$$

The difference stems from the fact, that we introduced a normed null-polynomial of minimal degree by defining

$$p(x) = \prod_{i=1}^{s(n)} (x + i),$$

whereas Kempner uses

$$\tilde{p}(x) = \prod_{i=0}^{s(n)-1} (x - i).$$

Notice that the basic null-polynomial  $b_{l(n)}$  is a non-zero polynomial of minimal degree  $q(n)$  (see, e.g., [8, Theorem 8]). This fact is used in the following decomposition theorem. With the notations above we have:

**Theorem 8.** Every null-polynomial  $p$  in  $\mathbb{Z}_n[x]$  has a unique decomposition of the form

$$p(x) = \sum_{k=1}^{t(n)} q_k(x) b_k(x),$$

where  $q_k \in \mathbb{Z}_{n/\alpha_k}[x]$  has degree strictly less than  $\beta_{k-1} - \beta_k$  if  $k > 1$  and where  $\deg(q_1) = \deg(p) - \beta_1$ .

*Proof.* We start by proving the existence of a decomposition of the desired type.

In a first step, we can write

$$p(x) = q_1(x) b_1(x) + p_1(x)$$

with  $q_1 \in \mathbb{Z}_n[x]$ ,  $\deg(q_1) = \deg(p) - \beta_1$ , and  $\deg(p_1) < \beta_1$ , by dividing the polynomials with remainder (observe that  $b_1$  is normed).

Now, we assume by induction that the decomposition has the form

$$p(x) = \sum_{k=1}^l q_k(x) b_k(x) + p_l(x)$$

with  $\deg(p_l) < \beta_l$ . Then, the next step is carried out as follows:  $p_l$  is a null-polynomial in  $\mathbb{Z}_n[x]$  of the form

$$p_l(x) = a_0 + a_1x + \dots + a_{\beta_l-1}x^{\beta_l-1}.$$

Hence, by [Theorem 5](#), it follows that

$$a_i(\beta_l - 1)! \equiv 0 \pmod n$$

for all  $i \in \{0, 1, \dots, \beta_l - 1\}$ . Since  $\beta_{l+1} = s((n, (\beta_l - 1)!)) < \beta_l$ , this implies

$$\alpha_{l+1} \mid a_i$$

for all  $i \in \{0, 1, \dots, \beta_l - 1\}$ . Hence, we can divide the polynomial  $p_l$  by  $b_{l+1}$  with remainder and obtain

$$p_l(x) = q_{l+1}(x)b_{l+1}(x) + p_{l+1}(x)$$

with  $\deg(p_{l+1}) < \beta_{l+1}$ ,  $\deg(q_{l+1}) < \beta_l - \beta_{l+1}$  and  $q_{l+1} \in \mathbb{Z}_{n/\alpha_{l+1}}[x]$ . This iterative process ends as soon as  $\deg(p_{l+1}) < q(n)$ , since then, it follows that  $p_{l+1} \equiv 0 \pmod n$  by [\[8, Theorem 8\]](#).

Now, we assume by contradiction that there exist two different decompositions of  $p$ , say

$$0 \equiv \sum_{k=1}^{t(n)} b_k(q_k - \tilde{q}_k) \pmod n \tag{7}$$

with a smallest index  $k_0$  with  $q_{k_0} \neq \tilde{q}_{k_0}$ . Let  $i$  denote the highest power  $i$  in  $q_{k_0}$  and  $\tilde{q}_{k_0}$  with different coefficients  $a_i \neq \tilde{a}_i$  in  $\mathbb{Z}_{n/\alpha_{k_0}}$ . Then, according to the construction of the basic null-polynomials  $b_k$ , the coefficient of the highest power of  $x$  on the right-hand side of (7) is  $\alpha_{k_0}(a_i - \tilde{a}_i)$ . By (7), we have

$$\alpha_{k_0} \underbrace{(a_i - \tilde{a}_i)}_{\in \mathbb{Z}_{n/\alpha_{k_0}}} \equiv 0 \pmod n$$

which implies that  $a_i \equiv \tilde{a}_i \pmod{(n/\alpha_{k_0})}$ , and this is a contradiction. □

### 2.3. The number of polyfunctions

The result of the previous section allows now to compute the cardinality of the ring  $G(\mathbb{Z}_n)$ .

**Corollary 9.** *The number  $\Psi(n)$  of polyfunctions over  $\mathbb{Z}_n$  is given by*

$$\Psi(n) = \prod_{k=1}^{t(n)} (n, \beta_k!)^{\beta_k - \beta_{k-1}}$$

with the convention  $\beta_0 := 0$ .

*Proof.* We consider the additive group  $F(n)$  of polynomials in  $\mathbb{Z}_n[x]$  of degree strictly less than  $s(n)$  and the normal subgroup  $N(n)$  of all null-polynomials in  $F(n)$ . The additive group of polyfunctions over  $\mathbb{Z}_n$  is then isomorphic to the quotient  $F(n)/N(n)$ . All cosets have the cardinality of the set of null-polynomials of degree strictly less than  $s(n)$ , namely, according to [Theorem 8](#),

$$|N(n)| = \prod_{i=2}^{t(n)} \left(\frac{n}{\alpha_i}\right)^{\beta_{i-1} - \beta_i}.$$

On the other hand, the number of polynomials of degree strictly less than  $s(n)$  is  $|F(n)| = n^{\beta_1}$ . Division  $|F(n)|/|N(n)|$  gives the claimed formula. □



**Example 10.** Let us come back to Example 6 with  $n=90$ : The formula in Corollary 9 gives  $\Psi(90) = (90, 6!)^6(90, 5!)^{-1}(90, 3!)^{-2}(90, 2!)^{-1} = 246037500$  for the number of polyfunctions over  $\mathbb{Z}_{90}$ . □

In the case when  $n$  equals the power of a prime number the formula for  $\Psi$  takes a particularly simple form. Since  $\Psi$  will be shown to be multiplicative, it is actually enough to know the values of  $\Psi(p^m)$  for  $p$  prime (see Section 2.3.1).

