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Research Article

Topological Structures via Bipolar Hypersoft Sets

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In this article, we introduce bipolar hypersoft topological spaces over the collection of bipolar hypersoft sets. It is proven that a bipolar hypersoft topological space gives a parametrized family of hypersoft topological spaces, but the converse does not hold in general, and this is shown with the help of an example. Furthermore, we give a condition on a given parametrized family of hypersoft topologies, which assure that there is a bipolar hypersoft topology whose induced family of hypersoft topologies is the given family. The notions of bipolar hypersoft neighborhood, bipolar hypersoft subspace, and bipolar hypersoft limit points are introduced. Finally, we define bipolar hypersoft interior, bipolar hypersoft closure, bipolar hypersoft exterior, and bipolar hypersoft boundary, and the relations between them, differing from the relations on hypersoft topology, are investigated.

1. Introduction

Many challenges in engineering, artificial intelligence, economics, environmental research, social science, and other fields include ambiguous or unclear data. As a result, traditional methods depending on the specific instance may be ineffective in solving or modeling them. Many theories have been developed to address these issues in light of this. Soft set theory was introduced by Molodtsov [1] as a revolutionary idea for dealing with uncertainty. In comparison to earlier ideas, he demonstrated how effective soft sets are at solving complex issues. Many scholars have since researched soft set theory's properties, operations, and applications (see, for example, [2–8]). The relevance of topology and its multiple applications, particularly in physics, economics, and computer science, have piqued researchers' interest in the topological structure of soft sets. There were two definitions of soft topological spaces presented. Shabir and Naz [9] were the first to establish the concept of soft topological space on a universe set. Çağman et al. [10] used a soft set to demonstrate the notion of soft topological space. Following that, several academics concentrated on soft topological spaces [11–26].

Shabir and Naz [27] first used bipolar soft set to combine bipolarity [28] with soft set theory [1]. According to Dubois and Prade [28], our decision-making is built upon two sides, positive and negative, and we choose based on which is more powerful. Some definitions, operations, and applications on bipolar soft sets were studied in [29–32]. The notion of bipolar soft topological spaces over an ordinary set was first introduced by Shabir and Bakhtawar [33], along with studies into bipolar soft connectedness and bipolar soft compactness. The notions of interior and closure operators, basis, and subspace in bipolar soft topological spaces were studied by Ozturk [34]. By redefining bipolar soft topological spaces on a bipolar soft set, Fadel [35] has expanded the definition of bipolar soft topological spaces introduced in [33]. They have covered the key concepts and properties, as well as some illustrative examples. For additional work on the topological structures on bipolar soft sets, one may study [36, 37].

In 2018, Smarandache [38] transformed the argument mapping F into a multiargument mapping to turn generalized soft sets into hypersoft sets. This notion is more adaptable than soft set and more suited to challenges involving decision-making. Under the hypersoft set environment, some vital basics (e.g., elementary properties, set theoretic operations, basic laws, relations, functions, and matrices) as well as applications are conceptualized in [39–46]. Musa and Asaad [47] initiated the study of hypersoft topological spaces. They defined hypersoft topology as a collection of hypersoft sets over the universe X. So, they defined elementary notions of hypersoft topological spaces such as hypersoft open sets, hypersoft closed sets, hypersoft neighborhood, hypersoft limit points, and hypersoft subspace. Further, hypersoft closure, hypersoft interior, and hypersoft boundary were studied, and some of their basic properties were investigated.

Recently, Musa and Asaad [48] introduced the notion of bipolar hypersoft sets, which was created by merging the bipolarity [28] and hypersoft set [38]. It was distinguished by two hypersoft sets, one of which provided positive data and the other negative data. In their paper, they defined some basic operations such as bipolar hypersoft subset, bipolar hypersoft complement, and bipolar hypersoft difference. Furthermore, aggregate operations such as intersection, union, AND-operation, and OR-operation of two bipolar hypersoft sets with their characteristics were explored and illustrated with examples.

The following is how the rest of the paper is laid out: Section 2 contains some basic definitions on bipolar hypersoft sets. In Section 3, we introduce bipolar hypersoft topological spaces, which are defined over an initial universe with a fixed set of parameters. Then, we discuss some basic properties of bipolar hypersoft topological spaces and define bipolar hypersoft neighborhood, bipolar hypersoft subspace, and bipolar hypersoft limit points. In Section 4, bipolar hypersoft interior, bipolar hypersoft closure, bipolar hypersoft exterior, and bipolar hypersoft boundary are defined. Also, the relationship between these concepts is investigated. In Section 5, we summarize the main results and make suggestions for further studies.

2. Preliminaries

Here, we recall some basic terminologies regarding bipolar hypersoft sets and hypersoft topological spaces.

Throughout this work, we use X as an initial universe, $P(X)$ as the power set of X, and $E_1, E_2, \ldots, E_n$ the pairs of disjoint sets of parameters. For $i = 1, 2, \ldots, n$, let $A_i, B_i \subseteq E_i$.

To keep things simple, for $E_1 \times E_2 \times \cdots \times E_n$, $A_1 \times A_2 \times \cdots \times A_n$, and $B_1 \times B_2 \times \cdots \times B_n$ we use the notations E, A, and B, respectively. Let A, $B \subseteq E$ and each element in A, B, and E is obviously an n-tuple element.

Definition 1. [48] A bipolar hypersoft set over X is defined as (Λ, Ω, A) where Λ and Ω are mappings given by $\Lambda\colon A \longrightarrow P(X)$ and $\Omega\colon \neg A \longrightarrow P(X)$ such that $\Lambda(\alpha) \cap \Omega(\neg\alpha) = \phi$ for all $\alpha \in A$.

A bipolar hypersoft set (Λ, Ω, A) can be represented as follows:

$$(\Lambda, \Omega, A) = \{(\alpha, \Lambda(\alpha), \Omega(\neg\alpha))\colon \alpha \in A \text{ and } \Lambda(\alpha) \cap \Omega(\neg\alpha) = \phi\}. \tag{1}$$

Definition 2. [48] Let (Λ_1, Ω_1, A) and (Λ_2, Ω_2, B) be two bipolar hypersoft sets over a common universe X, we say that (Λ_1, Ω_1, A) is a bipolar hypersoft subset of (Λ_2, Ω_2, B) if for all $\alpha \in A$:

(1) $A \subseteq B$ and;

(2) $\Lambda_1(\alpha) \subseteq \Lambda_2(\alpha)$ and $\Omega_2(\neg\alpha) \subseteq \Omega_1(\neg\alpha)$.

We symbolize it by $(\Lambda_1, \Omega_1, A) \mathrel{\tilde{\tilde{\sqsubseteq}}} (\Lambda_2, \Omega_2, B)$.

If (Λ_2, Ω_2, B) is a bipolar hypersoft subset of (Λ_1, Ω_1, A), then (Λ_1, Ω_1, A) is said to be a bipolar hypersoft superset of (Λ_2, Ω_2, B). It is symbolized by $(\Lambda_1, \Omega_1, A) \mathrel{\tilde{\tilde{\sqsupseteq}}} (\Lambda_2, \Omega_2, B)$.

Definition 3. [48] (Λ_1, Ω_1, A) and (Λ_2, Ω_2, B) are said to be bipolar hypersoft equal if $(\Lambda_1, \Omega_1, A) \mathrel{\tilde{\tilde{\sqsubseteq}}} (\Lambda_2, \Omega_2, B)$ and $(\Lambda_2, \Omega_2, A) \mathrel{\tilde{\tilde{\sqsubseteq}}} (\Lambda_1, \Omega_1, B)$.

Definition 4. [48] $(\Lambda, \Omega, A)^c = (\Lambda^c, \Omega^c, A)$ is defined as the bipolar hypersoft complement of (Λ, Ω, A), where Λ^c and Ω^c are mappings given by $\Lambda^c(\alpha) = \Omega(\neg\alpha)$ and $\Omega^c(\neg\alpha) = \Lambda(\alpha)$ for all $\alpha \in A$.

Definition 5. [48] We called (Φ, Ψ, A) over X a relative null bipolar hypersoft set if for all $\alpha \in A$, $\Phi(\alpha) = \phi$ and $\Psi(\neg\alpha) = X$.

An absolute null bipolar hypersoft set over X is denoted by (Φ, Ψ, E).

Definition 6. [48] We called (Ψ, Φ, A) over X a relative whole bipolar hypersoft set if for all $\alpha \in A$, $\Psi(\alpha) = X$ and $\Phi(\neg\alpha) = \phi$.

An absolute whole bipolar hypersoft set over X is denoted by (Ψ, Φ, E).

Definition 7. [48] A bipolar hypersoft set (Γ, Θ, C), where $C = A \cap B$, is defined as the difference of (Λ_1, Ω_1, A) and (Λ_2, Ω_2, B) over a common universe X if for all $\alpha \in C$:

$\Gamma(\alpha) = \Lambda_1(\alpha) \cap \Lambda_2^c(\alpha) = \Lambda_1(\alpha) \cap \Omega_2(\neg\alpha)$ and $\Theta(\neg\alpha) = \Omega_1(\neg\alpha) \cup \Omega_2^c(\neg\alpha) = \Omega_1(\neg\alpha) \cup \Lambda_2(\alpha)$.

We symbolize it by $(\Lambda_1, \Omega_1, A) \backslash (\Lambda_2, \Omega_2, B) = (\Gamma, \Theta, C)$.

Definition 8. [48] A bipolar hypersoft set (Γ, Θ, C), where $C = A \cap B$, is defined as the union of (Λ_1, Ω_1, A) and (Λ_2, Ω_2, B) over a common universe X if for all $\alpha \in C$:

$$\Gamma(\alpha) = \Lambda_1(\alpha) \cup \Lambda_2(\alpha) \text{ and } \Theta(\neg\alpha) = \Omega_1(\neg\alpha) \cap \Omega_2(\neg\alpha). \tag{2}$$

We symbolize it by $(\Lambda_1, \Omega_1, A) \mathbin{\tilde{\tilde{\sqcup}}} (\Lambda_2, \Omega_2, B) = (\Gamma, \Theta, C)$.

Definition 9. [48] A bipolar hypersoft set (Γ, Θ, C), where $C = A \cap B$, is defined as the intersection of (Λ_1, Ω_1, A) and (Λ_2, Ω_2, B) over a common universe X if for all $\alpha \in C$:

$$\Gamma(\alpha) = \Lambda_1(\alpha) \cap \Lambda_2(\alpha) \text{ and } \Theta(\neg\alpha) = \Omega_1(\neg\alpha) \cup \Omega_2(\neg\alpha). \tag{3}$$

We symbolize it by $(\Lambda_1, \Omega_1, A) \mathbin{\tilde{\tilde{\sqcap}}} (\Lambda_2, \Omega_2, B) = (\Gamma, \Theta, C)$.

Definition 10. [47] If $T_{\mathscr{H}}$ is a collection of hypersoft sets over X, then $T_{\mathscr{H}}$ is a hypersoft topology on X if:

(1) (Φ, E), (Ψ, E) belong to $T_{\mathscr{H}}$.
(2) The intersection of any two hypersoft sets in $T_{\mathscr{H}}$ belongs to $T_{\mathscr{H}}$.
(3) The union of any number of hypersoft sets in $T_{\mathscr{H}}$ belongs to $T_{\mathscr{H}}$.

Then, $(X, T_{\mathscr{H}}, E)$ is called a hypersoft topological space over X.

Definition 11. [47] Suppose that $(X, T_{\mathscr{H}}, E)$ is a hypersoft space over X, then the members of $T_{\mathscr{H}}$ are said to be hypersoft open sets in X.

Definition 12. [47] Suppose that $(X, T_{\mathscr{H}}, E)$ is a hypersoft space over X. A hypersoft set (Λ, E) over X is said to be a hypersoft closed set in X, if $(\Lambda, E)^c$ belongs to $T_{\mathscr{H}}$.

3. Bipolar Hypersoft Topological Spaces

In this section, we introduce the concept of bipolar hypersoft topological spaces and study its relation with hypersoft topological spaces. Then, bipolar hypersoft subspace, bipolar hypersoft neighborhood, and bipolar hypersoft limit point are characterized.

Definition 13. Let (Λ, Ω, E) be a bipolar hypersoft set over X and $x \in X$. Then $x \in (\Lambda, \Omega, E)$ if $x \in \Lambda(\alpha)$ for all $\alpha \in E$. If $x \in (\Lambda, \Omega, E)$, then automatically $x \notin \Omega(\neg\alpha)$ for all $\neg\alpha \in \neg E$.

For any $x \in X$, $x \notin (\Lambda, \Omega, E)$, if $x \notin \Lambda(\alpha)$ for some $\alpha \in E$.

Definition 14. (Λ_1, Ω_1, E) and (Λ_2, Ω_2, E) are said to be disjoint bipolar hypersoft sets if for all $\alpha \in E$, $\Lambda_1(\alpha) \cap \Lambda_2(\alpha) = \phi$. We denote it by $(\Lambda_1, \Omega_1, E) \tilde{\sqcap} (\Lambda_2, \Omega_2, E) = (\Phi, \Omega, E)$ where $\Phi(\alpha) = \phi$ for all $\alpha \in E$ and $\Omega(\neg\alpha) \subseteq X$ for all $\neg\alpha \in \neg E$.

Definition 15. Let $\Upsilon \subseteq X$. Then, the bipolar hypersoft set (Y, Φ, E) over X defined by $Y(\alpha) = \Upsilon$ for all $\alpha \in E$ and $\Phi(\neg\alpha) = \phi$ for all $\neg\alpha \in \neg E$.

