*International J.Math. Combin. Vol.1 (2009), 08-17*

# Topological Multi-groups and Multi-fields

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**Abstract**: Topological groups, particularly, Lie groups are very important in differential geometry, analytic mechanics and theoretical physics. Applying Smarandache multi-spaces, topological spaces, particularly, manifolds and groups were generalized to combinatorial manifolds and multi-groups underlying a combinatorial structure in references. Then *whether can we generalize their combination, i.e., topological group or Lie group to a multiple one*? The answer is YES. In this paper, we show how to generalize topological groups and the homomorphism theorem for topological groups to multiple ones. By applying the classification theorem of topological fields, the topological multi-fields are classified in this paper.

**Key Words**: Smarandache multi-space, combinatorial system, topological group, topological multi-group, topological multi-field.

**AMS(2000)**: 05E15, 08A02, 15A03, 20E07, 51M15.

## §1. Introduction

In the reference [9], we formally introduced the conceptions of Smarandachely systems and combinatorial systems as follows:

**Definition** 1.1 *A rule in a mathematical system $(\Sigma;\mathcal{R})$ is said to be Smarandachely denied if it behaves in at least two different ways within the same set $\Sigma$, i.e., validated and invalided, or only invalided but in multiple distinct ways.*

*A Smarandache system $(\Sigma;\mathcal{R})$ is a mathematical system which has at least one Smarandachely denied rule in $\mathcal{R}$.*

**Definition** 1.2 *For an integer $m \geq 2$, let $(\Sigma_1;\mathcal{R}_1)$, $(\Sigma_2;\mathcal{R}_2)$, $\cdots$, $(\Sigma_m;\mathcal{R}_m)$ be $m$ mathematical systems different two by two. A Smarandache multi-space is a pair $(\widetilde{\Sigma};\widetilde{\mathcal{R}})$ with*

$$\widetilde{\Sigma} = \bigcup_{i=1}^{m} \Sigma_i, \quad and \quad \widetilde{\mathcal{R}} = \bigcup_{i=1}^{m} \mathcal{R}_i.$$

**Definition** 1.3 *A combinatorial system $\mathscr{C}_G$ is a union of mathematical systems $(\Sigma_1;\mathcal{R}_1)$,$(\Sigma_2;\mathcal{R}_2)$, $\cdots$, $(\Sigma_m;\mathcal{R}_m)$ for an integer $m$, i.e.,*

[1]Received December 12, 2008. Accepted January 8, 2009.

$$\mathscr{C}_G = (\bigcup_{i=1}^{m} \Sigma_i;\ \bigcup_{i=1}^{m} \mathcal{R}_i)$$

*with an underlying connected graph structure $G$, where*

$$V(G) = \{\Sigma_1, \Sigma_2, \cdots, \Sigma_m\},$$
$$E(G) = \{\ (\Sigma_i, \Sigma_j) \mid \Sigma_i \bigcap \Sigma_j \neq \emptyset, 1 \leq i, j \leq m\}.$$

These notions enable us to establish combinatorial theory on geometry, particularly, *combinatorial differential geometry* in [8], also those of combinatorial theory for other sciences [7], for example, algebra systems, etc..

By definition, a *topological group* is nothing but the combination of a group associated with a topological space structure, i.e., an algebraic system $(\mathscr{H}; \circ)$ with conditions following hold ([16]):

$(i)$ $(\mathscr{H}; \circ)$ is a group;

$(ii)$ $\mathscr{H}$ is a topological space;

$(iii)$ the mapping $(a, b) \to a \circ b^{-1}$ is continuous for $\forall a, b \in \mathscr{H}$,

Application of topological group, particularly, Lie groups shows its importance to differential geometry, analytic mechanics, theoretical physics and other sciences. Whence, it is valuable to generalize topological groups to a multiple one by algebraic multi-systems.