**2.3.1. The case  $n = p^m$ ,  $p$  prime**

At this point it is useful to include a general remark on rings of polyfunctions: If  $R$  and  $S$  are commutative rings with unit element, then  $G(R \oplus S)$  and  $G(R) \oplus G(S)$  are isomorphic as rings in the obvious way. In particular, since  $\mathbb{Z}_n \oplus \mathbb{Z}_m \cong \mathbb{Z}_{nm}$  if  $m$  and  $n$  are relatively prime, we have that

$$G(\mathbb{Z}_{nm}) \cong G(\mathbb{Z}_n) \oplus G(\mathbb{Z}_m)$$

if  $(m, n) = 1$ . Therefore, we may confine ourselves to the case  $n = p^m$ ,  $p$  prime, without loss of generality.

This observation gives rise to the following version of Corollary 9, see also [10].

**Proposition 11.** *Let  $\Psi(n)$  denote the number of polyfunctions over  $\mathbb{Z}_n$  and  $s$  the Smarandache function. Then,*

- (i) *the function  $\Psi$  is multiplicative, i.e. if  $(m, n) = 1$  then  $\Psi(mn) = \Psi(m)\Psi(n)$ , and*
- (ii) *for a prime number  $p$  and  $m \in \mathbb{N}$  there holds*

$$\Psi(p^m) = \exp_p \left( \sum_{k=1}^m s(p^k) \right),$$

where we write  $\exp_p a := p^a$  for typographical reasons.

**Example 12.** Before we prove Proposition 11, we come back to Example 10, where  $n=90$ . By (i) in Proposition 11, we have

$$\Psi(90) = \Psi(2)\Psi(3^2)\Psi(5)$$

and the factors are by (ii)  $\Psi(2) = 2^2$ ,  $\Psi(3^2) = 3^{3+6}$  and  $\Psi(5) = 5^5$ . The product of these numbers is  $\Psi(90) = 4 \cdot 19683 \cdot 3125 = 246037500$  in accordance with the calculation in Example 10.

At this point, it is useful to introduce one more quantity which will play a role in the proof of Proposition 11 and which is going to be used in the description of the algebraic structure of the ring of polyfunctions over  $\mathbb{Z}_n$  (see Section 3.2). For prime numbers  $p$  and integers  $k \geq 0$ , we define

$$e_p(k) := \max\{x \in \mathbb{N}_0 : p^x | k!\}.$$

Notice that  $e_p(k) = j$  for  $jp \leq k < (j+1)p$  if  $k < p^2$ . But the next number is  $e_p(p^2) = p + 1$ .

*Proof of Proposition 11.*

- (i) The multiplicativity follows immediately from the remark preceding the proposition.
- (ii) The basic null-polynomials of degree strictly less than  $s(p^m)$  are in this case (see (6)) given by

$$b_k(x) = p^{m-e_p(k)} \prod_{i=1}^k (x - i)$$

for  $k = p, 2p, 3p, \dots, s(p^m) - p$ . Thus the number of null-polynomials in  $\mathbb{Z}_{p^m}[x]$  of degree strictly less than  $s(p^m)$  is

$$\prod_{k=1}^{\frac{s(p^m)}{p}-1} p^{pe_p(pk)},$$

and the total number of polynomials in  $\mathbb{Z}_{p^m}[x]$  of degree strictly less than  $s(p^m)$  is

$$p^{ms(p^m)}.$$

Division of both numbers yields the number of polyfunctions over  $\mathbb{Z}_{p^m}$ , namely

$$\Psi(p^m) = \exp_p \left( p \sum_{k=0}^{\frac{s(p^m)}{p}-1} (m - e_p(pk)) \right).$$

Hence, the claim is proved if we verify that for all  $m \in \mathbb{N}$  there holds

$$p \sum_{k=0}^{\frac{s(p^m)}{p}-1} (m - e_p(pk)) = \sum_{k=1}^m s(p^k). \tag{8}$$

Obviously, (8) is true for  $m = 1$ . Moreover  $s(p^{m+1}) - s(p^m)$  is either 0 or  $p$ . Using this, it is easy to see, that (8) holds for  $m + 1$  if it is correct for  $m$ , and the claim follows by induction.  $\square$

**Remark 13.**

- (i) The formula in (ii) above is particularly simple in the case  $m \leq p$ : We observe that  $s(p^k) = kp$  for  $k \leq p$ . Thus

$$\sum_{k=1}^m s(p^k) = p \binom{m+1}{2} \text{ and } \Psi(p^m) = \exp_p \left( p \binom{m+1}{2} \right)$$

for  $m \leq p$ .

- (ii) While the present approach for counting the number of polyfunctions in  $\mathbb{Z}_n$  consists in finding a unique representative for each null-polynomial, in [9, Theorem 5, p. 8], each polyfunction is shown to have a unique representative. An alternative proof of Theorem 11 is then given in [9, Theorem 6, p. 9] by counting these representatives. Moreover, a very short formula for  $\Psi(n)$  is given in [9, Theorem 9, p. 10] in terms of the Smarandache function, the Mangoldt function, and the Dirichlet convolution.
- (iii) Not only the formula for  $\Psi(n)$  looks particularly pleasant for  $n = p^m$ , also the decomposition of the additive group  $F(n)$  takes its simplest form for powers of prime numbers. As mentioned earlier in this section, it is sufficient to know the structure of  $F(n)$  for  $n = p^m$ . In this case, the decomposition in Theorem 14 simplifies to

$$F(p^m) \cong p \bigoplus_{k=0}^{s(p^m)/p-1} \mathbb{Z}_{p^{m-e_p(pk)}}.$$

Here and throughout Section 3, we will use the notation

$$nG = \bigoplus_{i=1}^n G$$

for the  $n$ -fold direct product of a group  $G$  with itself, where  $n \in \mathbb{N}$ .