Definition 16. Let $\Upsilon \subseteq X$. Then, the subbipolar hypersoft set $(\Lambda_{\Upsilon}, \Omega_{\Upsilon}, E)$ over Υ of a bipolar hypersoft set (Λ, Ω, E) over X is defined as

$\Lambda_{\Upsilon}(\alpha) = \Upsilon \cap \Lambda(\alpha)$ for all $\alpha \in E$ and $\Omega_{\Upsilon}(\neg\alpha) = \Upsilon \cap \Omega(\neg\alpha)$ for all $\neg\alpha \in \neg E$.

Definition 17. If $T_{\mathscr{BH}}$ is a collection of bipolar hypersoft sets over X, then $T_{\mathscr{BH}}$ is a bipolar hypersoft topology on X if:

(1) (Φ, Ψ, E), (Ψ, Φ, E) belong to $T_{\mathscr{BH}}$.
(2) The intersection of any two bipolar hypersoft sets in $T_{\mathscr{BH}}$ belongs to $T_{\mathscr{BH}}$.
(3) The union of any number of bipolar hypersoft sets in $T_{\mathscr{BH}}$ belongs to $T_{\mathscr{BH}}$.

Then, $(X, T_{\mathscr{BH}}, E, \neg E)$ is called a bipolar hypersoft topological space over X.

Definition 18. Suppose that $(X, T_{\mathscr{BH}}, E, \neg E)$ is a bipolar hypersoft space over X; then the members of $T_{\mathscr{BH}}$ are said to be bipolar hypersoft open sets in X.

Definition 19. Suppose that $(X, T_{\mathscr{BH}}, E, \neg E)$ is a bipolar hypersoft space over X. A bipolar hypersoft set (Λ, Ω, E) over X is said to be a bipolar hypersoft closed set in X, if $(\Lambda, \Omega, E)^c$ belongs to $T_{\mathscr{BH}}$.

Proposition 1. *Let $(X, T_{\mathscr{BH}}, E, \neg E)$ be a bipolar hypersoft space over X. Then:*

(1) (Φ, Ψ, E), (Ψ, Φ, E) are bipolar hypersoft closed set over X.
(2) The union of any two bipolar hypersoft closed sets is a bipolar hypersoft closed set over X.
(3) The intersection of any number of bipolar hypersoft closed sets is a bipolar hypersoft closed set over X.

proof. Follows from Definition 17 and De Morgan's laws. □

Definition 20. Suppose that X is an initial universe, E is the parameters set, $\neg E$ is the not set of E, and $T_{\mathscr{BH}} = \{(\Phi, \Psi, E), (\Psi, \Phi, E)\}$. Then, $T_{\mathscr{BH}}$ is called the bipolar hypersoft indiscrete topology on X and $(X, T_{\mathscr{BH}}, E, \neg E)$ is said to be a bipolar hypersoft indiscrete space over X.

Definition 21. Suppose that X is an initial universe, E is the parameters set, $\neg E$ is the not set of E, and $T_{\mathscr{BH}}$ is the collection of all bipolar hypersoft sets, which can be defined over X. Then, $T_{\mathscr{BH}}$ is called the bipolar hypersoft discrete topology on X and $(X, T_{\mathscr{BH}}, E, \neg E)$ is said to be a bipolar hypersoft discrete space over X.

The following proposition shows that a bipolar hypersoft topological space gives a parametrized family of hypersoft topological spaces.

Proposition 2. *Suppose that $(X, T_{\mathscr{BH}}, E, \neg E)$ is a bipolar hypersoft space over X. Then, the following collections define hypersoft topology on X.*

(1) $T_{\mathscr{H}} = \left\{(\Lambda, E) \mid (\Lambda, \Omega, E) \tilde{\tilde{\in}} T_{\mathscr{BH}}\right\}$.
(2) $\neg T_{\mathscr{H}} = \left\{(\Omega, \neg E) \mid (\Lambda, \Omega, E) \tilde{\tilde{\in}} T_{\mathscr{BH}}\right\}$ (provided that X is finite).

Proof. (1) Suppose that $(X, T_{\mathscr{BH}}, E, \neg E)$ is a bipolar hypersoft space over X. Then:

(i) (Φ, Ψ, E), $(\Psi, \Phi, E) \tilde{\tilde{\in}} T_{\mathscr{BH}}$ imply that (Φ, E), $(\Psi, E) \tilde{\in} T_{\mathscr{H}}$.
(ii) Let (Λ_1, E), $(\Lambda_2, E) \tilde{\in} T_{\mathscr{H}}$. Since (Λ_1, Ω_1, E), $(\Lambda_2, \Omega_2, E) \tilde{\tilde{\in}} T_{\mathscr{BH}}$ then $(\Lambda_1, \Omega_1, E) \tilde{\tilde{\sqcap}} (\Lambda_2, \Omega_2, E) \tilde{\tilde{\in}} T_{\mathscr{BH}}$ this implies that $(\Lambda_1, E) \tilde{\sqcap} (\Lambda_2, E) \tilde{\in} T_{\mathscr{H}}$.

(iii) Let $\{(\Lambda_i, E)|i \in I\}$ belong to $T_{\mathscr{H}}$. Since $(\Lambda_i, \Omega_i, E) \tilde{\tilde{\in}} T_{\mathscr{B}\mathscr{H}}$ for all $i \in I$ so that $\tilde{\tilde{\sqcup}}_{i\in I}(\Lambda_i, \Omega_i, E) \tilde{\tilde{\in}} T_{\mathscr{B}\mathscr{H}}$ thus $\tilde{\sqcup}_{i\in I}(\Lambda_i, E) \tilde{\in} T_{\mathscr{H}}$.

Hence, $T_{\mathscr{H}}$ defines a hypersoft topology over X.

(2) The proof is similar to part (1). □

Remark 1. The converse of Proposition 2 is incorrect as the next example shows.

Example 1. Let $X = \{\chi_1, \chi_2, \chi_3, \chi_4\}$, $E_1 = \{\omega_1, \omega_2\}$, $E_2 = \{\omega_3\}$, and $E_3 = \{\omega_4\}$. Assume that $T_{\mathscr{H}} = \{(\Phi, E), (\Psi, E), (\Lambda_1, E), (\Lambda_2, E), (\Lambda_3, E), (\Lambda_4, E)\}$ and $\neg T_{\mathscr{H}} = \{(\Omega_1, \neg E), (\Omega_2, \neg E), (\Omega_3, \neg E), (\Omega_4, \neg E)\}$ are two hypersoft topologies on X where the hypersoft sets (Λ_1, E), (Λ_2, E), (Λ_3, E), (Λ_4, E), $(\Omega_1, \neg E)$, $(\Omega_2, \neg E)$, $(\Omega_3, \neg E)$, and $(\Omega_4, \neg E)$ are defined as follows:

$$\begin{aligned}
(\Lambda_1, E) &= \{((\omega_1, \omega_3, \omega_4), \{\chi_2\}), (\omega_2, \omega_3, \omega_4), \{\chi_1\})\},\\
(\Lambda_2, E) &= \{((\omega_1, \omega_3, \omega_4), \{\chi_1, \chi_2\}), (\omega_2, \omega_3, \omega_4), \{\chi_3\})\},\\
(\Lambda_3, E) &= \{((\omega_1, \omega_3, \omega_4), \{\chi_2\}), ((\omega_2, \omega_3, \omega_4), \phi)\},\\
(\Lambda_4, E) &= \{((\omega_1, \omega_3, \omega_4), \{\chi_1, \chi_2\}), (\omega_2, \omega_3, \omega_4), \{\chi_1, \chi_3\})\},
\end{aligned} \tag{4}$$

and

$$\begin{aligned}
(\Omega_1, \neg E) &= \{(\neg(\omega_1, \omega_3, \omega_4), \{\chi_1, \chi_3, \chi_4\}), (\neg(\omega_2, \omega_3, \omega_4), \{\chi_2, \chi_4\})\},\\
(\Omega_2, \neg E) &= \{(\neg(\omega_1, \omega_3, \omega_4), \phi), (\neg(\omega_2, \omega_3, \omega_4), \{\chi_2\})\},\\
(\Omega_3, \neg E) &= \{(\neg(\omega_1, \omega_3, \omega_4), \{\chi_1\}), (\neg(\omega_2, \omega_3, \omega_4), \{\chi_2\})\},\\
(\Omega_4, \neg E) &= \{(\neg(\omega_1, \omega_3, \omega_4), \{\chi_3, \chi_4\}), (\neg(\omega_2, \omega_3, \omega_4), \{\chi_2, \chi_4\})\},
\end{aligned} \tag{5}$$

Then,

$T_{\mathscr{B}\mathscr{H}} = \{(\Phi, \Psi, E), (\Psi, \Phi, E), (\Lambda_1, \Omega_1, E), (\Lambda_2, \Omega_2, E), (\Lambda_3, \Omega_3, E), (\Lambda_4, \Omega_4, E)\}$ where (Λ_1, Ω_1, E), (Λ_2, Ω_2, E), (Λ_3, Ω_3, E), and (Λ_4, Ω_4, E) are bipolar hypersoft sets over X, defined as follows:

$$\begin{aligned}
(\Lambda_1, \Omega_1, E) &= \{((\omega_1, \omega_3, \omega_4), \{\chi_2\}, \{\chi_1, \chi_3, \chi_4\}), ((\omega_2, \omega_3, \omega_4), \{\chi_1\}, \{\chi_2, \chi_4\})\},\\
(\Lambda_2, \Omega_2, E) &= \{((\omega_1, \omega_3, \omega_4), \{\chi_1, \chi_2\}, \phi\}), ((\omega_2, \omega_3, \omega_4), \{\chi_3\}, \{\chi_2\})\},\\
(\Lambda_3, \Omega_3, E) &= \{((\omega_1, \omega_3, \omega_4), \{\chi_2\}, \{\chi_1\}), ((\omega_2, \omega_3, \omega_4), \phi, \{\chi_2\})\},\\
(\Lambda_4, \Omega_4, E) &= \{((\omega_1, \omega_3, \omega_4), \{\chi_1, \chi_2\}, \{\chi_3, \chi_4\}), ((\omega_2, \omega_3, \omega_4), \{\chi_1, \chi_3\}, \{\chi_2, \chi_4\})\}.
\end{aligned} \tag{6}$$

If we take

$$(\Lambda_1, \Omega_1, E) \tilde{\tilde{\sqcap}} (\Lambda_2, \Omega_2, E) = (\Lambda, \Omega, E), \tag{7}$$

then

$$(\Lambda, \Omega, E) = \{((\omega_1, \omega_3, \omega_4), \{\chi_2\}, \{\chi_1, \chi_3, \chi_4\}), ((\omega_2, \omega_3, \omega_4), \phi, \{\chi_2, \chi_4\})\}, \tag{8}$$

but $(\Lambda, \Omega, E) \tilde{\tilde{\notin}} T_{\mathscr{B}\mathscr{H}}$. Hence, $T_{\mathscr{B}\mathscr{H}}$ is not a bipolar hypersoft topology on X.

The following proposition demonstrates when the converse of Proposition 2 will be true.

Proposition 3. *Suppose that $(X, T_{\mathscr{H}}, E)$ is a hypersoft space over X. Then, $T_{\mathscr{B}\mathscr{H}}$ consisting of bipolar hypersoft sets (Λ, Ω, E) such that $(\Lambda, E) \tilde{\in} T_{\mathscr{H}}$ and $\Omega(\neg\alpha) = X \backslash \Lambda(\alpha)$ for all $\neg\alpha \in \neg E$, defines a bipolar hypersoft topology over X.*

Proof

(i) Since $(\Phi, E) \tilde{\in} T_{\mathscr{H}}$, then $\Psi(\neg\alpha) = X \backslash \Phi(\alpha) = X \backslash \phi = X$ and hence $(\Phi, \Psi, E) \tilde{\tilde{\in}} T_{\mathscr{B}\mathscr{H}}$. Also, since $(\Psi, E) \tilde{\in} T_{\mathscr{H}}$ then $\Phi(\neg\alpha) = X \backslash \Psi(\alpha) = X \backslash X = \phi$ and hence $(\Psi, \Phi, E) \tilde{\tilde{\in}} T_{\mathscr{B}\mathscr{H}}$.

(ii) Let $(\Lambda_1, \Omega_1, E), (\Lambda_2, \Omega_2, E) \tilde{\tilde{\in}} T_{\mathscr{B}\mathscr{H}}$. Then $(\Lambda_1, E) \tilde{\in} T_{\mathscr{H}}$ and $\Omega_1(\neg\alpha) = X \backslash \Lambda_1(\alpha)$. Also, $(\Lambda_2, E) \tilde{\in} T_{\mathscr{H}}$ and $\Omega_2(\neg\alpha) = X \backslash \Lambda_2(\alpha)$. Now, since $T_{\mathscr{H}}$ is a hypersoft topology, then $(\Lambda_1, E) \tilde{\sqcap} (\Lambda_2, E) \tilde{\in} T_{\mathscr{H}}$. Let $(\Lambda, E) = (\Lambda_1, E) \tilde{\sqcap} (\Lambda_2, E)$; then $\Omega(\neg\alpha) = X \backslash \Lambda(\alpha) = X \backslash (\Lambda_1(\alpha) \tilde{\sqcap} \Lambda_2(\alpha)) = (X \backslash \Lambda_1(\alpha)) \tilde{\sqcup} (X \backslash \Lambda_2(\alpha)) = \Omega_1(\neg\alpha) \tilde{\sqcup} \Omega_2(\neg\alpha)$. Hence, $(\Lambda_1, \Omega_1, E) \tilde{\tilde{\sqcap}} (\Lambda_2, \Omega_2, E) \tilde{\tilde{\in}} T_{\mathscr{B}\mathscr{H}}$.