**Definition** 1.4 *A topological multi-group $(\mathscr{S}_G; \mathscr{O})$ is an algebraic multi-system $(\widetilde{\mathscr{A}}; \mathscr{O})$ with $\widetilde{\mathscr{A}} = \bigcup_{i=1}^{m} \mathscr{H}_i$ and $\mathscr{O} = \bigcup_{i=1}^{m} \{\circ_i\}$ with conditions following hold:*

*$(i)$ $(\mathscr{H}_i; \circ_i)$ is a group for each integer $i$, $1 \leq i \leq m$, namely, $(\mathscr{H}, \mathscr{O})$ is a multi-group;*

*$(ii)$ $\widetilde{\mathscr{A}}$ is a combinatorially topological space $\mathscr{S}_G$, i.e., a combinatorial topological space underlying a structure $G$;*

*$(iii)$ the mapping $(a, b) \to a \circ b^{-1}$ is continuous for $\forall a, b \in \mathscr{H}_i$, $\forall \circ \in \mathcal{O}_i$, $1 \leq i \leq m$.*

A *combinatorial Euclidean space* is a combinatorial system $\mathscr{C}_G$ of Euclidean spaces $\mathbf{R}^{n_1}$, $\mathbf{R}^{n_2}$, $\cdots$, $\mathbf{R}^{n_m}$ with an underlying structure $G$, denoted by $\mathscr{E}_G(n_1, \cdots, n_m)$ and abbreviated to $\mathscr{E}_G(r)$ if $n_1 = \cdots = n_m = r$. It is obvious that a topological multi-group is a topological group if $m = 1$ in Definition 1.4. Examples following show the existence of topological multi-groups.

**Example** 1.1 Let $\mathbf{R}^{n_i}, 1 \leq i \leq m$ be Euclidean spaces with an additive operation $+_i$ and scalar multiplication $\cdot$ determined by

$$(\lambda_1 \cdot x_1, \lambda_2 \cdot x_2, \cdots, \lambda_{n_i} \cdot x_{n_i}) +_i (\zeta_1 \cdot y_1, \zeta_2 \cdot y_2, \cdots, \zeta_{n_i} \cdot y_{n_i})$$
$$= (\lambda_1 \cdot x_1 + \zeta_1 \cdot y_1, \lambda_2 \cdot x_2 + \zeta_2 \cdot y_2, \cdots, \lambda_{n_i} \cdot x_{n_i} + \zeta_{n_i} \cdot y_{n_i})$$

for $\forall \lambda_l, \zeta_l \in \mathbf{R}$, where $1 \leq \lambda_l, \zeta_l \leq n_i$. Then each $\mathbf{R}^{n_i}$ is a continuous group under $+_i$. Whence, the algebraic multi-system $(\mathscr{E}_G(n_1, \cdots, n_m); \mathscr{O})$ is a topological multi-group with a underlying structure $G$ by definition, where $\mathscr{O} = \bigcup_{i=1}^{m}\{+_i\}$. Particularly, if $m = 1$, i.e., an $n$-dimensional Euclidean space $\mathbf{R}^n$ with the vector additive $+$ and multiplication $\cdot$ is a topological group.

**Example** 1.2 Notice that there is function $\kappa : M_{n\times n} \to \mathbf{R}^{n^2}$ from real $n \times n$-matrices $M_{n\times n}$ to $\mathbf{R}$ determined by

$$\kappa : \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ a_{21} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots \\ a_{n1} & \cdots & a_{n\times n} \end{pmatrix} \to \begin{pmatrix} a_{11} & \cdots & a_{1n}, \cdots, a_{n1} & \cdots & a_{n\times n} \end{pmatrix}$$

Denoted all $n \times n$-matrices by $\mathbf{M}(n, \mathbf{R})$. Then the general linear group of degree $n$ is defined by

$$GL(n, \mathbf{R}) = \{\ M \in \mathbf{M}(n, \mathbf{R}) \mid \mathrm{det}M \neq 0\ \},$$

where $\mathrm{det}M$ is the determinant of $M$. It can be shown that $GL(n, \mathbf{R})$ is a topological group. In fact, since the function $\mathrm{det} : M_{n\times n} \to \mathbf{R}$ is continuous, $\mathrm{det}^{-1}\mathbf{R} \setminus \{0\}$ is open in $\mathbf{R}^{n^2}$, and hence an open subset of $\mathbf{R}^{n^2}$.

We show the mappings $\phi : GL(n, \mathbf{R} \times GL(n, \mathbf{R})) \to GL(n, \mathbf{R})$ and $\psi : GL(n, \mathbf{R}) \to GL(n, \mathbf{R})$ determined by $\phi(a, b) = ab$ and $\psi(a) = a^{-1}$ are both continuous for $a, b \in GL(n, \mathbf{R})$. Let $a = (a_{ij})_{n\times n}$ and $b = (b_{ij})_{n\times n} \in \mathbf{M}(n, \mathbf{R})$. By definition, we know that

$$ab = ((ab)_{ij}) = (\sum_{k=1}^{n} a_{ik}b_{kj}).$$

Whence, $\phi(a, b) = ab$ is continuous. Similarly, let $\psi(a) = (\psi_{ij})_{n\times n}$. Then we know that

$$\psi_{ij} = \frac{a^*_{ij}}{\mathrm{det}a}$$

is continuous, where $a^*_{ij}$ is the cofactor of $a_{ij}$ in the determinant $\mathrm{det}a$. Therefore, $GL(n, \mathbf{R})$ is a topological group.