### 3. Algebraic properties of the ring of polyfunctions

#### 3.1. The additive group of polyfunctions

Let  $F(n)$  denote the additive group of polyfunctions over  $\mathbb{Z}_n$  and  $F_k(n)$  the subgroup of polyfunctions which have a representative of degree less than or equal to  $k$ . Using the notation of Section 2.2, we have the following result:

**Theorem 14.** *The group  $F(n)$  is isomorphic to*

$$\bigoplus_{j=1}^{t(n)} (\beta_j - \beta_{j+1}) \mathbb{Z}_{\alpha_{j+1}}$$

with the convention  $\beta_{t(n)+1} := 0$  and  $\alpha_{t(n)+1} := n$ .

We prepare the proof by the following lemma:

**Lemma 15.** *Let  $\beta_j \leq k + 1 < \beta_{j-1}$ ,  $k \geq 0$ ,  $2 \leq j \leq t(n) + 1$ . Then there holds:*

- (i) *Every element in the quotient  $F(n)/F_k(n)$  has order less than or equal to  $\alpha_j$ .*
- (ii) *The polyfunction represented by  $x^{k+1}$  has the order  $\alpha_j$  in  $F(n)/F_k(n)$ .*

*Proof* of the Lemma.

- (i) We have, that in  $F(n)/F_k(n)$

$$\alpha_j x^{k+1} = \alpha_j x^{\beta_j} x^{k+1-\beta_j} = \underbrace{b_j(x)}_{=0 \text{ for all } x \in \mathbb{Z}_n} x^{k+1-\beta_j} = 0$$

since  $\beta_j \leq k + 1$ . Here,  $b_j$  is a basic null-polynomial (see Section 2.2). Now, every  $f \in F(n)/F_k(n)$  contains  $x^{k+1}$  as a factor and hence  $\text{ord}(f) \leq \alpha_j$ .

- (ii) Suppose  $\alpha x^{k+1} = 0$  in  $F(n)/F_k(n)$  for some  $\alpha$  in  $\mathbb{Z}_n$ . Then, by Theorem 5,  $\alpha(k + 1)! \equiv 0 \pmod n$ . Hence,  $\alpha$  is a multiple of

$$\frac{n}{(n, (k + 1)!)} > \frac{n}{(n, \beta_{j-1}!)} = \alpha_{j-1}$$

since  $k + 1 < \beta_{j-1}$ . Thus we have

$$\frac{n}{(n, (k + 1)!)} \geq \alpha_j$$

(see Remark 7) and hence  $\alpha \notin \{1, 2, \dots, \alpha_j - 1\}$ . □

Now, Theorem 14 follows from Lemma 15 by iteration: First, we observe that  $1 \in F(n)$  has the (maximal) order  $n = \alpha_{t(n)+1}$ . Thus

$$F(n) \cong \mathbb{Z}_n \oplus F(n)/F_0(n)$$

since finite Abelian groups split off a maximal cyclic subgroup. Now, we proceed iteratively and split in each step

$$F(n)/F_k(n) \cong \mathbb{Z}_{\alpha_j} \oplus F(n)/F_{k+1}(n)$$

by using Lemma 15. The process stops as soon as  $k + 1 = s(n)$ , and by collecting the quotients we obtain the claimed decomposition. □

**Example 16.** We revisit [Examples 6, 10, and 12](#) respectively in order to compute the decomposition of  $F(90)$ . With the notational conventions of [Theorem 14](#) we have:

$j$	1	2	3	4	5
$\alpha_j$	1	3	15	45	90
$\beta_j$	6	5	3	2	0

In a first step, we decompose

$$F(90) \cong \mathbb{Z}_{90} \oplus F(90)/F_0(90).$$

If  $k = 0$ , we have  $\beta_5 < k + 1 < \beta_4$  and hence  $F(90)/F_0(90)$  splits off a cyclic subgroup of order  $\alpha_5 = 90$  and hence  $F(90)/F_0(90) \cong \mathbb{Z}_{90} \oplus F(90)/F_1(90)$ .

If  $k = 1$ , we have  $\beta_4 \leq k + 1 < \beta_3$  and hence  $F(90)/F_1(90)$  splits off a cyclic subgroup of order  $\alpha_4$  and hence  $F(90)/F_1(90) \cong \mathbb{Z}_{45} \oplus F(90)/F_2(90)$ .

If  $k = 2, 3$ , we have  $\beta_3 \leq k + 1 < \beta_2$  so we might split off twice the subgroup  $\mathbb{Z}_{15}$  and hence  $F(90)/F_2(90) \cong \mathbb{Z}_{15} \oplus \mathbb{Z}_{15} \oplus F(90)/F_4(90)$ .

Finally, if  $k = 4$ , it holds that  $\beta_2 \leq k + 1 < \beta_1$  and we find  $F(90)/F_4(90) \cong \mathbb{Z}_3$  and the process ends. This leads to the desired decomposition

$$F(90) \cong \mathbb{Z}_3 \oplus 2\mathbb{Z}_{15} \oplus \mathbb{Z}_{45} \oplus 2\mathbb{Z}_{90}$$

and we find again  $|F(90)| = 3 \cdot 15^2 \cdot 45 \cdot 90^2 = 246037500$  in accordance with [Examples 10 and 12](#).