(iii) Let $\{(\Lambda_i, \Omega_i, E)|i \in I\} \tilde{\tilde{\in}} T_{\mathscr{B}\mathscr{H}}$. Then $\{(\Lambda_i, E)|i \in I\} \tilde{\in} T_{\mathscr{H}}$ and $\Omega_i(\neg\alpha) = X \backslash \Lambda_i(\alpha)$. Now, since $T_{\mathscr{H}}$ is a hypersoft topology, then $\tilde{\sqcup}_{i\in I}(\Lambda_i, E) \tilde{\in} T_{\mathscr{H}}$. Let $(\Lambda, E) = \tilde{\sqcup}_{i\in I}(\Lambda_i, E)$ then $\Omega(\neg\alpha) = X \backslash \Lambda(\alpha) = X \backslash (\tilde{\sqcup}_{i\in I}\Lambda_i(\alpha)) = \tilde{\sqcap}_{i\in I}\Omega_i(\neg\alpha)$. Thus $\tilde{\tilde{\sqcup}}_{i\in I}(\Lambda_i, \Omega_i, E) \tilde{\tilde{\in}} T_{\mathscr{B}\mathscr{H}}$. Therefore, the proof is completed. □

Definition 22. Let $(X, T_{\mathscr{B}\mathscr{H}_1}, E, \neg E)$ and $(X, T_{\mathscr{B}\mathscr{H}_2}, E, \neg E)$ be two bipolar hypersoft topological spaces over X. If $T_{\mathscr{B}\mathscr{H}_1} \tilde{\tilde{\sqsubseteq}} T_{\mathscr{B}\mathscr{H}_2}$, then $T_{\mathscr{B}\mathscr{H}_2}$ is said to be finer than $T_{\mathscr{B}\mathscr{H}_1}$. If $T_{\mathscr{B}\mathscr{H}_1} \tilde{\tilde{\sqsubseteq}} T_{\mathscr{B}\mathscr{H}_2}$ or $T_{\mathscr{B}\mathscr{H}_2} \tilde{\tilde{\sqsubseteq}} T_{\mathscr{B}\mathscr{H}_1}$, then $T_{\mathscr{B}\mathscr{H}_1}$ and $T_{\mathscr{B}\mathscr{H}_2}$ are said to be comparable bipolar hypersoft topologies over X.

Proposition 4. *Let $(X, T_{\mathscr{BH}_1}, E, \neg E)$ and $(X, T_{\mathscr{BH}_2}, E, \neg E)$ be two bipolar hypersoft topological spaces on X, and then $(X, T_{\mathscr{BH}_1} \tilde{\tilde{\sqcap}} T_{\mathscr{BH}_2}, E, \neg E)$ is a bipolar hypersoft topological space over X.*

Proof

(i) (Φ, Ψ, E), (Ψ, Φ, E) belong to $T_{\mathscr{BH}_1} \tilde{\tilde{\sqcap}} T_{\mathscr{BH}_2}$.

(ii) Let (Λ_1, Ω_1, E), $(\Lambda_2, \Omega_2, E) \tilde{\tilde{\in}} T_{\mathscr{BH}_1} \tilde{\tilde{\sqcap}} T_{\mathscr{BH}_2}$. Then (Λ_1, Ω_1, E), $(\Lambda_2, \Omega_2, E) \tilde{\tilde{\in}} T_{\mathscr{BH}_1}$ and (Λ_1, Ω_1, E), $(\Lambda_2, \Omega_2, E) \tilde{\tilde{\in}} T_{\mathscr{BH}_2}$. Since $(\Lambda_1, \Omega_1, E) \tilde{\tilde{\sqcap}} (\Lambda_2, \Omega_2, E) \tilde{\tilde{\in}} T_{\mathscr{BH}_1}$ and $(\Lambda_1, \Omega_1, E) \tilde{\tilde{\sqcap}} (\Lambda_2, \Omega_2, E) \tilde{\tilde{\in}} T_{\mathscr{BH}_2}$, so $(\Lambda_1, \Omega_1, E) \tilde{\tilde{\sqcap}} (\Lambda_2, \Omega_2, E) \tilde{\tilde{\in}} T_{\mathscr{BH}_1} \tilde{\tilde{\sqcap}} T_{\mathscr{BH}_2}$.

(iii) Let $\{(\Lambda_i, \Omega_i, E | i \in I)\}$ be a family of bipolar hypersoft sets in $T_{\mathscr{BH}_1} \tilde{\tilde{\sqcap}} T_{\mathscr{BH}_2}$. Then $(\Lambda_i, \Omega_i, E) \tilde{\tilde{\in}} T_{\mathscr{BH}_1}$ and $(\Lambda_i, \Omega_i, E) \tilde{\tilde{\in}} T_{\mathscr{BH}_2}$, for all $i \in I$, so $\sqcup_{i \in I} (\Lambda_i, \Omega_i, E) \tilde{\tilde{\in}} T_{\mathscr{BH}_1}$ and $\tilde{\tilde{\sqcup}}_{i \in I} (\Lambda_i, \Omega_i, E) \tilde{\tilde{\in}} T_{\mathscr{BH}_2}$. Therefore, $\sqcup_{i \in I} (\Lambda_i, \Omega_i, E) \tilde{\tilde{\in}} T_{\mathscr{BH}_1} \tilde{\tilde{\sqcap}} T_{\mathscr{BH}_2}$.

Thus $T_{\mathscr{BH}_1} \tilde{\tilde{\sqcap}} T_{\mathscr{BH}_2}$ defines a bipolar hypersoft topology on X and $(X, T_{\mathscr{BH}_1} \tilde{\tilde{\sqcap}} T_{\mathscr{BH}_2}, E, \neg E)$ is a bipolar hypersoft topological space over X. □

Remark 2. The union of any two bipolar hypersoft topologies on X may not be a bipolar hypersoft topology on X.

Example 2. Let $X = \{\chi_1, \chi_2, \chi_3, \chi_4\}$, $E_1 = \{\omega_1, \omega_2\}$, $E_2 = \{\omega_3\}$, and $E_3 = \{\omega_4\}$. Let $T_{\mathscr{BH}_1} = \{(\Phi, \Psi, E), (\Psi, \Phi, E), (\Lambda_1, \Omega_1, E), (\Lambda_2, \Omega_2, E), (\Lambda_3, \Omega_3, E)\}$ and $T_{\mathscr{BH}_2} = \{(\Phi, \Psi, E), (\Psi, \Phi, E), (\Gamma_1, \Theta_1, E), (\Gamma_2, \Theta_2, E), (\Gamma_3, \Theta_3, E)\}$ be two bipolar hypersoft topologies defined on X where (Λ_1, Ω_1, E), (Λ_2, Ω_2, E), (Λ_3, Ω_3, E), (Γ_1, Θ_1, E), (Γ_2, Θ_2, E), and (Γ_3, Θ_3, E) are bipolar hypersoft sets over X, defined as follows:

$$\begin{aligned}
(\Lambda_1, \Omega_1, E) &= \{((\omega_1, \omega_3, \omega_4), \{\chi_3, \chi_4\}, \{\chi_2\}), ((\omega_2, \omega_3, \omega_4), \{\chi_2, \chi_3\}, \{\chi_1\})\},\\
(\Lambda_2, \Omega_2, E) &= \{((\omega_1, \omega_3, \omega_4), \{\chi_1, \chi_2, \chi_3\}, \phi\}), ((\omega_2, \omega_3, \omega_4), \{\chi_1, \chi_4\}, \{\chi_2, \chi_3\})\},\\
(\Lambda_3, \Omega_3, E) &= \{((\omega_1, \omega_3, \omega_4), \{\chi_3\}, \{\chi_2\}), ((\omega_2, \omega_3, \omega_4), \phi, \{\chi_1, \chi_2, \chi_3\})\},
\end{aligned} \tag{9}$$

and

$$\begin{aligned}
(\Gamma_1, \Theta_1, E) &= \{((\omega_1, \omega_3, \omega_4), \{\chi_3, \chi_4\}, \{\chi_1\}), ((\omega_2, \omega_3, \omega_4), \{\chi_1, \chi_3, \chi_4\}, \{\chi_2\})\},\\
(\Gamma_2, \Theta_2, E) &= \{((\omega_1, \omega_3, \omega_4), \{\chi_1, \chi_2\}, \{\chi_3\}), ((\omega_2, \omega_3, \omega_4), \{\chi_2, \chi_4\}, \{\chi_1\})\},\\
(\Gamma_3, \Theta_3, E) &= \{((\omega_1, \omega_3, \omega_4), \phi, \{\chi_1, \chi_3\}), ((\omega_2, \omega_3, \omega_4), \{\chi_4\}, \{\chi_1, \chi_2\})\},
\end{aligned} \tag{10}$$

Then,

$T_{\mathscr{BH}_1} \tilde{\tilde{\sqcup}} T_{\mathscr{BH}_2} = \{(\Phi, \Psi, E), (\Psi, \Phi, E), (\Lambda_1, \Omega_1, E), (\Lambda_2, \Omega_2, E), (\Lambda_3, G_3, E), (\Gamma_1, \Theta_1, E), (\Gamma_2, \Theta_2, E), (\Gamma_3, \Theta_3, E)\}$.

If we take

$$(\Lambda_1, \Omega_1, E) \tilde{\tilde{\sqcup}} (\Gamma_1, \Theta_1, E) = (\Lambda, \Omega, E), \tag{11}$$

then

$$(\Lambda, \Omega, E) = \{((\omega_1, \omega_3, \omega_4), \{\chi_3, \chi_4\}, \phi), ((\omega_2, \omega_3, \omega_4), X, \phi)\}, \tag{12}$$

but $(\Lambda, \Omega, E) \widetilde{\widetilde{\notin}} T_{\mathscr{BH}_1} \tilde{\tilde{\sqcup}} T_{\mathscr{BH}_2}$. Hence, $T_{\mathscr{BH}_1} \tilde{\tilde{\sqcup}} T_{\mathscr{BH}_2}$ is not a bipolar hypersoft topology on X.

Definition 23. Suppose that $(X, T_{\mathscr{BH}}, E, \neg E)$ is a bipolar hypersoft space over X, and (Λ, Ω, E) is a bipolar hypersoft set over X and $x \in X$. Then, (Λ, Ω, E) is said to be a bipolar hypersoft neighborhood of x if there exists a bipolar hypersoft open set (Γ, Θ, E) such that $x \in (\Gamma, \Theta, E) \tilde{\tilde{\sqsubseteq}} (\Lambda, \Omega, E)$.

Proposition 5. *Suppose that $(X, T_{\mathscr{BH}}, E, \neg E)$ is a bipolar hypersoft space over X; then:*

(1) If (Λ, Ω, E) is a bipolar hypersoft neighborhood of $x \in X$, then $x \in (\Lambda, \Omega, E)$.

(2) Each $x \in X$ has a bipolar hypersoft neighborhood.

(3) If (Λ, Ω, E) and (Γ, Θ, E) are bipolar hypersoft neighborhoods of some $x \in X$, then $(\Lambda, \Omega, E) \tilde{\tilde{\sqcap}} (\Gamma, \Theta, E)$ is also a bipolar hypersoft neighborhood of x.

(4) If (Λ, Ω, E) is a bipolar hypersoft neighborhood of $x \in X$ and $(\Lambda, \Omega, E) \tilde{\tilde{\sqsubseteq}} (\Gamma, \Theta, E)$, then (Γ, Θ, E) is also a bipolar hypersoft neighborhood of $x \in X$.

Proof

(1) It follows from Definition 23.

(2) For any $x \in X$, $x \in (\Psi, \Phi, E)$ and since $(\Psi, \Phi, E) \tilde{\tilde{\in}} T_{\mathscr{BH}}$, so $x \in (\Psi, \Phi, E) \tilde{\tilde{\sqsubseteq}} (\Psi, \Phi, E)$. Thus, (Ψ, Φ, E) is a bipolar hypersoft neighborhood of x.

(3) Let (Λ, Ω, E) and (Γ, Θ, E) be the bipolar hypersoft neighborhoods of $x \in X$, then there exist (Λ_1, Ω_1, E) and $(\Lambda_2, \Omega_2, E) \tilde{\tilde{\in}} T_{\mathscr{BH}}$ such that $x \in (\Lambda_1, \Omega_1, E) \tilde{\tilde{\sqsubseteq}} (\Lambda, \Omega, E)$ and $x \in (\Lambda_2, \Omega_2, E) \tilde{\tilde{\sqsubseteq}} (\Gamma, \Theta, E)$. Now $x \in (\Lambda_1, \Omega_1, E)$ and $x \in (\Lambda_2, \Omega_2, E)$ imply that $x \in$

$(\Lambda_1,\Omega_1,E)\ \tilde{\tilde{\sqcap}}\ (\Lambda_2,\Omega_2,E)$ and $(\Lambda_1,\Omega_1,E)\ \tilde{\tilde{\sqcap}}\ (\Lambda_2,\Omega_2,E)\ \tilde{\tilde{\in}}\ T_{\mathscr{BH}}$. So, we have $x \in (\Lambda_1,\Omega_1,E)\ \tilde{\tilde{\sqcap}}\ (\Lambda_2,\Omega_2,E)\ \tilde{\tilde{\sqsubseteq}}\ (\Lambda,\Omega,E)\tilde{\tilde{\sqcap}}(\Gamma,\Theta,E)$. Thus, $(\Lambda,\Omega,E)\tilde{\tilde{\sqcap}}(\Gamma,\Theta,E)$ is a bipolar hypersoft neighborhood of x.