Now for integers $n_1, n_2, \cdots, n_m \geq 1$, let $\mathscr{E}_G(GL_{n_1}, \cdots, GL_{n_m})$ be a multi-group consisting of $GL(n_1, \mathbf{R})$, $GL(n_2, \mathbf{R})$, $\cdots$, $GL(n_m, \mathbf{R})$ underlying a combinatorial structure $G$. Then it is itself a combinatorial space. Whence, $\mathscr{E}_G(GL_{n_1}, \cdots, GL_{n_m})$ is a topological multi-group.

Conversely, a combinatorial space of topological groups is indeed a topological multi-group by definition. This means that there are innumerable such multi-groups.

## §2. Topological multi-subgroups

A topological space $S$ is *homogenous* if for $\forall a, b \in S$, there exists a continuous mapping $f : S \to S$ such that $f(b) = a$. We have a simple characteristic following.

**Theorem** 2.1 *If a topological multi-group $(\mathscr{S}_G; \mathscr{O})$ is arcwise connected and associative, then it is homogenous.*

*Proof* Notice that $\mathscr{S}_G$ is arcwise connected if and only if its underlying graph $G$ is connected. For $\forall a, b \in \mathscr{S}_G$, without loss of generality, assume $a \in \mathscr{H}_0$ and $b \in \mathscr{H}_s$ and

$$P(a,b) = \mathscr{H}_0\mathscr{H}_1 \cdots \mathscr{H}_s, \quad s \geq 0,$$

a path from $\mathscr{H}_0$ to $\mathscr{H}_s$ in the graph $G$. Choose $c_1 \in \mathscr{H}_0 \cap \mathscr{H}_1$, $c_2 \in \mathscr{H}_1 \cap \mathscr{H}_2, \cdots$, $c_s \in \mathscr{H}_{s-1} \cap \mathscr{H}_s$. Then

$$a \circ_0 c_1 \circ_1 c_1^{-1} \circ_2 c_2 \circ_3 c_3 \circ_4 \cdots \circ_{s-1} c_s^{-1} \circ_s b^{-1}$$

is well-defined and

$$a \circ_0 c_1 \circ_1 c_1^{-1} \circ_2 c_2 \circ_3 c_3 \circ_4 \cdots \circ_{s-1} c_s^{-1} \circ_s b^{-1} \circ_s b = a.$$

Let $L = a \circ_0 c_1 \circ_1 c_1^{-1} \circ_2 c_2 \circ_3 c_3 \circ_4 \cdots \circ_{s-1} c_s^{-1} \circ_s b^{-1} \circ_s$. Then $L$ is continuous by the definition of topological multi-group. We finally get a continuous mapping $L : \mathscr{S}_G \to \mathscr{S}_G$ such that $L(b) = Lb = a$. Whence, $(\mathscr{S}_G; \mathscr{O})$ is homogenous. □

**Corollary** 6.4.1 *A topological group is homogenous if it is arcwise connected.*

A multi-subsystem $(\mathscr{L}_H; \mathcal{O})$ of $(\mathscr{S}_G; \mathscr{O})$ is called a *topological multi-subgroup* if it itself is a topological multi-group. Denoted by $\mathscr{L}_H \leq \mathscr{S}_G$. A criterion on topological multi-subgroups is shown in the following.

**Theorem** 2.2 *A multi-subsystem $(\mathscr{L}_H; \mathcal{O}_1)$ is a topological multi-subgroup of $(\mathscr{S}_G; \mathscr{O})$, where $\mathcal{O}_1 \subset \mathcal{O}$ if and only if it is a multi-subgroup of $(\mathscr{S}_G; \mathscr{O})$ in algebra.*

*Proof* The necessity is obvious. For the sufficiency, we only need to prove that for any operation $\circ \in \mathcal{O}_1$, $a \circ b^{-1}$ is continuous in $\mathscr{L}_H$. Notice that the condition $(iii)$ in the definition of topological multi-group can be replaced by:

*for any neighborhood $N_{\mathscr{S}_G}(a \circ b^{-1})$ of $a \circ b^{-1}$ in $\mathscr{S}_G$, there always exist neighborhoods $N_{\mathscr{S}_G}(a)$ and $N_{\mathscr{S}_G}(b^{-1})$ of $a$ and $b^{-1}$ such that $N_{\mathscr{S}_G}(a) \circ N_{\mathscr{S}_G}(b^{-1}) \subset N_{\mathscr{S}_G}(a \circ b^{-1})$, where $N_{\mathscr{S}_G}(a) \circ N_{\mathscr{S}_G}(b^{-1}) = \{x \circ y | \forall x \in N_{\mathscr{S}_G}(a), y \in N_{\mathscr{S}_G}(b^{-1})\}$*

by the definition of mapping continuity. Whence, we only need to show that for any neighborhood $N_{\mathscr{L}_H}(x \circ y^{-1})$ in $\mathscr{L}_H$, where $x, y \in \mathscr{L}_H$ and $\circ \in \mathcal{O}_1$, there exist neighborhoods $N_{\mathscr{L}_H}(x)$ and $N_{\mathscr{L}_H}(y^{-1})$ such that $N_{\mathscr{L}_H}(x) \circ N_{\mathscr{L}_H}(y^{-1}) \subset N_{\mathscr{L}_H}(x \circ y^{-1})$ in $\mathscr{L}_H$. In fact, each neighborhood $N_{\mathscr{L}_H}(x \circ y^{-1})$ of $x \circ y^{-1}$ can be represented by a form $N_{\mathscr{S}_G}(x \circ y^{-1}) \cap \mathscr{L}_H$. By assumption, $(\mathscr{S}_G; \mathscr{O})$ is a topological multi-group, we know that there are neighborhoods $N_{\mathscr{S}_G}(x)$, $N_{\mathscr{S}_G}(y^{-1})$ of $x$ and $y^{-1}$ in $\mathscr{S}_G$ such that $N_{\mathscr{S}_G}(x) \circ N_{\mathscr{S}_G}(y^{-1}) \subset N_{\mathscr{S}_G}(x \circ y^{-1})$. Notice that $N_{\mathscr{S}_G}(x) \cap \mathscr{L}_H$, $N_{\mathscr{S}_G}(y^{-1}) \cap \mathscr{L}_H$ are neighborhoods of $x$ and $y^{-1}$ in $\mathscr{L}_H$. Now let $N_{\mathscr{L}_H}(x) = N_{\mathscr{S}_G}(x) \cap \mathscr{L}_H$ and $N_{\mathscr{L}_H}(y^{-1}) = N_{\mathscr{S}_G}(y^{-1}) \cap \mathscr{L}_H$. Then we get that $N_{\mathscr{L}_H}(x) \circ N_{\mathscr{L}_H}(y^{-1}) \subset N_{\mathscr{L}_H}(x \circ y^{-1})$ in $\mathscr{L}_H$, i.e., the mapping $(x, y) \to x \circ y^{-1}$ is continuous. Whence, $(\mathscr{L}_H; \mathcal{O}_1)$ is a topological multi-subgroup. □

Particularly, for the topological groups, we know the following consequence.

**Corollary** 2.2 *A subset of a topological group $(\Gamma; \circ)$ is a topological subgroup if and only if it is a subgroup of $(\Gamma; \circ)$ in algebra.*

## §3. Homomorphism theorem on topological multi-subgroups

For two topological multi-groups $(\mathscr{S}_{G_1};\mathscr{O}_1)$ and $(\mathscr{S}_{G_2};\mathscr{O}_2)$, a mapping $\omega:(\mathscr{S}_{G_1};\mathscr{O}_1)\to(\mathscr{S}_{G_2};\mathscr{O}_2)$ is a *homomorphism* if it satisfies the following conditions:

(1) $\omega$ is a homomorphism from multi-groups $(\mathscr{S}_{G_1};\mathscr{O}_1)$ to $(\mathscr{S}_{G_2};\mathscr{O}_2)$, namely, for $\forall a,b\in\mathscr{S}_{G_1}$ and $\circ\in\mathcal{O}_1$, $\omega(a\circ b)=\omega(a)\omega(\circ)\omega(b)$;

(2) $\omega$ is a continuous mapping from topological spaces $\mathscr{S}_{G_1}$ to $\mathscr{S}_{G_1}$, i.e., for $\forall x\in\mathscr{S}_{G_1}$ and a neighborhood $U$ of $\omega(x)$, $\omega^{-1}(U)$ is a neighborhood of $x$.