**Remark 17.** Since it turns out that it is sufficient to know the structure of  $F(p^m)$  for prime numbers  $p$  (see [Section 2.3.1](#)), observe that in this case, the decomposition described in [Theorem 14](#) takes a particularly simple form (see [Remark 13](#), item (iii)).

### 3.2. The ring of polyfunctions

In this section, we use the shorthand notation  $G(n)$  for  $G(\mathbb{Z}_n)$ , i.e. the ring of polyfunctions over  $\mathbb{Z}_n$ . We recall that  $G(mn) \cong G(m) \oplus G(n)$  if  $(m, n) = 1$ , and hence we may restrict ourselves to investigate the structure of  $G(n)$  in the case  $n = p^m$  for  $p$  prime. Let  $I_{p,m}$  be the ideal of polynomials in  $\mathbb{Z}_{p^m}[x]$  defined by

$$I_{p,m} = \{f \in \mathbb{Z}_{p^m}[x] : f(kp) = 0 \text{ for all } k\}.$$

Then, we have the following decomposition:

**Theorem 18.**

- (i)  $G(p^m) \cong p \mathbb{Z}_{p^m}[x]/I_{p,m}$ .
- (ii)  $\mathbb{Z}_{p^m}[x]/I_{p,m}$  is not decomposable.

*Proof.* We proceed in several steps:

Step 1: For  $j \in \{0, 1, \dots, p - 1\}$  let

$$R_j(p^m) := \{f \in G(p^m) : f(k) = 0 \text{ if } k \not\equiv j \pmod{p}\}.$$

It is clear that  $R_j(p^m)$  is an ideal of  $G(p^m)$  and that  $R_i(p^m) \cap R_j(p^m) = \{0\}$  if  $i \neq j$ .

Step 2: We show that  $G(p^m) \cong \bigoplus_{j=0}^{p-1} R_j(p^m)$ .

To see this, we define

$$\varepsilon_0(x) := 1 - x^{m\varphi(p^m)},$$

where  $\varphi$  denotes Euler's  $\varphi$ -function. Then we have

$$\varepsilon_0(k) \equiv \begin{cases} 0 & \text{if } k \not\equiv 0 \pmod p \\ 1 & \text{if } k \equiv 0 \pmod p \end{cases} \pmod{p^m}.$$

Moreover, for  $\varepsilon_j(x) := \varepsilon_0(x - j)$ , we have similarly

$$\varepsilon_j(k) \equiv \begin{cases} 0 & \text{if } k \not\equiv j \pmod p \\ 1 & \text{if } k \equiv j \pmod p \end{cases} \pmod{p^m}.$$

Hence, for  $f \in G(p^m)$ , we have  $f\varepsilon_j \in R_j(p^m)$  and

$$f = \sum_{j=0}^{p-1} f\varepsilon_j.$$

Then,

$$\Phi_0 : G(p^m) \rightarrow \bigoplus_{j=0}^{p-1} R_j(p^m), f \mapsto (f\varepsilon_0, f\varepsilon_1, \dots, f\varepsilon_{p-1})$$

is a ring isomorphism (the ring operations  $+$  and  $\cdot$  are, as usual, defined componentwise).

Step 3: We show that  $R_j(p^m) \cong R_0(p^m)$  for  $j \in \{0, 1, \dots, p - 1\}$ .

The map

$$\Phi_1 : R_0(p^m) \rightarrow R_j(p^m), f \mapsto g,$$

where  $g(x) := f(x - j)$ ,  $x \in \mathbb{Z}_{p^m}$  is a ring isomorphism. Hence, according to the second step, we have that

$$G(p^m) \cong pR_0(p^m).$$

Step 4: We show that  $R_0(p^m) \cong \mathbb{Z}_{p^m}[x]/I_{p,m}$ .

To see this, we consider the map

$$\Phi_2 : \mathbb{Z}_{p^m}[x] \rightarrow R_0(p^m), f \mapsto f\varepsilon_0.$$

$\Phi_2$  is a surjective ring homomorphism. If  $f \in \ker(\Phi_2)$ , then  $\Phi_2(f)(k) = 0$  for all  $k \in \mathbb{Z}_{p^m}$  and hence  $f(jp)\varepsilon_0(jp) = f(jp) = 0$  for all  $j$ . This implies that  $f \in I_{p,m}$ . Arguing in the opposite direction, we conclude that  $f \in I_{p,m}$  implies that  $f \in \ker(\Phi_2)$ .

Now, (i) follows from the third and the fourth step and it remains to prove (ii). This is done in the last step:

Step 5: We show, that  $R_0(p^m)$  is not decomposable:

Let  $f \in R_0(p^m)$  be such that  $f^2 = f$ . In particular, this means  $f^2(jp) = f(jp)$  for all  $j$ . Hence,  $f(jp) \in \{0, 1\}$  for all  $j$ . Observe, that

$$f(jp) \equiv f(0) \pmod p$$

and hence

$$f(k) = 0 \quad \text{for all } k \in \mathbb{Z}_{p^m}$$

or

$$f(k) = \begin{cases} 0 & \text{if } k \not\equiv 0 \pmod p, \\ 1 & \text{if } k \equiv 0 \pmod p. \end{cases}$$

It follows that only two elements  $f \in R_0(p^m)$  with the property  $f^2 = f$  exist. In a decomposable ring there are at least four elements with  $f^2 = f$ . This completes the proof.  $\square$