(4) Let (Λ,Ω,E) be a bipolar hypersoft neighborhood of $x\in X$ and $(\Lambda,\Omega,E)\ \tilde{\tilde{\sqsubseteq}}\ (\Gamma,\Theta,E)$. By definition, there exists a bipolar hypersoft open set (Λ_1,Ω_1,E) such that $x\in(\Lambda_1,\Omega_1,E)\ \tilde{\tilde{\sqsubseteq}}\ (\Lambda,\Omega,E)\ \tilde{\tilde{\sqsubseteq}}\ (\Gamma,\Theta,E)$. Thus, $x\in(\Lambda_1,\Omega_1,E)\ \tilde{\tilde{\sqsubseteq}}\ (\Gamma,\Theta,E)$. Hence, (Γ,Θ,E) is a bipolar hypersoft neighborhood of x. □

Proposition 6. *Suppose that $(X,T_{\mathscr{BH}},E,\neg E)$ is a bipolar hypersoft space over X and (Λ,Ω,E) is any bipolar hypersoft open set over X; then (Λ,Ω,E) is a bipolar hypersoft neighborhood of each point of $\cap_{\alpha\in E}\Lambda(\alpha)$, that is, of each of its points.*

Proof. Let $(\Lambda,\Omega,E)\ \tilde{\tilde{\in}}\ T_{\mathscr{BH}}$. For $x\in\cap_{\alpha\in E}\Lambda(\alpha)$, we obtain $x\in\Lambda(\alpha)$ for each $\alpha\in E$. Therefore, $x\in(\Lambda,\Omega,E)\ \tilde{\tilde{\sqsubseteq}}\ (\Lambda,\Omega,E)$ and hence (Λ,Ω,E) is a bipolar hypersoft neighborhood of x. □

Remark 3. The converse of Proposition 6 is incorrect as the next example shows.

Example 3. Consider $T_{\mathscr{BH}_1}$ given in Example 2 and let (Λ,Ω,E) be any bipolar hypersoft set defined as follows:

$$(\Lambda,\Omega,E)=\{((\omega_1,\omega_3,\omega_4),\{\chi_1,\chi_3,\chi_4\},\{\chi_2\}),\ \cdot\ ((\omega_2,\omega_3,\omega_4),\{\chi_2,\chi_3\},\{\chi_1\})\},\tag{13}$$

Then, (Λ,Ω,E) is a bipolar hypersoft neighborhood of each point of $\cap_{\alpha\in E}\Lambda(\alpha)$, that is, of each of its points, but it is not a bipolar hypersoft open set.

Definition 24. Suppose that $(X,T_{\mathscr{BH}},E,\neg E)$ is a bipolar hypersoft space over X and (Λ,Ω,E) is a bipolar hypersoft set over X. A point $x\in X$ is called a bipolar hypersoft limit point of (Λ,Ω,E) if $(\Lambda,\Omega,E)\tilde{\tilde{\sqcap}}((\Gamma,\Theta,E)\backslash\{x\})\neq(\Phi,\Psi,E)$ for every bipolar hypersoft open set (Γ,Θ,E) containing x. The set of all bipolar hypersoft limit points of (Λ,Ω,E) is denoted by $(\Lambda,\Omega,E)^d$.

Proposition 7. *Suppose that $(X,T_{\mathscr{BH}},E,\neg E)$ is a bipolar hypersoft space over X and (Λ_1,Ω_1,E), (Λ_2,Ω_2,E) are bipolar hypersoft sets over X. Then,*

(1) $(\Lambda_1,\Omega_1,E)\tilde{\tilde{\sqsubseteq}}(\Lambda_2,\Omega_2,E)$ implies $(\Lambda_1,\Omega_1,E)^d\tilde{\tilde{\sqsubseteq}}(\Lambda_2,\Omega_2,E)^d$.

(2) $((\Lambda_1,\Omega_1,E)\tilde{\tilde{\sqcap}}(\Lambda_2,\Omega_2,E))^d\tilde{\tilde{\sqsubseteq}}(\Lambda_1,\Omega_1,E)^d\tilde{\tilde{\sqcap}}(\Lambda_2,\Omega_2,E)^d$.

(3) $((\Lambda_1,\Omega_1,E)\tilde{\tilde{\sqcup}}(\Lambda_2,\Omega_2,E))^d=(\Lambda_1,\Omega_1,E)^d\tilde{\tilde{\sqcup}}(\Lambda_2,\Omega_2,E)^d$.

Proof

(1) Let $x\in(\Lambda_1,\Omega_1,E)^d$ so that x is a bipolar hypersoft limit point of (Λ_1,Ω_1,E). Then, by definition $(\Lambda_1,\Omega_1,E)\ \tilde{\tilde{\sqcap}}\ ((\Gamma,\Theta,E)\backslash\{x\})\neq(\Phi,\Psi,E)$ for every bipolar hypersoft open set (Γ,Θ,E) containing x. But $(\Lambda_1,\Omega_1,E)\ \tilde{\tilde{\sqsubseteq}}\ (\Lambda_2,\Omega_2,E)$, and it follows that $(\Lambda_2,\Omega_2,E)\ \tilde{\tilde{\sqcap}}\ ((\Gamma,\Theta,E)\backslash\{x\})\neq(\Phi,\Psi,E)$. Thus, $x\in(\Lambda_2,\Omega_2,E)^d$. Therefore, $(\Lambda_1,\Omega_1,E)^d\ \tilde{\tilde{\sqsubseteq}}\ (\Lambda_2,\Omega_2,E)^d$.

(2) Since $(\Lambda_1,\Omega_1,E)\ \tilde{\tilde{\sqcap}}\ (\Lambda_2,\Omega_2,E)\ \tilde{\tilde{\sqsubseteq}}\ (\Lambda_1,\Omega_1,E)$ and $(\Lambda_1,\Omega_1,E)\ \tilde{\tilde{\sqcap}}\ (\Lambda_2,\Omega_2,E)\ \tilde{\tilde{\sqsubseteq}}\ (\Lambda_2,\Omega_2,E)$, it follows from (1.) that $((\Lambda_1,\Omega_1,E)\ \tilde{\tilde{\sqcap}}\ (\Lambda_2,\Omega_2,E))^d\ \tilde{\tilde{\sqsubseteq}}\ (\Lambda_1,\Omega_1,E)^d$ and $((\Lambda_1,\Omega_1,E)\ \tilde{\tilde{\sqcap}}\ (\Lambda_2,\Omega_2,E))^d\ \tilde{\tilde{\sqsubseteq}}\ (\Lambda_2,\Omega_2,E)^d$. So, $((\Lambda_1,\Omega_1,E)\tilde{\tilde{\sqcap}}(\Lambda_2,\Omega_2,E))^d\ \tilde{\tilde{\sqsubseteq}}\ (\Lambda_1,\Omega_1,E)^d\ \tilde{\tilde{\sqcap}}(\Lambda_2,\Omega_2,E)^d$.

(3) Since $(\Lambda_1,\Omega_1,E)\ \tilde{\tilde{\sqsubseteq}}\ (\Lambda_1,\Omega_1,E)\ \tilde{\tilde{\sqcup}}\ (\Lambda_2,\Omega_2,E)$ and $(\Lambda_2,\Omega_2,E)\ \tilde{\tilde{\sqsubseteq}}\ (\Lambda_1,\Omega_1,E)\ \tilde{\tilde{\sqcup}}\ (\Lambda_2,\Omega_2,E)$, by (1.), we have $(\Lambda_1,\Omega_1,E)^d\ \tilde{\tilde{\sqsubseteq}}\ ((\Lambda_1,\Omega_1,E)\ \tilde{\tilde{\sqcup}}\ (\Lambda_2,\Omega_2,E))^d$ and $(\Lambda_2,\Omega_2,E)^d\ \tilde{\tilde{\sqsubseteq}}\ ((\Lambda_1,\Omega_1,E)\ \tilde{\tilde{\sqcup}}\ (\Lambda_2,\Omega_2,E))^d$. So, $(\Lambda_1,\Omega_1,E)^d\ \tilde{\tilde{\sqcup}}\ (\Lambda_2,\Omega_2,E)^d\ \tilde{\tilde{\sqsubseteq}}\ ((\Lambda_1,\Omega_1,E)\ \tilde{\tilde{\sqcup}}\ (\Lambda_2,\Omega_2,E))^d$. Now, let $x\in((\Lambda_1,\Omega_1,E)\ \tilde{\tilde{\sqcup}}\ (\Lambda_2,\Omega_2,E))^d$. Then, $((\Lambda_1,\Omega_1,E)\ \tilde{\tilde{\sqcup}}\ (\Lambda_2,\Omega_2,E))\ \tilde{\tilde{\sqcap}}\ ((\Gamma,\Theta,E)\backslash\{x\})\neq(\Phi,\Psi,E)$ for every bipolar hypersoft open set (Γ,Θ,E) containing x. Therefore, $(\Lambda_1,\Omega_1,E)\ \tilde{\tilde{\sqcap}}\ ((\Gamma,\Theta,E)\backslash\{x\})\neq(\Phi,\Psi,E)$ or $(\Lambda_2,\Omega_2,E)\ \tilde{\tilde{\sqcap}}\ ((\Gamma,\Theta,E)\backslash\{x\})\neq(\Phi,\Psi,E)$. Thus, $x\in(\Lambda_1,\Omega_1,E)^d$ or $x\in(\Lambda_2,\Omega_2,E)^d$ and then $x\in(\Lambda_1,\Omega_1,E)^d\ \tilde{\tilde{\sqcup}}\ (\Lambda_2,\Omega_2,E)^d$. Therefore, $((\Lambda_1,\Omega_1,E)\ \tilde{\tilde{\sqcup}}\ (\Lambda_2,\Omega_2,E)^d)\ \tilde{\tilde{\sqsubseteq}}\ (\Lambda_1,\Omega_1,E)^d\ \tilde{\tilde{\sqcup}}\ (\Lambda_2,\Omega_2,E)^d$. Now, we have $((\Lambda_1,\Omega_1,E)\ \tilde{\tilde{\sqcup}}(\Lambda_2,\Omega_2,E))^d=(\Lambda_1,\Omega_1,E)^d\ \tilde{\tilde{\sqcup}}(\Lambda_2,\Omega_2,E)^d$. □

Remark 4. The equality in Proposition 7 (2.) does not hold in general as shown in the next example.

Example 4. Let us consider the bipolar hypersoft topological space $(X,T_{\mathscr{BH}_1},E,\neg E)$ in Example 2 and let (Λ,Ω,E) and (Γ,Θ,E) be bipolar hypersoft sets defined as follows:

$$\begin{aligned}(\Lambda,\Omega,E)&=\{((\omega_1,\omega_3,\omega_4),\{\chi_1\},\{\chi_2,\chi_4\}),((\omega_2,\omega_3,\omega_4),\{\chi_1,\chi_3\},\{\chi_2\})\},\\(\Gamma,\Theta,E)&=\{((\omega_1,\omega_3,\omega_4),\{\chi_2\},\{\chi_1,\chi_3\}),((\omega_2,\omega_3,\omega_4),\{\chi_2\},\{\chi_1,\chi_3,\chi_4\})\},\end{aligned}\tag{14}$$

Then, $(\Lambda, \Omega, E)^d = (\Gamma, \Theta, E)^d = \{\chi_1, \chi_3\}$. But $(\Lambda, \Omega, E) \tilde{\tilde{\sqcap}} (\Gamma, \Theta, E) = (\Phi, \Psi, E)$ and $((\Lambda, \Omega, E) \tilde{\tilde{\sqcap}} (\Gamma, \Theta, E))^d = (\Phi, \Psi, E)^d = \phi$. Hence, $((\Lambda_1, \Omega_1, E) \tilde{\tilde{\sqcap}} (\Lambda_2, \Omega_2, E))^d \neq (\Lambda_1, \Omega_1, E)^d \tilde{\tilde{\sqcap}} (\Lambda_2, \Omega_2, E)^d$.

Definition 25. Let $(X, T_{\mathscr{B}\mathscr{H}}, E, \neg E)$ be a bipolar hypersoft space over X and let Υ be a nonempty subset of X. Then,

$$T_{\mathscr{B}\mathscr{H}_{\Upsilon}} = \left\{ (\Lambda_{\Upsilon}, \Omega_{\Upsilon}, E) \,|\, (\Lambda, \Omega, E) \tilde{\tilde{\in}} T_{\mathscr{B}\mathscr{H}} \right\}, \tag{15}$$

is said to be the relative bipolar hypersoft topology on Υ and $(\Upsilon, T_{\mathscr{B}\mathscr{H}_{\Upsilon}}, E, \neg E)$ is called a bipolar hypersoft subspace of $(X, T_{\mathscr{B}\mathscr{H}}, E, \neg E)$.

It is easy to verify that $T_{\mathscr{B}\mathscr{H}_{\Upsilon}}$ is a bipolar hypersoft topology on Υ.

Example 5. Any bipolar hypersoft subspace of a bipolar hypersoft indiscrete topological space is a bipolar hypersoft indiscrete topological space.

Example 6. Any bipolar hypersoft subspace of a bipolar hypersoft discrete topological space is a bipolar hypersoft discrete topological space.