Furthermore, if $\omega:(\mathscr{S}_{G_1};\mathscr{O}_1)\to(\mathscr{S}_{G_2};\mathscr{O}_2)$ is an isomorphism in algebra and a homeomorphism in topology, then it is called an *isomorphism*, particularly, an *automorphism* if $(\mathscr{S}_{G_1};\mathscr{O}_1)=(\mathscr{S}_{G_2};\mathscr{O}_2)$ between topological multi-groups $(\mathscr{S}_{G_1};\mathscr{O}_1)$ and $(\mathscr{S}_{G_2};\mathscr{O}_2)$.

Let $(\mathscr{S}_G;\mathscr{O})$ be an associatively topological multi-subgroup and $(\mathscr{L}_H;\mathcal{O})$ one of its topological multi-subgroups with $\mathscr{S}_G=\bigcup\limits_{i=1}^{m}\mathscr{H}_i$, $\mathscr{L}_H=\bigcup_{i=1}^{m}\mathscr{G}_i$ and $\mathscr{O}=\bigcup\limits_{i=1}^{m}\{\circ_i\}$. In [8], we have know the following results on homomorphisms of multi-systems following.

**Lemma** 3.1([8]) *Let $(\mathscr{H},\widetilde{O})$ be an associative multi-operation system with a unit $1_\circ$ for $\forall\circ\in\widetilde{O}$ and $\mathscr{G}\subset\mathscr{H}$.*

*(i) If $\mathscr{G}$ is closed for operations in $\widetilde{O}$ and for $\forall a\in\mathscr{G},\circ\in\widetilde{O}$, there exists an inverse element $a_\circ^{-1}$ in $(\mathscr{G};\circ)$, then there is a representation pair $(R,\widetilde{P})$ such that the quotient set $\frac{\mathscr{H}}{\mathscr{G}}|_{(R,\widetilde{P})}$ is a partition of $\mathscr{H}$, i.e., for $a,b\in\mathscr{H},\forall\circ_1,\circ_2\in\widetilde{O}$, $(a\circ_1\mathscr{G})\cap(b\circ_2\mathscr{G})=\emptyset$ or $a\circ_1\mathscr{G}=b\circ_2\mathscr{G}$.*

*(ii) For $\forall\circ\in\widetilde{O}$, define an operation $\circ$ on $\frac{\mathscr{H}}{\mathscr{G}}|_{(R,\widetilde{P})}$ by*

$$(a\circ_1\mathscr{G})\circ(b\circ_2\mathscr{G})=(a\circ b)\circ_1\mathscr{G}.$$

*Then $(\frac{\mathscr{H}}{\mathscr{G}}|_{(R,\widetilde{P})};\widetilde{O})$ is an associative multi-operation system. Particularly, if there is a representation pair $(R,\widetilde{P})$ such that for $\circ'\in\widetilde{P}$, any element in $R$ has an inverse in $(\mathscr{H};\circ')$, then $(\frac{\mathscr{H}}{\mathscr{G}}|_{(R,\widetilde{P})},\circ')$ is a group.*

**Lemma** 3.2([8]) *Let $\omega$ be an onto homomorphism from associative systems $(\mathscr{H}_1;\widetilde{O}_1)$ to $(\mathscr{H}_2;\widetilde{O}_2)$ with $(\mathcal{I}(\widetilde{O}_2);\widetilde{O}_2)$ an algebraic system with unit $1_{\circ^-}$ for $\forall\circ^-\in\widetilde{O}_2$ and inverse $x^{-1}$ for $\forall x\in(\mathcal{I}(\widetilde{O}_2)$ in $((\mathcal{I}(\widetilde{O}_2);\circ^-)$. Then there are representation pairs $(R_1,\widetilde{P}_1)$ and $(R_2,\widetilde{P}_2)$, where $\widetilde{P}_1\subset\widetilde{O},\widetilde{P}_2\subset\widetilde{O}_2$ such that*

$$\frac{(\mathscr{H}_1;\widetilde{O}_1)}{(\widetilde{\mathrm{Ker}\omega};\widetilde{O}_1)}|_{(R_1,\widetilde{P}_1)}\cong\frac{(\mathscr{H}_2;\widetilde{O}_2)}{(\mathcal{I}(\widetilde{O}_2);\widetilde{O}_2)}|_{(R_2,\widetilde{P}_2)}$$

*if each element of $\widetilde{\mathrm{Ker}\omega}$ has an inverse in $(\mathscr{H}_1;\circ)$ for $\circ\in\widetilde{O}_1$.*

Whence, by Lemma 3.1, for any integer $i$, $1\le i\le m$, we get a quotient group $\mathscr{H}_i/\mathscr{G}_i$, i.e., a multi-subgroup $(\mathscr{S}_G/\mathscr{L}_H;\mathcal{O})=\bigcup\limits_{i=1}^{m}(\mathscr{H}_i/\mathscr{G}_i;\circ_i)$ on algebraic multi-groups.