We now want to investigate the structure of the ideal  $I_{p,m}$  in more detail. First, for  $m \in \mathbb{N}$  and a prime number  $p$ , we define

$$s^*(p^m) := \min\{x \in \mathbb{N} : p^m | p^x x!\}.$$

Then, for  $r \in \{1, 2, \dots, s^*(p^m) - 1\}$  let

$$e^*(r) := \max\{x \in \mathbb{N} : p^x | p^r r!\}$$

and

$$e^*(s^*(p^m)) := m.$$

**Remark 19.**  $s^*$  is connected with the Smarandache function by

$$p s^*(p^m) = s(p^m).$$

Let us assume, that  $f \in I_{p,m}$  :

$$f(x) = a_1x + a_2x^2 + \dots + a_r x^r.$$

Then,  $f(jp) \equiv 0 \pmod{p^m}$  for all  $j$  and hence, the polynomial

$$g(x) := a_1px + a_2p^2x^2 + \dots + a_r p^r x^r$$

is a null-polynomial over  $\mathbb{Z}_{p^m}$ . Hence, it follows from [Theorem 5](#) that

$$a_k p^k r! \equiv 0 \pmod{p^m}$$

for all  $k \in \{1, 2, \dots, r\}$ . From, this congruence, we immediately obtain the following conclusion.

**Proposition 20.**

- (i) If  $f \in I_{p,m}$  is normed, then  $\deg(f) \geq s^*(p^m)$ .
- (ii) If  $f \in I_{p,m}$ ,  $f(x) = a_1x + a_2x^2 + \dots + a_r x^r$ , with  $r \leq s^*(p^m)$ , then

$$p^{m-e^*(r)+r-k} | a_k$$

holds for all  $k \in \{1, 2, \dots, r\}$ .

Now, the polynomials in  $I_{p,m}$  can be decomposed similarly as the null-polynomials (see [Section 2.2](#) and (6)). The basic polynomials are in this case

$$b_k^*(x) := p^{m-e^*(k)} \prod_{j=1}^k (x + jp)$$

for  $k \in \{1, 2, \dots, s^*(p^m)\}$ . In fact, we have:

**Lemma 21.**  $b_k^* \in I_{p,m}$  for all  $k \in \{1, 2, \dots, s^*(p^m)\}$ .

*Proof.* We have

$$\begin{aligned} b_k^*(ip) &= p^{m-e^*(k)} \prod_{j=1}^k (ip + jp) \\ &= p^{m-e^*(k)} p^k \binom{i+k}{k} k! \end{aligned} \tag{9}$$

The right-hand side of (9) is congruent 0 modulo  $p^m$  for all  $j$  as is easily seen by treating separately the cases  $k < s^*(p^m)$  and  $k = s^*(p^m)$ . □

### 3.3. The units in $G(\mathbb{Z}_n)$

The previous results on the algebraic structure of the ring of polyfunctions over  $\mathbb{Z}_n$  allow now to answer more specific questions. As an example, we consider the multiplicative subgroup  $U_n$  of units in  $G(\mathbb{Z}_n)$  and ask for the size of  $U_{3^k}$ .

For this, we consider the set  $Q$  of polynomials in  $\mathbb{Z}_{3^k}[x]$  with degree strictly less than  $s(3^k) =: r + 1$ . A polynomial  $q \in Q$ ,  $q(x) = a_0 + a_1x + a_2x^2 + \dots + a_r x^r$  with  $a_i \in \mathbb{Z}_{3^k}$ , represents according to [9, Proposition 3, p. 5] an invertible polyfunction (i.e. a unit in  $G(\mathbb{Z}_{3^k})$ ) if and only if its image is contained in the multiplicative subgroup of units in  $\mathbb{Z}_{3^k}$ , that is

$$q(i) \not\equiv 0 \pmod{3} \quad \text{for } i = 0, 1, 2. \tag{10}$$

(Observe that  $q(x + 3j) \equiv q(x) \pmod{3}$  for all integers  $x$  and  $j$ .) Let

$$\Sigma_1 := \sum_{\substack{i=1 \\ i \text{ odd}}}^r a_i$$

and

$$\Sigma_2 := \sum_{\substack{i=2 \\ i \text{ even}}}^r a_i.$$

Then, we can rewrite (10) in the form

$$\left. \begin{aligned} a_0 &\not\equiv 0 \pmod{3} \\ a_0 + \Sigma_1 + \Sigma_2 &\not\equiv 0 \pmod{3} \\ a_0 + \Sigma_1 + 2\Sigma_2 &\not\equiv 0 \pmod{3} \end{aligned} \right\} \tag{11}$$

It is then easy to determine the total number  $X$  of solutions  $(a_0, a_1, \dots, a_r) \in \mathbb{Z}_{3^k}^{r+1}$  of (11):

$$X = 8 \cdot 3^{k(r+1)-3}.$$

Now, two polynomials in  $Q$  represent the same unit in  $G(\mathbb{Z}_{3^k})$  if and only if their difference is a null-polynomial of degree strictly less than  $s(3^k)$ . The number  $Y$  of such null-polynomials is according to Proposition 11 given by

$$Y = \frac{3^{ks(3^k)}}{\Psi(3^k)}.$$

Division of  $X$  by  $Y$  yields the following result:

**Proposition 22.**

$$|U_{3^k}| = \left(\frac{2}{3}\right)^3 \Psi(3^k) = \left(\frac{2}{3}\right)^3 \exp_3\left(\sum_{i=1}^k s(3^i)\right).$$

In other words, the fraction of units among all polyfunctions in  $G(\mathbb{Z}_{3^k})$  is  $\frac{8}{27}$ , independently of  $k$ .

Proposition 22 gives a flavor of a more general result: In Section 4.2, we will determine the number of units in the ring  $G_d(\mathbb{Z}_{p^m})$  of multivariate polyfunctions.