Proposition 8. *Assume that $(\Upsilon, T_{\mathscr{B}\mathscr{H}_{\Upsilon}}, E, \neg E)$ and $(Z, T_{\mathscr{B}\mathscr{H}_Z}, E, \neg E)$ are bipolar hypersoft subspaces of $(X, T_{\mathscr{B}\mathscr{H}}, E, \neg E)$ and $\Upsilon \subseteq Z$. Then, $(\Upsilon, T_{\mathscr{B}\mathscr{H}_{\Upsilon}}, E, \neg E)$ is a bipolar hypersoft subspace of $(Z, T_{\mathscr{B}\mathscr{H}_Z}, E, \neg E)$.*

Proof. Since $\Upsilon \subseteq Z$, then $\Upsilon = \Upsilon \cap Z$. From Definition 16, each bipolar hypersoft open set $(\Lambda_{\Upsilon}, \Omega_{\Upsilon}, E)$ is defined as $\Lambda_{\Upsilon}(\alpha) = \Upsilon \cap \Lambda(\alpha)$ and $\Omega_{\Upsilon}(\neg\alpha) = \Upsilon \cap \Omega(\neg\alpha)$, for each $\alpha \in E$ where (Λ, Ω, E) is a bipolar hypersoft open set of $(X, T_{\mathscr{B}\mathscr{H}}, E, \neg E)$. Now, for each $\alpha \in E$,

$$\begin{aligned}
&\Lambda_{\Upsilon}(\alpha) = (\Upsilon \cap Z) \cap \Lambda(\alpha),\ \Omega_{\Upsilon}(\neg\alpha) = (\Upsilon \cap Z) \cap \Omega(\neg\alpha),\\
&\Rightarrow \Lambda_{\Upsilon}(\alpha) = \Upsilon \cap (Z \cap \Lambda(\alpha)),\ \Omega_{\Upsilon}(\neg\alpha) = \Upsilon \cap (Z \cap \Omega(\neg\alpha)),\\
&\Rightarrow \Lambda_{\Upsilon}(\alpha) = \Upsilon \cap \Lambda_Z(\alpha),\ \Omega_{\Upsilon}(\neg\alpha) = \Upsilon \cap \Omega_Z(\neg\alpha),
\end{aligned} \tag{16}$$

where (Λ_Z, Ω_Z, E) is a bipolar hypersoft open set of Z. So, $(\Upsilon, T_{\mathscr{B}\mathscr{H}_{\Upsilon}}, E, \neg E)$ is a bipolar hypersoft subspace of $(Z, T_{\mathscr{B}\mathscr{H}_Z}, E, \neg E)$. □

4. Bipolar Hypersoft Closure, Bipolar Hypersoft Interior, Bipolar Hypersoft Exterior, and Bipolar Hypersoft Boundary

In this section, we introduce the notions of bipolar hypersoft closure, bipolar hypersoft interior, bipolar hypersoft exterior, and bipolar hypersoft boundary supported with some properties and relations between them.

Definition 26. Suppose that $(X, T_{\mathscr{B}\mathscr{H}}, E, \neg E)$ is a bipolar hypersoft space, and (Λ, Ω, E) is a bipolar hypersoft set over X. The bipolar hypersoft closure of (Λ, Ω, E) is denoted by $\overline{(\Lambda, \Omega, E)}$ and is defined as the intersection of all bipolar hypersoft closed supersets of (Λ, Ω, E).

In other words, $\overline{(\Lambda, \Omega, E)} = \tilde{\tilde{\sqcap}} \setminus \{(\Gamma, \Theta, E) |\ (\Gamma, \Theta, E)^c \tilde{\tilde{\in}} T_{\mathscr{B}\mathscr{H}},\ (\Gamma, \Theta, E) \tilde{\tilde{\sqsubseteq}} (\Lambda, \Omega, E) \setminus\}$.

Proposition 9. *Suppose that $(X, T_{\mathscr{B}\mathscr{H}}, E, \neg E)$ is a bipolar hypersoft space, and (Λ, Ω, E) is a bipolar hypersoft set over X. Then:*

(1) $\overline{(\Lambda, \Omega, E)}$ is the smallest bipolar hypersoft closed set containing (Λ, Ω, E).

(2) (Λ, Ω, E) is a bipolar hypersoft closed set if and only if $(\Lambda, \Omega, E) = \overline{(\Lambda, \Omega, E)}$.

Proof

(1) It follows from Definition 26.

(2) Suppose that (Λ, Ω, E) is a bipolar hypersoft closed set. So, (Λ, Ω, E) itself is the smallest bipolar hypersoft closed set over X containing (Λ, Ω, E) and hence $(\Lambda, \Omega, E) = \overline{(\Lambda, \Omega, E)}$. Conversely, let $(\Lambda, \Omega, E) = \overline{(\Lambda, \Omega, E)}$. By (1.), $\overline{(\Lambda, \Omega, E)}$ is a bipolar hypersoft closed, and (Λ, Ω, E) is also a bipolar hypersoft closed set over X. □

Proposition 10. *Suppose that $(X, T_{\mathscr{B}\mathscr{H}}, E, \neg E)$ is a bipolar hypersoft space over X and (Λ_1, Ω_1, E), (Λ_2, Ω_2, E) are bipolar hypersoft sets over X. Then:*

(1) $\overline{(\Phi, \Psi, E)} = (\Phi, \Psi, E)$ and $\overline{(\Psi, \Phi, E)} = (\Psi, \Phi, E)$.

(2) $(\Lambda_1, \Omega_1, E) \tilde{\tilde{\sqsubseteq}} \overline{(\Lambda_1, \Omega_1, E)}$.

(3) $((\Lambda_1, \Omega_1, E) \tilde{\tilde{\sqsubseteq}} (\Lambda_2, \Omega_2, E)$ implies $\overline{(\Lambda_1, \Omega_1, E)} \tilde{\tilde{\sqsubseteq}} \overline{(\Lambda_2, \Omega_2, E)}$.

(4) $\overline{((\Lambda_1, \Omega_1, E) \tilde{\tilde{\sqcup}} (\Lambda_2, \Omega_2, E))} = \overline{(\Lambda_1, \Omega_1, E)} \tilde{\tilde{\sqcup}} \overline{(\Lambda_2, \Omega_2, E)}$.

(5) $\overline{((\Lambda_1, \Omega_1, E) \tilde{\tilde{\sqcap}} (\Lambda_2, \Omega_2, E))} \tilde{\tilde{\sqsubseteq}} \overline{(\Lambda_1, \Omega_1, E)} \tilde{\tilde{\sqcap}} \overline{(\Lambda_2, \Omega_2, E)}$.

(6) $\overline{\overline{(\Lambda_1, \Omega_1, E)}} = \overline{(\Lambda_1, \Omega_1, E)}$.

Proof

(1) Since (Φ, Ψ, E) and (Ψ, Φ, E) are bipolar hypersoft closed sets, then by Proposition 9 (2.), we have $\overline{(\Phi, \Psi, E)} = (\Phi, \Psi, E)$ and $\overline{(\Psi, \Phi, E)} = (\Psi, \Phi, E)$.

(2) By Proposition 9 (1.), $\overline{(\Lambda_1, \Omega_1, E)}$ is the smallest bipolar hypersoft closed set containing (Λ_1, Ω_1, E) and so $(\Lambda_1, \Omega_1, E) \tilde{\tilde{\sqsubseteq}} \overline{(\Lambda_1, \Omega_1, E)}$.

(3) By (2.), $(\Lambda_2, \Omega_2, E) \tilde{\tilde{\sqsubseteq}} \overline{(\Lambda_2, \Omega_2, E)}$. Since $(\Lambda_1, \Omega_1, E) \tilde{\tilde{\sqsubseteq}} (\Lambda_2, \Omega_2, E)$, we have $(\Lambda_1, \Omega_1, E) \tilde{\tilde{\sqsubseteq}} \overline{(\Lambda_2, \Omega_2, E)}$. But $\overline{(\Lambda_2, \Omega_2, E)}$ is a bipolar hypersoft closed set. Thus, $\overline{(\Lambda_2, \Omega_2, E)}$ is a bipolar hypersoft closed set containing (Λ_1, Ω_1, E). Since $\overline{(\Lambda_1, \Omega_1, E)}$ is the smallest bipolar hypersoft closed set over X containing (Λ_1, Ω_1, E), so we have $\overline{(\Lambda_1, \Omega_1, E)} \tilde{\tilde{\sqsubseteq}} \overline{(\Lambda_2, \Omega_2, E)}$.

(4) Since $(\Lambda_1,\Omega_1,E) \tilde{\tilde{\sqsubseteq}} (\Lambda_1,\Omega_1,E) \tilde{\tilde{\sqcup}} (\Lambda_2,\Omega_2,E)$ and $(\Lambda_2,\Omega_2,E) \tilde{\tilde{\sqsubseteq}} (\Lambda_1,\Omega_1,E) \tilde{\tilde{\sqcup}} (\Lambda_2,\Omega_2,E)$, by (3.), $\overline{(\Lambda_1,\Omega_1,E)} \tilde{\tilde{\sqsubseteq}} \overline{((\Lambda_1,\Omega_1,E)\tilde{\tilde{\sqcup}}(\Lambda_2,\Omega_2,E))}$ and $\overline{(\Lambda_2,\Omega_2,E)} \tilde{\tilde{\sqsubseteq}} \overline{((\Lambda_1,\Omega_1,E)\tilde{\tilde{\sqcup}}(\Lambda_2,\Omega_2,E))}$. Hence, $\overline{(\Lambda_1,\Omega_1,E)} \tilde{\tilde{\sqcup}} \overline{(\Lambda_2,\Omega_2,E)} \tilde{\tilde{\sqsubseteq}} \overline{((\Lambda_1,\Omega_1,E)\tilde{\tilde{\sqcup}}(\Lambda_2,\Omega_2,E))}$. Now, since $\overline{(\Lambda_1,\Omega_1,E)}$ and $\overline{(\Lambda_2,\Omega_2,E)}$ are bipolar hypersoft closed sets, $\overline{(\Lambda_1,\Omega_1,E)} \tilde{\tilde{\sqcup}} \overline{(\Lambda_2,\Omega_2,E)}$ is also bipolar hypersoft closed. Also, $(\Lambda_1,\Omega_1,E) \tilde{\tilde{\sqsubseteq}} \overline{(\Lambda_1,\Omega_1,E)}$ and $(\Lambda_2,\Omega_2,E) \tilde{\tilde{\sqsubseteq}} \overline{(\Lambda_2,\Omega_2,E)}$ imply that $(\Lambda_1,\Omega_1,E) \tilde{\tilde{\sqcup}} (\Lambda_2,\Omega_2,E) \tilde{\tilde{\sqsubseteq}} \overline{(\Lambda_1,\Omega_1,E)} \tilde{\tilde{\sqcup}} \overline{(\Lambda_2,\Omega_2,E)}$. Thus, $\overline{(\Lambda_1,\Omega_1,E)} \tilde{\tilde{\sqcup}} \overline{(\Lambda_2,\Omega_2,E)}$ is a bipolar hypersoft closed containing $(\Lambda_1,\Omega_1,E) \tilde{\tilde{\sqcup}} (\Lambda_2,\Omega_2,E)$. Since $\overline{((\Lambda_1,\Omega_1,E)\tilde{\tilde{\sqcup}}(\Lambda_2,\Omega_2,E))}$ is the smallest bipolar hypersoft closed set containing $(\Lambda_1,\Omega_1,E) \tilde{\tilde{\sqcup}} (\Lambda_2,\Omega_2,E)$, we have $\overline{((\Lambda_1,\Omega_1,E)\tilde{\tilde{\sqcup}}(\Lambda_2,\Omega_2,E))} \tilde{\tilde{\sqsubseteq}} \overline{(\Lambda_1,\Omega_1,E)} \tilde{\tilde{\sqcup}} \overline{(\Lambda_2,\Omega_2,E)}$. So, $\overline{((\Lambda_1,\Omega_1,E)\tilde{\tilde{\sqcup}}(\Lambda_2,\Omega_2,E))} = \overline{(\Lambda_1,\Omega_1,E)} \tilde{\tilde{\sqcup}} \overline{(\Lambda_2,\Omega_2,E)}$.

(5) Since $(\Lambda_1,\Omega_1,E) \tilde{\tilde{\sqcap}} (\Lambda_2,\Omega_2,E) \tilde{\tilde{\sqsubseteq}} (\Lambda_1,\Omega_1,E)$ and $(\Lambda_1,\Omega_1,E) \tilde{\tilde{\sqcap}} (\Lambda_2,\Omega_2,E) \tilde{\tilde{\sqsubseteq}} (\Lambda_2,\Omega_2,E)$, then $\overline{((\Lambda_1,\Omega_1,E)\tilde{\tilde{\sqcap}}(\Lambda_2,\Omega_2,E))} \tilde{\tilde{\sqsubseteq}} \overline{(\Lambda_1,\Omega_1,E)}$ and $\overline{((\Lambda_1,\Omega_1,E)\tilde{\tilde{\sqcap}}(\Lambda_2,\Omega_2,E))} \tilde{\tilde{\sqsubseteq}} \overline{(\Lambda_2,\Omega_2,E)}$. Therefore, $\overline{((\Lambda_1,\Omega_1,E)\tilde{\tilde{\sqcap}}(\Lambda_2,\Omega_2,E))} \tilde{\tilde{\sqsubseteq}} \overline{(\Lambda_1,\Omega_1,E)} \tilde{\tilde{\sqcap}} \overline{(\Lambda_2,\Omega_2,E)}$.

(6) Since $\overline{(\Lambda_1,\Omega_1,E)}$ is a bipolar hypersoft closed set, therefore, by Proposition 9 (2.), we have $\overline{\overline{(\Lambda_1,\Omega_1,E)}} = \overline{(\Lambda_1,\Omega_1,E)}$. □

Remark 5. The equality does not hold in Proposition 10 (5.) as the following example shows.