Notice that for a topological space $S$ with an equivalent relation $\sim$ and a projection $\pi:S\to S/\sim=\{[x]|\forall y\in[x],y\sim x\}$, we can introduce a topology on $S/\sim$ by defining its opened sets to be subsets $V$ in $S/\sim$ such that $\pi^{-1}(V)$ is opened in $S$. Such topological space $S/\sim$

is called a *quotient space*. Now define a relation in $(\mathscr{S}_G;\mathscr{O})$ by $a\sim b$ for $a,b\in\mathscr{S}_G$ providing $b=h\circ a$ for an element $h\in\mathscr{L}_H$ and an operation $\circ\in\mathcal{O}$. It is easily to know that such relation is an equivalence. Whence, we also get an induced quotient space $\mathscr{S}_G/\mathscr{L}_H$.

**Theorem** 3.1 *Let $\omega:(\mathscr{S}_{G_1};\mathscr{O}_1)\to(\mathscr{S}_{G_2};\mathscr{O}_2)$ be an opened onto homomorphism from associatively topological multi-groups $(\mathscr{S}_{G_1};\mathscr{O}_1)$ to $(\mathscr{S}_{G_2};\mathscr{O}_2)$, i.e., it maps an opened set to an opened set. Then there are representation pairs $(R_1,\mathcal{P}_1)$ and $(R_2,\mathcal{P}_2)$ such that*

$$\frac{(\mathscr{S}_{G_1};\mathscr{O}_1)}{(\widetilde{\mathrm{Ker}\omega};\mathscr{O}_1)}|_{(R_1,\widetilde{P}_1)}\cong\frac{(\mathscr{S}_{G_2};\mathscr{O}_2)}{(\mathcal{I}(\widetilde{O}_2);\widetilde{O}_2)}|_{(R_2,\widetilde{P}_2)},$$

*where $\mathcal{P}_1\subset\mathscr{O}_1,\mathcal{P}_2\subset\mathscr{O}_2$, $\mathcal{I}(\mathscr{O}_2)=\{1_\circ,\circ\in\mathscr{O}_2\}$ and*

$$\widetilde{\mathrm{Ker}\omega}=\{\ a\in\mathscr{S}_{G_1}\mid\omega(a)=1_\circ\in\mathcal{I}(\mathscr{O}_2)\ \}.$$

*Proof* According to Lemma 3.2, we know that there are representation pairs $(R_1,\mathcal{P}_1)$ and $(R_2,\mathcal{P}_2)$ such that

$$\frac{(\mathscr{S}_{G_1};\mathscr{O}_1)}{(\widetilde{\mathrm{Ker}\omega};\mathscr{O}_1)}|_{(R_1,\widetilde{P}_1)}\overset{\sigma}{\cong}\frac{(\mathscr{S}_{G_2};\mathscr{O}_2)}{(\mathcal{I}(\widetilde{O}_2);\widetilde{O}_2)}|_{(R_2,\widetilde{P}_2)}$$

in algebra, where $\sigma(a\circ\mathrm{Ker}\omega)=\sigma(a)\circ^{-1}\mathcal{I}(\mathscr{O}_2)$ in the proof of Lemma 3.2. We only need to prove that $\sigma$ and $\sigma^{-1}$ are continuous.

On the First, for $x=\sigma(a)\circ^{-1}\mathcal{I}(\mathscr{O}_2)\in\frac{(\mathscr{S}_{G_2};\mathscr{O}_2)}{(\mathcal{I}(\widetilde{O}_2);\widetilde{O}_2)}|_{(R_2,\widetilde{P}_2)}$ let $\widehat{U}$ be a neighborhood of $\sigma^{-1}(x)$ in the space $\frac{(\mathscr{S}_{G_1};\mathscr{O}_1)}{(\widetilde{\mathrm{Ker}\omega};\mathscr{O}_1)}|_{(R_1,\widetilde{P}_1)}$, where $\widehat{U}$ is a union of $a\circ\mathrm{Ker}\omega$ for $a$ in an opened set $U$ and $\circ\in\widetilde{P}_1$. Since $\omega$ is opened, there is a neighborhood $\widehat{V}$ of $x$ such that $\omega(U)\supset\widehat{V}$, which enables us to find that $\sigma^{-1}(\widehat{V})\subset\widehat{U}$. In fact, let $\widehat{y}\in\widehat{V}$. Then there exists $y\in U$ such that $\omega(y)=\widehat{y}$. Whence, $\sigma^{-1}(\widehat{y})=y\circ\mathrm{Ker}\omega\in\widehat{U}$. Therefore, $\sigma^{-1}$ is continuous.