## 4. Polyfunctions in several variables

In order to keep the formulas short, we use the following multi-index notation: For  $\mathbf{k} = (k_1, k_2, \dots, k_d) \in \mathbb{N}_0^d$  and  $\mathbf{x} := (x_1, x_2, \dots, x_d) \in \mathbb{N}_0^d$  let

$$\mathbf{x}^{\mathbf{k}} := \prod_{i=1}^d x_i^{k_i}, \quad \mathbf{k}! := \prod_{i=1}^d k_i!, \quad |\mathbf{k}| := \sum_{i=1}^d k_i, \quad \text{and} \quad \binom{\mathbf{x}}{\mathbf{k}} := \prod_{i=1}^d \binom{x_i}{k_i}.$$

Recall that

$$G_d(R) = \{f : R^d \rightarrow R \mid \exists p \in R[x_1, x_2, \dots, x_d] \quad \forall x \in R^j \Rightarrow p(x) = f(x)\},$$

equipped with pointwise addition and multiplication denotes the ring of polyfunctions in  $d$  variables, whenever  $R$  is a commutative ring with unit element.

An alternative (but equivalent) construction is to define  $G_d(R)$  recursively as the ring of polyfunctions in one variable from  $R$  to  $G_{d-1}(R)$  by

$$G_d(R) = \{f : R \rightarrow G_{d-1}(R) \mid \exists p \in G_{d-1}(R)[x] \quad \forall x \in R \Rightarrow p(x) = f(x)\}.$$

#### 4.1. The number of multivariate polyfunctions on $\mathbb{Z}_n$

We recall a few facts and definitions from [9] in order to count the number of polyfunctions on  $\mathbb{Z}_n$  in  $d$  variables, and again it is enough to find a formula for  $n = p^m$  since we have the natural decomposition  $G_d(\mathbb{Z}_{ab}) \cong G_d(\mathbb{Z}_a) \oplus G_d(\mathbb{Z}_b)$  if  $(a, b) = 1$ . We define the set

$$S_d(n) := \{\mathbf{k} \in \mathbb{N}_0^d : n \nmid \mathbf{k}!\} \tag{12}$$

and let  $s_d(n) := |S_d(n)|$  be the generalization of the Smarandache function introduced in [9]. As for the case of one variable we define

$$e_p(\mathbf{k}) := \max\{x \in \mathbb{N}_0 : p^x \mid \mathbf{k}!\}.$$

**Definition 23.** Let  $a$  be an element of  $\mathbb{Z}_n$ . We say, the polynomial  $a\mathbf{x}^{\mathbf{k}} \in \mathbb{Z}_n[\mathbf{x}]$  is reducible (modulo  $n$ ) if a polynomial  $p(\mathbf{x}) \in \mathbb{Z}_n[\mathbf{x}]$  exists with  $\deg(p) < |\mathbf{k}|$  such that  $a\mathbf{x}^{\mathbf{k}} \equiv p(\mathbf{x}) \pmod n$  for all  $\mathbf{x} \in \mathbb{Z}_n^d$ . Moreover, we say that  $a\mathbf{x}^{\mathbf{k}}$  is weakly reducible if  $a\mathbf{x}^{\mathbf{k}} \equiv p(\mathbf{x}) \pmod n$  for all  $\mathbf{x} \in \mathbb{Z}_n^d$ , where  $p \in \mathbb{Z}_n[\mathbf{x}]$  is such that  $\deg(p) \leq |\mathbf{k}|$  (instead of  $\deg(p) < |\mathbf{k}|$ ) and such that  $\mathbf{x}^{\mathbf{k}}$  (or a multiple of it) does not appear as a monomial in  $p$ .

We will need the following lemma (see also [9, Lemma 4, p. 6]) which characterizes tuples  $\mathbf{k}$  for which  $a\mathbf{x}^{\mathbf{k}}$  is (weakly) reducible in  $\mathbb{Z}_n[\mathbf{x}]$ .

**Lemma 24.**

- (i) If  $a\mathbf{x}^{\mathbf{k}}$  is weakly reducible modulo  $n$ , then  $n \mid a\mathbf{k}!$ .
- (ii) If  $n \mid a\mathbf{k}!$ , then  $a\mathbf{x}^{\mathbf{k}}$  is reducible modulo  $n$ .

*Proof.*

(i) We assume, that  $p(\mathbf{x})$  reduces  $a\mathbf{x}^{\mathbf{k}}$  weakly. Hence,  $q(\mathbf{x}) := a\mathbf{x}^{\mathbf{k}} - p(\mathbf{x})$  is a null-polynomial in  $d$  variables over  $\mathbb{Z}_n$ . Let us define the following “integral” for functions  $f : \mathbb{Z}_n \rightarrow \mathbb{Z}_n$ :

$$\int_0^m f(x) d\mu(x) := \sum_{j=0}^m (-1)^{m-j} \binom{m}{j} f(j).$$

Now, we write  $q$  in the form

$$q(\mathbf{x}) = \sum_{\substack{\mathbf{l} \in \mathbb{N}_0^d \\ |\mathbf{l}| \leq |\mathbf{k}|}} q_{\mathbf{l}} \mathbf{x}^{\mathbf{l}}$$

for suitable coefficients  $q_{\mathbf{l}} \in \mathbb{Z}_n$ , with  $q_{\mathbf{k}} = a$ . Then, modulo  $n$ , we have



$$\begin{aligned}
 0 &= \int_0^{k_d} \int_0^{k_{d-1}} \dots \int_0^{k_1} q(\mathbf{x}) d\mu(x_1) \dots d\mu(x_{d-1}) d\mu(x_d) = \\
 &= \sum_{\substack{\mathbf{l} \in \mathbb{N}_0^d \\ |\mathbf{l}| \leq |\mathbf{k}|}} q_{\mathbf{l}} \int_0^{k_d} \int_0^{k_{d-1}} \dots \int_0^{k_1} \mathbf{x}^{\mathbf{l}} d\mu(x_1) \dots d\mu(x_{d-1}) d\mu(x_d).
 \end{aligned}$$