Example 7. Let us consider the bipolar hypersoft topological space $(X, T_{\mathscr{BH}_1}, E, \neg E)$ in Example 2 and let (Λ,Ω,E) and (Γ,Θ,E) be bipolar hypersoft sets defined as follows:

$$\begin{aligned}(\Lambda,\Omega,E) &= \{((\omega_1,\omega_3,\omega_4),\phi,X),((\omega_2,\omega_3,\omega_4),\phi,\{\chi_1,\chi_2,\chi_3\})\},\\ (\Gamma,\Theta,E) &= \{((\omega_1,\omega_3,\omega_4),\{\chi_2\},\{\chi_1,\chi_3\}),((\omega_2,\omega_3,\omega_4),\{\chi_2\},\{\chi_4\})\},\end{aligned} \tag{17}$$

Then, $\overline{(\Lambda,\Omega,E)} = (\Lambda_1,\Omega_1,E)^c$ and $\overline{(\Gamma,\Theta,E)} = (\Lambda_3,\Omega_3,E)^c$ and $\overline{(\Lambda,\Omega,E)} \tilde{\tilde{\sqcap}} \overline{(\Gamma,\Theta,E)} = (\Lambda_1,\Omega_1,E)^c$. Now, $(\Lambda,\Omega,E) \tilde{\tilde{\sqcap}} (\Gamma,\Theta,E) = (\Phi,\Psi,E)$ and $\overline{((\Lambda,\Omega,E)\tilde{\tilde{\sqcap}}(\Gamma,\Theta,E))} = \overline{(\Phi,\Psi,E)} = (\Phi,\Psi,E)$. Hence, $\overline{((\Lambda,\Omega,E)\tilde{\tilde{\sqcap}}(\Gamma,\Theta,E))} \neq \overline{(\Lambda,\Omega,E)} \tilde{\tilde{\sqcap}} \overline{(\Gamma,\Theta,E)}$.

Definition 27. Suppose that $(X, T_{\mathscr{BH}}, E, \neg E)$ is a bipolar hypersoft space over X, and (Λ,Ω,E) is a bipolar hypersoft set over X and $x \in X$. Then, x is said to be a bipolar hypersoft interior point of (Λ,Ω,E) if there exists a bipolar hypersoft open set (Γ,Θ,E) such that $x \in (\Gamma,\Theta,E) \tilde{\tilde{\sqsubseteq}} (\Lambda,\Omega,E)$.

Definition 28. Suppose that $(X, T_{\mathscr{BH}}, E, \neg E)$ is a bipolar hypersoft space over X. The bipolar hypersoft interior of bipolar hypersoft set (Λ,Ω,E) is denoted by $(\Lambda,\Omega,E)^o$ and is defined as the union of all bipolar hypersoft open set contained in (Λ,Ω,E).

In other words, $(\Lambda,\Omega,E)^o = \tilde{\tilde{\sqcup}}\, \{(\Gamma,\Theta,E) | (\Gamma,\Theta,E) \tilde{\tilde{\in}} T_{\mathscr{BH}}, (\Gamma,\Theta,E) \tilde{\tilde{\sqsubseteq}} (\Lambda,\Omega,E)\}$.

Proposition 11. *Suppose that $(X, T_{\mathscr{BH}}, E, \neg E)$ is a bipolar hypersoft space and (Λ,Ω,E) is a bipolar hypersoft set over X. Then:*

(1) $(\Lambda,\Omega,E)^o$ is the largest bipolar hypersoft open set contained in (Λ,Ω,E).

(2) (Λ,Ω,E) is a bipolar hypersoft open set if and only if $(\Lambda,\Omega,E) = (\Lambda,\Omega,E)^o$.

Proof

(1) It follows from Definition 28.

(2) Suppose that (Λ,Ω,E) is a bipolar hypersoft open set. Then, (Λ,Ω,E) is surely identical with the largest bipolar hypersoft open subset of (Λ,Ω,E). But by (1.), $(\Lambda,\Omega,E)^o$ is the largest bipolar hypersoft open subset of (Λ,Ω,E). Hence, $(\Lambda,\Omega,E) = (\Lambda,\Omega,E)^o$. Conversely, let $(\Lambda,\Omega,E) = (\Lambda,\Omega,E)^o$. By (1.), $(\Lambda,\Omega,E)^o$ is a bipolar hypersoft open set, and therefore, (Λ,Ω,E) is also bipolar hypersoft open set. □

Proposition 12. *Suppose that $(X, T_{\mathscr{BH}}, E, \neg E)$ is a bipolar hypersoft space over X and (Λ_1,Ω_1,E), (Λ_2,Ω_2,E) are bipolar hypersoft sets over X. Then:*

(1) $(\Phi,\Psi,E)^o = (\Phi,\Psi,E)$ and $(\Psi,\Phi,E)^o = (\Psi,\Phi,E)$.

(2) $(\Lambda_1,\Omega_1,E)^o \tilde{\tilde{\sqsubseteq}} (\Lambda_1,\Omega_1,E)$.

(3) $(\Lambda_1,\Omega_1,E) \tilde{\tilde{\sqsubseteq}} (\Lambda_2,\Omega_2,E)$ implies $(\Lambda_1,\Omega_1,E)^o \tilde{\tilde{\sqsubseteq}} (\Lambda_2,\Omega_2,E)^o$.

(4) $(\Lambda_1,\Omega_1,E)^o \tilde{\tilde{\sqcap}} (\Lambda_2,\Omega_2,E)^o = ((\Lambda_1,\Omega_1,E)\tilde{\tilde{\sqcap}}(\Lambda_2,\Omega_2,E))^o$.

(5) $(\Lambda_1,\Omega_1,E)^o \tilde{\tilde{\sqcup}} (\Lambda_2,\Omega_2,E)^o \tilde{\tilde{\sqsubseteq}} ((\Lambda_1,\Omega_1,E) \tilde{\tilde{\sqcup}} (\Lambda_2,\Omega_2,E))^o$.

(6) $((\Lambda_1,\Omega_1,E)^o)^o = (\Lambda_1,\Omega_1,E)^o$.

Proof

(1) Since (Φ,Ψ,E) and (Ψ,Φ,E) are bipolar hypersoft open sets, then by Proposition 11 (2.), we have $(\Phi,\Psi,E)^o = (\Phi,\Psi,E)$ and $(\Psi,\Phi,E)^o = (\Psi,\Phi,E)$.

(2) Let $x \in (\Lambda_1,\Omega_1,E)^o$; then x is a bipolar hypersoft interior point of (Λ_1,Ω_1,E). This implies that (Λ_1,Ω_1,E) is a bipolar hypersoft neighborhood of x. Then, $x \in (\Lambda_1,\Omega_1,E)$. Hence, $(\Lambda_1,\Omega_1,E)^o \tilde{\tilde{\sqsubseteq}} (\Lambda_1,\Omega_1,E)$.

(3) Let $x \in (\Lambda_1,\Omega_1,E)^o$. Then, x is a bipolar hypersoft interior point of (Λ_1,Ω_1,E) and so (Λ_1,Ω_1,E) is a bipolar hypersoft neighborhood of x. Since $(\Lambda_1,\Omega_1,E) \tilde{\tilde{\sqsubseteq}} (\Lambda_2,\Omega_2,E)$, (Λ_2,Ω_2,E) is also a bipolar hypersoft neighborhood of x. This implies that $x \in (\Lambda_2,\Omega_2,E)^o$. Thus, $(\Lambda_1,\Omega_1,E)^o \tilde{\tilde{\sqsubseteq}} (\Lambda_2,\Omega_2,E)^o$.

(4) Since $(\Lambda_1,\Omega_1,E) \tilde{\tilde{\sqcap}} (\Lambda_2,\Omega_2,E) \tilde{\tilde{\sqsubseteq}} (\Lambda_1,\Omega_1,E)$ and $(\Lambda_1,\Omega_1,E) \tilde{\tilde{\sqcap}} (\Lambda_2,\Omega_2,E) \tilde{\tilde{\sqsubseteq}} (\Lambda_2,\Omega_2,E)$, we have, by (3.), $((\Lambda_1,\Omega_1,E) \tilde{\tilde{\sqcap}} (\Lambda_2,\Omega_2,E))^o \tilde{\tilde{\sqsubseteq}} (\Lambda_1,\Omega_1,E)^o$ and $((\Lambda_1,\Omega_1,E) \tilde{\tilde{\sqcap}} (\Lambda_2,\Omega_2,E))^o \tilde{\tilde{\sqsubseteq}} (\Lambda_2,\Omega_2,E)^o$, implying $((\Lambda_1,\Omega_1,E) \tilde{\tilde{\sqcap}} (\Lambda_2,\Omega_2,E))^o \tilde{\tilde{\sqsubseteq}} (\Lambda_1,\Omega_1,E)^o \tilde{\tilde{\sqcap}} (\Lambda_2,\Omega_2,E)^o$. Now, let $x \in (\Lambda_1,\Omega_1,E)^o \tilde{\tilde{\sqcap}} (\Lambda_2,\Omega_2,E))^o$. Then, $x \in (\Lambda_1,\Omega_1,E)^o$ and $x \in (\Lambda_2,\Omega_2,E))^o$. Hence, x is a bipolar hypersoft interior point of each of the bipolar hypersoft sets (Λ_1,Ω_1,E) and (Λ_2,Ω_2,E). It follows that (Λ_1,Ω_1,E) and (Λ_2,Ω_2,E) are bipolar hypersoft neighborhoods of x so that their intersection $(\Lambda_1,\Omega_1,E) \tilde{\tilde{\sqcap}} (\Lambda_2,\Omega_2,E)$ is also a bipolar hypersoft neighborhood of x. Hence, $x \in ((\Lambda_1,\Omega_1,E) \tilde{\tilde{\sqcap}} (\Lambda_2,\Omega_2,E))^o$. Thus, $(\Lambda_1,\Omega_1,E)^o \tilde{\tilde{\sqcap}} (\Lambda_2,\Omega_2,E))^o \tilde{\tilde{\sqsubseteq}} ((\Lambda_1,\Omega_1,E) \tilde{\tilde{\sqcap}} (\Lambda_2,\Omega_2,E))^o$. Therefore, $(\Lambda_1,\Omega_1,E)^o \tilde{\tilde{\sqcap}} (\Lambda_2,\Omega_2,E)^o = ((\Lambda_1,\Omega_1,E) \tilde{\tilde{\sqcap}} (\Lambda_2,\Omega_2,E))^o$.

(5) By (3.), $(\Lambda_1,\Omega_1,E) \tilde{\tilde{\sqsubseteq}} (\Lambda_1,\Omega_1,E) \tilde{\tilde{\sqcup}} (\Lambda_2,\Omega_2,E)$ implies $(\Lambda_1,\Omega_1,E)^o \tilde{\tilde{\sqsubseteq}} ((\Lambda_1,\Omega_1,E) \tilde{\tilde{\sqcup}} (\Lambda_2,\Omega_2,E))^o$ and $(\Lambda_2,\Omega_2,E) \tilde{\tilde{\sqsubseteq}} (\Lambda_1,\Omega_1,E) \tilde{\tilde{\sqcup}} (\Lambda_2,\Omega_2,E)$ implies $(\Lambda_2,\Omega_2,E)^o \tilde{\tilde{\sqsubseteq}} ((\Lambda_1,\Omega_1,E) \tilde{\tilde{\sqcup}} (\Lambda_2,\Omega_2,E))^o$. Hence, $(\Lambda_1,\Omega_1,E)^o \tilde{\tilde{\sqcup}} (\Lambda_2,\Omega_2,E)^o \tilde{\tilde{\sqsubseteq}} ((\Lambda_1,\Omega_1,E) \tilde{\tilde{\sqcup}} (\Lambda_2,\Omega_2,E))^o$.

(6) By Proposition 11 (1.), $(\Lambda_1,\Omega_1,E)^o$ is the bipolar hypersoft open set. Hence, by (2.) of the same proposition $((\Lambda_1,\Omega_1,E)^o)^o = (\Lambda_1,\Omega_1,E)^o$. □

Remark 6. The equality does not hold in Proposition 12 (5.) as shown in the following example.

Example 8. Let us consider the bipolar hypersoft topological space $(X, T_{\mathscr{BH}_1}, E, \neg E)$ in Example 2 and let (Λ,Ω,E) and (Γ,Θ,E) be bipolar hypersoft sets defined as follows:

$$\begin{aligned}(\Lambda,\Omega,E) &= \{((\omega_1,\omega_3,\omega_4),\{\chi_1,\chi_3,\chi_4\},\{\chi_2\}),((\omega_2,\omega_3,\omega_4),\{\chi_2,\chi_3\},\{\chi_1\})\},\\ (\Gamma,\Theta,E) &= \{(\omega_1,\omega_3,\omega_4),X,\phi),((\omega_2,\omega_3,\omega_4),\{\chi_1,\chi_4\},\{\chi_2\})\},\end{aligned} \tag{18}$$

Then, $(\Lambda,\Omega,E)^o = (\Lambda_1,\Omega_1,E)$ and $(\Gamma,\Theta,E)^o = (\Lambda_2,\Omega_2,E)$ and $(\Lambda,\Omega,E)^o \tilde{\tilde{\sqcup}} (\Gamma,\Theta,E)^o = (\Lambda_3,\Omega_3,E)$. Now, $(\Lambda,\Omega,E) \tilde{\tilde{\sqcup}} (\Gamma,\Theta,E) = (\Psi,\Phi,E)$ and $((\Lambda,\Omega,E) \tilde{\tilde{\sqcup}} (\Gamma,\Theta,E))^o = (\Psi,\Phi,E)^o = (\Psi,\Phi,E)$. Hence, $((\Lambda,\Omega,E))t\tilde{\tilde{\sqcup}}n \Gamma,\Theta,E))^o \neq (\Lambda,\Omega,E)^o \tilde{\tilde{\sqcup}} (\Gamma,\Theta,E)^o$.