On the other hand, let $\widehat{V}$ be a neighborhood of $\sigma(x)\circ^{-1}\mathcal{I}(\mathscr{O}_2)$ in the space $\frac{(\mathscr{S}_{G_2};\mathscr{O}_2)}{(\mathcal{I}(\widetilde{O}_2);\widetilde{O}_2)}|_{(R_2,\widetilde{P}_2)}$ for $x\circ\mathrm{Ker}\omega$. By the continuity of $\omega$, we know that there is a neighborhood $U$ of $x$ such that $\omega(U)\subset\widehat{V}$. Denoted by $\widehat{U}$ the union of all sets $z\circ\mathrm{Ker}\omega$ for $z\in U$. Then $\sigma(\widehat{U})\subset\widehat{V}$ because of $\omega(U)\subset\widehat{V}$. Whence, $\sigma$ is also continuous. Combining the continuity of $\sigma$ and its inverse $\sigma^{-1}$, we know that $\sigma$ is also a homeomorphism from topological spaces $\frac{(\mathscr{S}_{G_1};\mathscr{O}_1)}{(\widetilde{\mathrm{Ker}\omega};\mathscr{O}_1)}|_{(R_1,\widetilde{P}_1)}$ to $\frac{(\mathscr{S}_{G_2};\mathscr{O}_2)}{(\mathcal{I}(\widetilde{O}_2);\widetilde{O}_2)}|_{(R_2,\widetilde{P}_2)}$. □

**Corollary** 3.1 *Let $\omega:(\mathscr{S}_G;\mathscr{O})\to(\mathscr{A};\circ)$ be a onto homomorphism from a topological multi-group $(\mathscr{S}_G;\mathscr{O})$ to a topological group $(\mathscr{A};\circ)$. Then there are representation pairs $(R,\widetilde{P})$, $\widetilde{P}\subset\mathscr{O}$ such that*

$$\frac{(\mathscr{S}_G;\mathscr{O})}{(\widetilde{\mathrm{Ker}\omega};\mathscr{O})}|_{(R,\widetilde{P})}\cong(\mathscr{A};\circ).$$

*Particularly, if $\mathscr{O}=\{\bullet\}$, i.e., $(\mathscr{S}_G;\bullet)$ is a topological group, then*

$$\mathscr{S}_G/\mathrm{Ker}\omega\cong(\mathscr{A};\circ).$$

## §4. Topological multi-fields

**Definition** 4.1 *A distributive multi-system* $(\widetilde{\mathscr{A}};\mathscr{O}_1 \hookrightarrow \mathscr{O}_2)$ *with* $\widetilde{\mathscr{A}} = \bigcup\limits_{i=1}^{m} \mathscr{H}_i$, $\mathscr{O}_1 = \bigcup\limits_{i=1}^{m}\{\cdot_i\}$ *and* $\mathscr{O}_2 = \bigcup\limits_{i=1}^{m}\{+_i\}$ *is called a topological multi-ring if*

*(i)* $(\mathscr{H}_i;+_i,\cdot_i)$ *is a ring for each integer* $i$, $1 \leq i \leq m$, *i.e.,* $(\mathscr{H},\mathscr{O}_1 \hookrightarrow \mathscr{O}_2)$ *is a multi-ring;*

*(ii)* $\widetilde{\mathscr{A}}$ *is a combinatorially topological space* $\mathscr{S}_G$*;*

*(iii) the mappings* $(a,b) \to a \cdot_i b^{-1}$, $(a,b) \to a +_i (-_i b)$ *are continuous for* $\forall a,b \in \mathscr{H}_i$, $1 \leq i \leq m$.