Observe that the only term which does not vanish in the above sum is

$$q_{\mathbf{k}} \int_0^{k_d} \int_0^{k_{d-1}} \dots \int_0^{k_1} \mathbf{x}^{\mathbf{k}} d\mu(x_1) \dots d\mu(x_{d-1}) d\mu(x_d) = a\mathbf{k}!.$$

In fact all other terms vanish by (3), since  $|\mathbf{l}| \leq |\mathbf{k}|$  and  $\mathbf{l} \neq \mathbf{k}$  implies that for some  $i \in \{0, 1, \dots, d\}$  we have  $l_i < k_i$  and therefore the integral with respect to  $x_i$  gives zero. This completes the proof of (i).

(ii) We assume, that  $n|a\mathbf{k}!$ . Then, the polynomial

$$q(\mathbf{x}) := a \prod_{i=1}^d \prod_{l=1}^{k_i} (x_i + l) = a\mathbf{k}! \prod_{i=1}^d \binom{x_i + k_i}{k_i} = a\mathbf{k}! \binom{\mathbf{x} + \mathbf{k}}{\mathbf{k}}$$

is a null-polynomial over  $\mathbb{Z}_n$  and the term of maximal degree is  $a\mathbf{x}^{\mathbf{k}}$ . Hence,  $q(\mathbf{x}) - a\mathbf{x}^{\mathbf{k}}$  reduces  $a\mathbf{x}^{\mathbf{k}}$ . □

As an immediate consequence, we have:

**Corollary 25.** *A monomial  $\mathbf{x}^{\mathbf{k}}$  is reducible modulo  $n$  if and only if it is weakly reducible.*

Furthermore it is proved in [9, Proposition 5, p. 8] that every polyfunction  $f \in G_d(\mathbb{Z}_{p^m})$  has a unique representative of the form

$$f(\mathbf{x}) = \sum_{\substack{\mathbf{k} \in \mathbb{N}_0^d \\ e_p(\mathbf{k}) < m}} \alpha_{\mathbf{k}} \mathbf{x}^{\mathbf{k}},$$

where  $\alpha_{\mathbf{k}} \in \{0, 1, \dots, p^{m-e_p(\mathbf{k})} - 1\}$ . Notice, that  $e_p(\mathbf{k}) < m$  if and only if  $\mathbf{k} \in S_d(p^m)$  and hence this representative can be written as

$$f(\mathbf{x}) = \sum_{\mathbf{k} \in S_d(p^m)} \alpha_{\mathbf{k}} \mathbf{x}^{\mathbf{k}}. \tag{13}$$

In the case of one variable, what the Smarandache function really does is counting the number of monomials  $x^k$ ,  $k \in \mathbb{N}_0$ , which are not reducible. Using the unique representative of a polyfunction above we can count the number of monomials  $\mathbf{x}^{\mathbf{k}}$ ,  $\mathbf{k} \in \mathbb{N}_0^d$ , which are not reducible and hence to find a formula for  $\Psi_d(n)$  which counts the number of polyfunctions in  $G_d(\mathbb{Z}_n)$ . In view of (13), we have for every coefficient  $\alpha_{\mathbf{k}}$  exactly  $p^{m-e_p(\mathbf{k})}$  choices and therefore we obtain:

**Proposition 26.** *The number of polyfunctions in  $G_d(\mathbb{Z}_{p^m})$  is given by*

$$\Psi_d(p^m) = \prod_{\substack{\mathbf{k} \in \mathbb{N}_0^d \\ e_p(\mathbf{k}) < m}} p^{m-e_p(\mathbf{k})}. \tag{14}$$

On the other hand it is shown in [9, Theorem 6, p. 9] that

$$\Psi_d(p^m) = \exp_p \left( \sum_{k=1}^m s_d(p^k) \right). \tag{15}$$

The equivalence of the two formulas (14) and (15) can be established by a similar induction argument as in the [proof of Proposition 11](#). However, it is much more instructive, to give a direct algebraic argument: We consider the surjective homomorphism  $H$  of rings defined by

$$H : G_d(\mathbb{Z}_{p^{m+1}}) \rightarrow G_d(\mathbb{Z}_{p^m}), \quad f \mapsto H(f) := h \circ f \circ h^*. \tag{16}$$

Here,

$$h : \mathbb{Z}_{p^{m+1}} \rightarrow \mathbb{Z}_{p^m}, \quad [x]_{p^{m+1}} \mapsto [x]_{p^m},$$

where  $[x]_n$  denotes the coset of  $x \in \mathbb{Z}$  modulo  $n$ . Similarly,

$$h^* : \mathbb{Z}_{p^m}^d \rightarrow \mathbb{Z}_{p^{m+1}}^d, \quad [x]_{p^m} \mapsto [x]_{p^{m+1}}$$

where  $[x]_{p^m} = [(x_1, \dots, x_d)]_{p^m} := ([x_1]_{p^m}, \dots, [x_d]_{p^m})$  for  $0 \leq x_i < p^m$ . Then,

$$\Psi_d(p^{m+1}) = |G_d(\mathbb{Z}_{p^{m+1}})| = |G_d(\mathbb{Z}_{p^m})| |\ker H|$$

and the equivalence of (14) and (15) is proved if we can show that

$$|\ker H| = p^{s_d(p^{m+1})}. \tag{17}$$

In view of (13), every polyfunction  $f \in G_d(\mathbb{Z}_{p^{m+1}})$  has a unique representation