Proposition 13. *Suppose that $(X, T_{\mathscr{BH}}, E, \neg E)$ is a bipolar hypersoft space over X and (Λ,Ω,E) is a bipolar hypersoft set over X. Then, $(\Lambda,\Omega,E)^o \tilde{\tilde{\sqsubseteq}} (\Lambda,\Omega,E) \tilde{\tilde{\sqsubseteq}} \overline{(\Lambda,\Omega,E)}$.*

Proof. It follows from Proposition 10 (2.) and Proposition12 (2.). □

Proposition 14. *Suppose that $(X, T_{\mathscr{BH}}, E, \neg E)$ is a bipolar hypersoft space over X and (Λ_1,Ω_1,E), (Λ_2,Ω_2,E) are bipolar hypersoft sets over X. Then:*

(1) $(\overline{(\Lambda_1,\Omega_1,E)})^c = ((\Lambda_1,\Omega_1,E)^c)^o$.

(2) $((\Lambda_1,\Omega_1,E)^o)^c = \overline{((\Lambda_1,\Omega_1,E)^c)}$.

(3) $\overline{(\Lambda_1,\Omega_1,E)} = (((\Lambda_1,\Omega_1,E)^c)^o)^c$.

(4) $(\Lambda_1,\Omega_1,E)^o = (\overline{(\Lambda_1,\Omega_1,E^c)})^c$.

(5) $((\Lambda_1,\Omega_1,E)\backslash(\Lambda_2,\Omega_2,E))^o \tilde{\tilde{\sqsubseteq}} (\Lambda_1,\Omega_1,E)^o \backslash (\Lambda_2,\Omega_2,E)^o$.

Proof. From the definitions of bipolar hypersoft closure and bipolar hypersoft interior, we have:

(1) $\overline{(\Lambda_1,\Omega_1,E)} = \tilde{\tilde{\sqcap}} \{(\Gamma,\Theta,E) \mid (\Gamma,\Theta,E)^c \tilde{\tilde{\in}} T_{\mathscr{BH}}, (\Gamma,\Theta,E) \tilde{\tilde{\sqsubseteq}} (\Lambda_1,\Omega_1,E)\}$. $(\overline{(\Lambda_1,\Omega_1,E)})^c = [\tilde{\tilde{\sqcap}} \{(\Gamma,\Theta,E) \mid (\Gamma,\Theta,E)^c \tilde{\tilde{\in}} T_{\mathscr{BH}}, (\Gamma,\Theta,E) \tilde{\tilde{\sqsubseteq}} (\Lambda_1,\Omega_1,E)\}]^c$. $(\overline{(\Lambda_1,\Omega_1,E)})^c = \tilde{\tilde{\sqcup}} \{(\Gamma,\Theta,E)^c \mid (\Gamma,\Theta,E)^c \tilde{\tilde{\in}} T_{\mathscr{BH}}, (\Gamma,\Theta,E)^c \tilde{\tilde{\sqsubseteq}} (\Lambda_1,\Omega_1,E)^c\} = ((\Lambda_1,\Omega_1,E)^c)^o$.

(2) $(\Lambda_1,\Omega_1,E)^o = \tilde{\tilde{\sqcup}} \{(\Gamma,\Theta,E) \mid (\Gamma,\Theta,E) \tilde{\tilde{\in}} T_{\mathscr{BH}}, (\Gamma,\Theta,E) \tilde{\tilde{\sqsubseteq}} (\Lambda_1,\Omega_1,E)\}$. $((\Lambda_1,\Omega_1,E)^o)^c = [\tilde{\tilde{\sqcup}} \{(\Gamma,\Theta,E) \mid (\Gamma,\Theta,E) \tilde{\tilde{\in}} T_{\mathscr{BH}}, (\Gamma,\Theta,E) \tilde{\tilde{\sqsubseteq}} (\Lambda_1,\Omega_1,E)\}]^c$. $((\Lambda_1,\Omega_1,E)^o)^c = \tilde{\tilde{\sqcap}} \{(\Gamma,\Theta,E)^c \mid (\Gamma,\Theta,E) \tilde{\tilde{\in}} T_{\mathscr{BH}}, (\Lambda_1,\Omega_1,E)^c \tilde{\tilde{\sqsubseteq}} (\Gamma,\Theta,E)^c\} = \overline{((\Lambda_1,\Omega_1,E)^c)}$.

(3) Obtained from (1.) by taking the bipolar hypersoft complement.

(4) Obtained from (2.) by taking the bipolar hypersoft complement.

(5) $((\Lambda_1,\Omega_1,E) \setminus (\Lambda_2,\Omega_2,E))^o = ((\Lambda_1,\Omega_1,E) \tilde{\tilde{\sqcap}} (\Lambda_2,\Omega_2,E)^c)^o = (\Lambda_1,\Omega_1,E)^o \tilde{\tilde{\sqcap}} ((\Lambda_2,\Omega_2,E)^c)^o = (\Lambda_1,\Omega_1,E)^o \tilde{\tilde{\sqcap}} (\overline{(\Lambda_2,\Omega_2,E)})^c = (\Lambda_1,\Omega_1,E)^o \setminus \overline{(\Lambda_2,\Omega_2,E)} \tilde{\tilde{\sqsubseteq}} (\Lambda_1,\Omega_1,E)^o \setminus (\Lambda_2,\Omega_2,E)^o$. □

Definition 29. Suppose that $(X, T_{\mathscr{B}\mathscr{H}}, E, \neg E)$ is a bipolar hypersoft space over X and (Λ,Ω,E) is a bipolar hypersoft set over X. A point $x \in X$ is said to be a bipolar hypersoft exterior point of (Λ,Ω,E) if and only if it is a bipolar hypersoft interior point of $(\Lambda,\Omega,E)^c$, that is, if and only if there exists a bipolar hypersoft open set (Γ,Θ,E) such that $x \in (\Gamma,\Theta,E) \tilde{\tilde{\sqsubseteq}} (\Lambda,\Omega,E)^c$. The set of all bipolar hypersoft exterior points of (Λ,Ω,E) is called the bipolar hypersoft exterior of (Λ,Ω,E) and is denoted by $(\Lambda,\Omega,E)^e$.

Thus, $(\Lambda,\Omega,E)^e = ((\Lambda,\Omega,E)^c)^o$. It follows that $((\Lambda,\Omega,E)^c)^e = (((\Lambda,\Omega,E)^c)^c)^o = (\Lambda,\Omega,E)^o$.

We also have $(\Lambda,\Omega,E) \tilde{\tilde{\sqcap}} (\Lambda,\Omega,E)^e = (\Phi,\Omega,E)$; that is, no point of (Λ,Ω,E) can be a bipolar hypersoft exterior point of (Λ,Ω,E).

Remark 7. Since $(\Lambda,\Omega,E)^e$ is the bipolar hypersoft interior of $(\Lambda,\Omega,E)^c$, it follows that $(\Lambda,\Omega,E)^e$ is the bipolar hypersoft open and is the largest bipolar hypersoft open set contained in $(\Lambda,\Omega,E)^c$.

Proposition 15. *Suppose that $(X, T_{\mathscr{B}\mathscr{H}}, E, \neg E)$ is a bipolar hypersoft space and (Λ,Ω,E) is a bipolar hypersoft set over X. Then,*

$$(\Lambda,\Omega,E)^e = \tilde{\tilde{\sqcup}}\left\{(\Gamma,\Theta,E) \mid (\Gamma,\Theta,E) \tilde{\tilde{\in}} T_{\mathscr{B}\mathscr{H}}, (\Gamma,\Theta,E) \tilde{\tilde{\sqsubseteq}} (\Lambda,\Omega,E)^c\right\}. \quad (19)$$

Proof. From the definitions of bipolar hypersoft interior and bipolar hypersoft exterior, we have

$$((\Lambda,\Omega,E)^c)^o = \tilde{\tilde{\sqcup}}\left\{(\Gamma,\Theta,E) \mid (\Gamma,\Theta,E) \tilde{\tilde{\in}} T_{\mathscr{B}\mathscr{H}}, (\Gamma,\Theta,E) \tilde{\tilde{\sqsubseteq}} (\Lambda,\Omega,E)^c\right\} = (\Lambda,\Omega,E)^e. \quad (20)$$

□

Proposition 16. *Suppose that $(X, T_{\mathscr{B}\mathscr{H}}, E, \neg E)$ is a bipolar hypersoft space over X and (Λ_1,Ω_1,E), (Λ_2,Ω_2,E) are bipolar hypersoft sets over X. Then:*

(1) $(\Psi,\Phi,E)^e = (\Phi,\Psi,E)$ and $(\Phi,\Psi,E)^e = (\Psi,\Phi,E)$.

(2) $(\Lambda_1,\Omega_1,E)^e \tilde{\tilde{\sqsubseteq}} (\Lambda_1,\Omega_1,E)^c$.

(3) $(\Lambda_1,\Omega_1,E)^e = (((\Lambda_1,\Omega_1,E)^e)^c)^e$.

(4) $(\Lambda_1,\Omega_1,E) \tilde{\tilde{\sqsubseteq}} (\Lambda_2,\Omega_2,E)$ implies $(\Lambda_2,\Omega_2,E)^e \tilde{\tilde{\sqsubseteq}} (\Lambda_1,\Omega_1,E)^e$.

(5) $(\Lambda_1,\Omega_1,E)^o \tilde{\tilde{\sqsubseteq}} ((\Lambda_1,\Omega_1,E)^e)^e$.

(6) $((\Lambda_1,\Omega_1,E) \tilde{\tilde{\sqcup}} (\Lambda_2,\Omega_2,E))^e = (\Lambda_1,\Omega_1,E)^e \tilde{\tilde{\sqcap}} (\Lambda_2,\Omega_2,E)^e$.

(7) $((\Lambda_1,\Omega_1,E) \tilde{\tilde{\sqcap}} (\Lambda_2,\Omega_2,E))^e \tilde{\tilde{\sqsubseteq}} (\Lambda_1,\Omega_1,E)^e \tilde{\tilde{\sqcup}} (\Lambda_2,\Omega_2,E)^e$.

Proof

(1) $(\Psi,\Phi,E)^e = ((\Psi,\Phi,E)^c)^o = (\Phi,\Psi,E)^o = (\Phi,\Psi,E)$.

$$(\Phi,\Psi,E)^e = \left((\Phi,\Psi,E)^c\right)^o = (\Psi,\Phi,E)^o = (\Psi,\Phi,E). \quad (21)$$

(2) By definition, $(\Lambda_1,\Omega_1,E)^e = ((\Lambda_1,\Omega_1,E)^c)^o$ and by Proposition 12 (2.), we have $((\Lambda_1,\Omega_1,E)^c)^o \tilde{\tilde{\sqsubseteq}} (\Lambda_1,\Omega_1,E)^c$. Hence, $(\Lambda_1,\Omega_1,E)^e \tilde{\tilde{\sqsubseteq}} (\Lambda_1,\Omega_1,E)^c$.

(3) $(((\Lambda_1,\Omega_1,E)^e)^c)^e = ((((\Lambda_1,\Omega_1,E)^c)^o)^c)^e = (((((\Lambda_1,\Omega_1,E)^c)^o)^c)^c)^o = (((\Lambda_1,\Omega_1,E)^c)^o)^o = ((\Lambda_1,\Omega_1,E)^c)^o = (\Lambda_1,\Omega_1,E)^e$.

(4) $(\Lambda_1,\Omega_1,E) \tilde{\tilde{\sqsubseteq}} (\Lambda_2,\Omega_2,E)$ then $(\Lambda_2,\Omega_2,E)^c \tilde{\tilde{\sqsubseteq}} (\Lambda_1,\Omega_1,E)^c$, implying $((\Lambda_2,\Omega_2,E)^c)^o \tilde{\tilde{\sqsubseteq}} ((\Lambda_1,\Omega_1,E)^c)^o$. So, $(\Lambda_2,\Omega_2,E)^e \tilde{\tilde{\sqsubseteq}} (\Lambda_1,\Omega_1,E)^e$.

(5) By (2.), we have $(\Lambda_1,\Omega_1,E)^e \tilde{\tilde{\sqsubseteq}} (\Lambda_1,\Omega_1,E)^c$. Then, (4.) gives $((\Lambda_1,\Omega_1,E)^c)^e \tilde{\tilde{\sqsubseteq}} ((\Lambda_1,\Omega_1,E)^e)^e$. But, $(\Lambda_1,\Omega_1,E)^o = ((\Lambda_1,\Omega_1,E)^c)^e$. Hence, $(\Lambda_1,\Omega_1,E)^o \tilde{\tilde{\sqsubseteq}} ((\Lambda_1,\Omega_1,E)^e)^e$.

(6) $((\Lambda_1,\Omega_1,E) \tilde{\tilde{\sqcup}} (\Lambda_2,\Omega_2,E))^e = ((\Lambda_1,\Omega_1,E) \tilde{\tilde{\sqcup}} (\Lambda_2,\Omega_2,E)^c)^o = ((\Lambda_1,\Omega_1,E)^c \tilde{\tilde{\sqcap}} (\Lambda_2,\Omega_2,E)^c)^o = ((\Lambda_1,\Omega_1,E)^c)^o \tilde{\tilde{\sqcap}} ((\Lambda_2,\Omega_2,E)^c)^o = (\Lambda_1,\Omega_1,E)^e \tilde{\tilde{\sqcap}} (\Lambda_2,\Omega_2,E)^e$.