Denoted by $(\mathscr{S}_G;\mathscr{O}_1 \hookrightarrow \mathscr{O}_2)$ a topological multi-ring. A topological multi-ring $(\mathscr{S}_G;\mathscr{O}_1 \hookrightarrow \mathscr{O}_2)$ is called a *topological divisible multi-ring* or *multi-field* if the previous condition $(i)$ is replaced by $(\mathscr{H}_i;+_i,\cdot_i)$ *is a divisible ring or field for each integer* $1 \leq i \leq m$. Particularly, if $m = 1$, then a topological multi-ring, divisible multi-ring or multi-field is nothing but a topological ring, divisible ring or field. Some examples of topological fields are presented in the following.

**Example** 4.1 A 1-dimensional Euclidean space $\mathbf{R}$ is a topological field since $\mathbf{R}$ is itself a field under operations additive $+$ and multiplication $\times$.

**Example** 4.2 A 2-dimensional Euclidean space $\mathbf{R}^2$ is isomorphic to a topological field since for $\forall(x,y) \in \mathbf{R}^2$, it can be endowed with a unique complex number $x + iy$, where $i^2 = -1$. It is well-known that all complex numbers form a field.

**Example** 4.3 A 4-dimensional Euclidean space $\mathbf{R}^4$ is isomorphic to a topological field since for each point $(x,y,z,w) \in \mathbf{R}^4$, it can be endowed with a unique quaternion number $x+iy+jz+kw$, where

$$ij = -ji = k,\ jk = -kj = i,\ ki = -ik = j,$$

and

$$i^2 = j^2 = k^2 = -1.$$

We know all such quaternion numbers form a field.

For topological fields, we have known a classification theorem following.

**Lemma** 4.1([12]) *A locally compacted topological field is isomorphic to one of the following:*

$(i)$ *Euclidean real line* $\mathbf{R}$*, the real number field;*

$(ii)$ *Euclidean plane* $\mathbf{R}^2$*, the complex number field;*

$(iii)$ *Euclidean space* $\mathbf{R}^4$*, the quaternion number field.*

Applying Lemma 4.1 and the definition of combinatorial Euclidean space, we can determine these topological multi-fields underlying any connected graph $G$ following.

**Theorem** 4.1 *For any connected graph $G$, a locally compacted topological multi-field $(\mathscr{S}_G;\mathscr{O}_1 \hookrightarrow \mathscr{O}_2)$ is isomorphic to one of the following:*

*(i) Euclidean space $\mathbf{R}$, $\mathbf{R}^2$ or $\mathbf{R}^4$ endowed respectively with the real, complex or quaternion number for each point if $|G| = 1$;*

*(ii) combinatorially Euclidean space $\mathscr{E}_G(2,\cdots,2,4,\cdots,4)$ with coupling number, i.e., the dimensional number $l_{ij} = 1, 2$ or $3$ of an edge $(\mathbf{R}^i,\mathbf{R}^j) \in E(G)$ only if $i = j = 4$, otherwise $l_{ij} = 1$ if $|G| \geq 2$.*

*Proof* By the definition of topological multi-field $(\mathscr{S}_G;\mathscr{O}_1 \hookrightarrow \mathscr{O}_2)$, for an integer $i$, $1 \leq i \leq m$, $(\mathscr{H}_i;+_i,\cdot_i)$ is itself a locally compacted topological field. Whence, $(\mathscr{S}_G;\mathscr{O}_1 \hookrightarrow \mathscr{O}_2)$ is a topologically combinatorial multi-field consisting of locally compacted topological fields. According to Lemma 4.1, we know there must be

$$(\mathscr{H}_i;+_i,\cdot_i) \cong \mathbf{R},\ \mathbf{R}^2,\ \text{or } \mathbf{R}^4$$

for each integer $i$, $1 \leq i \leq m$. Let the coordinate system of $\mathbf{R},\mathbf{R}^2,\mathbf{R}^4$ be $x,(y_1,y_2)$ and $(z_1,z_2,z_3,z_4)$. If $|G| = 1$, then it is just the classifying in Theorem 6.4.4. Now let $|G| \geq 2$. For $\forall(\mathbf{R}^i,\mathbf{R}^j) \in E(G)$, we know that $\mathbf{R}^i \setminus \mathbf{R}^j \neq \emptyset$ and $\mathbf{R}^j \setminus \mathbf{R}^i \neq \emptyset$ by the definition of combinatorial space. Whence, $i, j = 2$ or $4$. If $i = 2$ or $j = 2$, then $l_{ij} = 1$ because of $1 \leq l_{ij} < 2$, which means $l_{ij} \geq 2$ only if $i = j = 4$. This completes the proof. □

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