$$f(\mathbf{x}) = \sum_{\mathbf{k} \in S_d(p^{m+1})} \alpha_{\mathbf{k}} \mathbf{x}^{\mathbf{k}},$$

where  $\alpha_{\mathbf{k}} \in \{0, 1, \dots, p^{m+1-e_p(\mathbf{k})} - 1\}$ . Since every number in this set can be written in a unique way as

$$\alpha_{\mathbf{k}} = \sum_{\{i \leq m+1 : \mathbf{k} \in S_d(p^i)\}} p^{m+1-i} \alpha_{\mathbf{k}i},$$

where  $\alpha_{\mathbf{k}i} \in \mathbb{Z}_p$ , all coefficients can be described as  $ip^{m+1-e_p(\mathbf{k})}$ ,  $\mathbf{k} \in S_d(p^{m+1})$  and  $i = 0, 1, \dots, p - 1$  (see also [9, Proposition 5, p. 8]).

Observe, that  $f \in \ker H$  if and only if  $f(\mathbf{x}) \equiv 0 \pmod{p^m}$ , i.e. exactly if  $pf$  vanishes as a function  $\mathbb{Z}_{p^{m+1}}^d \rightarrow \mathbb{Z}_{p^{m+1}}$ .

Now, for each  $\mathbf{k} \in S_d(p^{m+1})$  and every  $a_i := ip^{m-e_p(\mathbf{k})}$ ,  $i = 0, 1, \dots, p - 1$ , the monomial  $a_i p \mathbf{x}^{\mathbf{k}}$  is reducible modulo  $p^{m+1}$  by [Lemma 24](#) since  $p^{m+1} | a_i p \mathbf{k}!$ . This implies that  $p^m | a_i \mathbf{k}!$  and hence the monomial  $a_i \mathbf{x}^{\mathbf{k}}$  is reducible modulo  $p^m$ , i.e. there exists a polynomial  $q_{i,\mathbf{k}}(\mathbf{x})$  of degree strictly less than  $|\mathbf{k}|$  which agrees modulo  $p^m$  with  $a_i \mathbf{x}^{\mathbf{k}}$ . Thus,  $a_i \mathbf{x}^{\mathbf{k}} - q_{i,\mathbf{k}}(\mathbf{x})$  represent polyfunctions in  $\ker H$ . By the considerations above, every  $f \in \ker H$  has therefore a unique representation of the form

$$f(\mathbf{x}) = \sum_{\mathbf{k} \in S_d(p^{m+1})} a_i \mathbf{x}^{\mathbf{k}} - q_{i,\mathbf{k}}(\mathbf{x}), \quad i \in \{0, 1, \dots, p - 1\}$$

and hence  $|\ker H| = p^{|S_d(p^{m+1})|} = p^{s_d(p^{m+1})}$ , as claimed. □

#### 4.2. The number of units in $G_d(\mathbb{Z}_{p^m})$

We end this discussion by coming back to the question of units in the ring of polyfunctions (see [Proposition 22](#)). We denote by  $U_{p^m}^d$  the multiplicative subgroup of units in  $G_d(\mathbb{Z}_{p^m})$  and continue

to use the notation  $\Psi_d(p^m) = |G_d(\mathbb{Z}_{p^m})|$ . We refer here to the formula  $\Psi_d(p^m) = \exp_p(\sum_{k=1}^m s_d(p^k))$  from Proposition 26, where  $s_d(p^k)$  is defined in (12). Then the following proposition holds

**Proposition 27.**

$$|U_{p^m}^d| = \left(\frac{p-1}{p}\right)^{dp} \Psi_d(p^m).$$

*Proof.* Using [9, Proposition 3, p. 5], we know that the elements in  $U_{p^m}^d$  are precisely the unit-valued polyfunctions in  $G_d(\mathbb{Z}_{p^m})$ . Note that every function  $\mathbb{Z}_p^d \rightarrow \mathbb{Z}$  is a polyfunction hence  $|G_d(\mathbb{Z}_p)| = p^{dp}$  and since there are  $p-1$  units in  $\mathbb{Z}_p$ , we have

$$|U_p^d| = (p-1)^{dp}.$$

We use again the map

$$H : G_d(\mathbb{Z}_{p^{m+1}}) \rightarrow G_d(\mathbb{Z}_{p^m}), \quad f \mapsto H(f) = h \circ f \circ h^*$$

as defined after (16). Now

$$f \in U_{p^{m+1}}^d \iff H(f) \in U_{p^m}^d.$$

Indeed,  $f \in U_{p^{m+1}}^d$  if and only if  $f \circ h^*$  is unit valued with values in  $\mathbb{Z}_{p^{m+1}}$  if and only if  $((f \circ h^*)(\mathbf{x}), p^{m+1}) = 1$  if and only if  $(H(f)(\mathbf{x}), p^{m+1}) = 1$  if and only if  $(H(f)(\mathbf{x}), p^m) = 1$  (see also [2, Remark 12.1, Lemmas 7 and 8]). We conclude that

$$|U_{p^{m+1}}^d| = |\ker H| |U_{p^m}^d|$$

and it follows from the proof of Proposition 26 that  $|\ker H| = p^{s_d(p^{m+1})}$ . So, inductively

$$|U_{p^m}^d| = \prod_{i=2}^m p^{s_d(p^i)} |U_p^d|$$

and since  $|U_p^d| = (p-1)^{dp}$  we find using the formula for  $\Psi_d(p^m)$  of Proposition 26

$$|U_{p^m}^d| = (p-1)^{dp} \exp_p\left(\sum_{i=2}^m s_d(p^i)\right) = \left(\frac{p-1}{p}\right)^{dp} \Psi_d(p^m). \quad \square$$

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