(7) $((\Lambda_1,\Omega_1,E) \tilde{\tilde{\sqcap}} (\Lambda_2,\Omega_2,E))^e = ((\Lambda_1,\Omega_1,E) \tilde{\tilde{\sqcap}} (\Lambda_2,\Omega_2,E)^c)^o = ((\Lambda_1,\Omega_1,E)^c \tilde{\tilde{\sqcup}} (\Lambda_2,\Omega_2,E)^c)^o \tilde{\tilde{\sqsubseteq}} ((\Lambda_1,\Omega_1,E)^c)^o \tilde{\tilde{\sqcap}} ((\Lambda_2,\Omega_2,E)^c)^o = (\Lambda_1,\Omega_1,E)^e \tilde{\tilde{\sqcap}} (\Lambda_2,\Omega_2,E)^e$. □

Definition 30. Suppose that $(X, T_{\mathscr{B}\mathscr{H}}, E, \neg E)$ is a bipolar hypersoft space over X; then, bipolar hypersoft boundary of bipolar hypersoft set (Λ,Ω,E) over X is denoted by $(\Lambda,\Omega,E)^b$ and is defined as $(\Lambda,\Omega,E)^b = \overline{(\Lambda,\Omega,E)} \sqcap \overline{(\Lambda,\Omega,E)^c}$.

Remark 8. From Definition 30 it follows that the bipolar hypersoft sets (Λ,Ω,E) and $(\Lambda,\Omega,E)^c$ have the same bipolar hypersoft boundary.

Proposition 17. *Suppose that $(X, T_{\mathscr{B}\mathscr{H}}, E, \neg E)$ is a bipolar hypersoft space and (Λ,Ω,E) is a bipolar hypersoft set over X. Then:*

(1) $(\Lambda,\Omega,E)^b \tilde{\tilde{\sqsubseteq}} \overline{(\Lambda,\Omega,E)}$.
(2) $(\Lambda,\Omega,E)^b = \overline{(\Lambda,\Omega,E)} \backslash (\Lambda,\Omega,E)^o$.
(3) $((\Lambda,\Omega,E)^b)^c = (\Lambda,\Omega,E)^o \tilde{\tilde{\sqcup}} (\Lambda,\Omega,E)^e$.
(4) $(\Lambda,\Omega,E)^o \tilde{\tilde{\sqsubseteq}} (\Lambda,\Omega,E) \backslash (\Lambda,\Omega,E)^b$.
(5) $((\Lambda,\Omega,E)^o)^b \tilde{\tilde{\sqsubseteq}} (\Lambda,\Omega,E)^b$.
(6) $(\overline{(\Lambda,\Omega,E)})^b \tilde{\tilde{\sqsubseteq}} (\Lambda,\Omega,E)^b$.

Proof

(1) By definition, $(\Lambda,\Omega,E)^b = \overline{(\Lambda,\Omega,E)} \tilde{\tilde{\sqcap}} \overline{(\Lambda,\Omega,E)^c}$. Hence, $(\Lambda,\Omega,E)^b \tilde{\tilde{\sqsubseteq}} \overline{(\Lambda,\Omega,E)}$.
(2) $(\Lambda,\Omega,E)^b = \overline{(\Lambda,\Omega,E)} \tilde{\tilde{\sqcap}} \overline{(\Lambda,\Omega,E)^c} = \overline{(\Lambda,\Omega,E)} \tilde{\tilde{\sqcap}} ((\Lambda,\Omega,E)^o)^c = \overline{(\Lambda,\Omega,E)} \backslash (\Lambda,\Omega,E)^o$.
(3) $((\Lambda,\Omega,E)^b)^c = [\overline{(\Lambda,\Omega,E)} \tilde{\tilde{\sqcap}} \overline{(\Lambda,\Omega,E)^c}]^c = (\overline{(\Lambda,\Omega,E)})^c \tilde{\tilde{\sqcup}} (\overline{(\Lambda,\Omega,E)^c})^c = ((\Lambda,\Omega,E)^c)^o \tilde{\tilde{\sqcup}} (\Lambda,\Omega,E)^o = (\Lambda,\Omega,E)^e \tilde{\tilde{\sqcup}} (\Lambda,\Omega,E)^o$.
(4) $(\Lambda,\Omega,E) \backslash (\Lambda,\Omega,E)^b = (\Lambda,\Omega,E) \tilde{\tilde{\sqcap}} ((\Lambda,\Omega,E)^b)^c = (\Lambda,\Omega,E) \tilde{\tilde{\sqcap}} ((\Lambda,\Omega,E)^o \tilde{\tilde{\sqcup}} (\Lambda,\Omega,E)^e) = ((\Lambda,\Omega,E) \tilde{\tilde{\sqcap}} (\Lambda,\Omega,E)^o) \tilde{\tilde{\sqcup}} ((\Lambda,\Omega,E) \tilde{\tilde{\sqcap}} (\Lambda,\Omega,E)^e) = (\Lambda,\Omega,E)^o \tilde{\tilde{\sqcup}} (\Phi,\Omega,E) \tilde{\tilde{\sqcap}} (\Lambda,\Omega,E)^o$.
(5) $((\Lambda,\Omega,E)^o)^b = \overline{(\Lambda,\Omega,E)^o} \tilde{\tilde{\sqcap}} \overline{((\Lambda,\Omega,E)^o)^c} = \overline{(\Lambda,\Omega,E)^o} \tilde{\tilde{\sqcap}} \overline{\overline{(\Lambda,\Omega,E)^c}} \tilde{\tilde{\sqsubseteq}} \overline{(\Lambda,\Omega,E)} \tilde{\tilde{\sqcup}} \overline{(\Lambda,\Omega,E)^c} = (\Lambda,\Omega,E)^b$.
(6) $(\overline{(\Lambda,\Omega,E)})^b = \overline{\overline{(\Lambda,\Omega,E)}} \tilde{\tilde{\sqcap}} \overline{(\overline{(\Lambda,\Omega,E)})^c} \tilde{\tilde{\sqsubseteq}} \overline{(\Lambda,\Omega,E)} \tilde{\tilde{\sqcup}} \overline{(\Lambda,\Omega,E)^c} = (\Lambda,\Omega,E)^b$. □

Remark 9. The next example shows that the equality of Proposition 17 (4.) is incorrect, whereas it is equal in hypersoft topology.

Example 9. Let $X = \{\chi_1,\chi_2,\chi_3\}$, $E_1 = \{\omega_1,\omega_2\}$, $E_2 = \{\omega_3\}$, and $E_3 = \{\omega_4\}$. Let $T_{\mathscr{BH}} = \{(\Phi,\Psi,E), (\Psi,\Phi,E), (\Lambda_1,\Omega_1,E), (\Lambda_2,\Omega_2,E), (\Lambda_3,\Omega_3,E)\}$ be a bipolar hypersoft topology defined on X where (Λ_1,Ω_1,E), (Λ_2,Ω_2,E), and (Λ_3,Ω_3,E) are bipolar hypersoft sets over X, defined as follows:

$$\begin{aligned}(\Lambda_1,\Omega_1,E) &= \{((\omega_1,\omega_3,\omega_4),\{\chi_3\},\{\chi_1\}),((\omega_2,\omega_3,\omega_4),\{\chi_3\},\{\chi_1,\chi_2\})\},\\ (\Lambda_2,\Omega_2,E) &= \{((\omega_1,\omega_3,\omega_4),\phi,\{\chi_2,\chi_3\}),((\omega_2,\omega_3,\omega_4),\{\chi_1\},\{\chi_3\})\},\\ (\Lambda_3,\Omega_3,E) &= \{((\omega_1,\omega_3,\omega_4),\{\chi_3\},\phi),((\omega_2,\omega_3,\omega_4)\{\chi_1,\chi_3\},\phi)\}.\end{aligned} \tag{22}$$

Let (Λ,Ω,E) be any bipolar hypersoft set defined as

$$(\Lambda,\Omega,E) = \{((\omega_1,\omega_3,\omega_4),\{\chi_1\},\{\chi_3\}),((\omega_2,\omega_3,\omega_4),\{\chi_1\},\{\chi_3\})\}, \tag{23}$$

Then,

$$\begin{aligned}(\Lambda,\Omega,E)^o &= \{((\omega_1,\omega_3,\omega_4),\phi,\{\chi_2,\chi_3\}),((\omega_2,\omega_3,\omega_4),\{\chi_1\},\{\chi_3\})\},\\ (\Lambda,\Omega,E)^b &= \{((\omega_1,\omega_3,\omega_4),\phi,\{\chi_3\}),((\omega_2,\omega_3,\omega_4),\phi,\{\chi_1,\chi_3\})\},\end{aligned} \tag{24}$$

Hence,

$$(\Lambda,\Omega,E)^o \neq (\Lambda,\Omega,E) \backslash (\Lambda,\Omega,E)^b. \tag{25}$$

Proposition 18. *Suppose that $(X,T_{\mathscr{BH}},E,\neg E)$ is a bipolar hypersoft space over X and (Λ_1,Ω_1,E), (Λ_2,Ω_2,E) are bipolar hypersoft sets over X. Then:*

(1) $((\Lambda_1,\Omega_1,E) \tilde{\tilde{\sqcup}} (\Lambda_2,\Omega_2,E))^b \tilde{\tilde{\sqsubseteq}} (\Lambda_1,\Omega_1,E)^b \tilde{\tilde{\sqcup}} (\Lambda_2,\Omega_2,E)^b$.
(2) $((\Lambda_1,\Omega_1,E))t\tilde{\tilde{\sqcap}}n\Lambda_2,\Omega_2,E))^b \tilde{\tilde{\sqsubseteq}} (\Lambda_1,\Omega_1,E)^b \tilde{\tilde{\sqcup}} (\Lambda_2,\Omega_2,E)^b$.

Proof

(1) $((\Lambda_1,\Omega_1,E) \tilde{\tilde{\sqcup}} (\Lambda_2,\Omega_2,E))^b = \overline{[(\Lambda_1,\Omega_1,E) \tilde{\tilde{\sqcup}} (\Lambda_2,\Omega_2,E)]} \tilde{\tilde{\sqcap}} \overline{[((\Lambda_1,\Omega_1,E) \tilde{\tilde{\sqcup}} (\Lambda_2,\Omega_2,E))^c]} = [\overline{(\Lambda_1,\Omega_1,E)} \tilde{\tilde{\sqcup}} \overline{(\Lambda_2,\Omega_2,E)}] \tilde{\tilde{\sqcap}} \overline{[(\Lambda_1,\Omega_1,E)^c \tilde{\tilde{\sqcap}} (\Lambda_2,\Omega_2,E)^c]} \tilde{\tilde{\sqsubseteq}} [\overline{(\Lambda_1,\Omega_1,E)} \tilde{\tilde{\sqcup}} \overline{(\Lambda_2,\Omega_2,E)}] \tilde{\tilde{\sqcap}} [\overline{(\Lambda_1,\Omega_1,E)^c} \tilde{\tilde{\sqcap}} \overline{(\Lambda_2,\Omega_2,E)^c}] = [\overline{(\Lambda_1,\Omega_1,E)} \tilde{\tilde{\sqcap}} (\overline{(\Lambda_1,\Omega_1,E)^c} \tilde{\tilde{\sqcap}} \overline{(\Lambda_2,\Omega_2,E)^c})] \tilde{\tilde{\sqcup}} [\overline{(\Lambda_2,\Omega_2,E)} \tilde{\tilde{\sqcap}} (\overline{(\Lambda_1,\Omega_1,E)^c} \tilde{\tilde{\sqcap}} \overline{(\Lambda_2,\Omega_2,E)^c})] = [(\overline{(\Lambda_1,\Omega_1,E)} \tilde{\tilde{\sqcap}} \overline{(\Lambda_1,\Omega_1,E)^c}) \tilde{\tilde{\sqcap}} \overline{(\Lambda_2,\Omega_2,E)^c}] \tilde{\tilde{\sqcup}} [(\overline{(\Lambda_2,\Omega_2,E)} \tilde{\tilde{\sqcap}} \overline{(\Lambda_2,\Omega_2,E)^c}) \tilde{\tilde{\sqcap}} \overline{(\Lambda_1,\Omega_1,E)^c}] \tilde{\tilde{\sqsubseteq}} [(\Lambda_1,\Omega_1,E)^b \tilde{\tilde{\sqcap}} \overline{(\Lambda_2,\Omega_2,E)^c}] \tilde{\tilde{\sqcup}} [(\Lambda_2,\Omega_2,E)^b \tilde{\tilde{\sqcap}} \overline{(\Lambda_1,\Omega_1,E)^c}] \tilde{\tilde{\sqsubseteq}} (\Lambda_1,\Omega_1,E)^b \tilde{\tilde{\sqcup}} (\Lambda_2,\Omega_2,E)^b$.
(2) $((\Lambda_1,\Omega_1,E) \tilde{\tilde{\sqcap}} (\Lambda_2,\Omega_2,E))^b = [\overline{(\Lambda_1,\Omega_1,E) \tilde{\tilde{\sqcap}} (\Lambda_2,\Omega_2,E)}] \tilde{\tilde{\sqcap}} [\overline{((\Lambda_1,\Omega_1,E) \tilde{\tilde{\sqcap}} (\Lambda_2,\Omega_2,E))^c}] \tilde{\tilde{\sqsubseteq}} [\overline{(\Lambda_1,\Omega_1,E)} \tilde{\tilde{\sqcap}} \overline{(\Lambda_2,\Omega_2,E)}] \tilde{\tilde{\sqcap}} [\overline{((\Lambda_1,\Omega_1,E) \tilde{\tilde{\sqcap}} (\Lambda_2,\Omega_2,E))^c}] = [\overline{(\Lambda_1,\Omega_1,E)} \tilde{\tilde{\sqcap}} \overline{(\Lambda_2,\Omega_2,E)}] \tilde{\tilde{\sqcap}} \overline{(\Lambda_1,\Omega_1,E)^c} \tilde{\tilde{\sqcup}} \overline{(\Lambda_2,\Omega_2,E)^c} = [\overline{(\Lambda_1,\Omega_1,E)} \tilde{\tilde{\sqcap}} \overline{(\Lambda_2,\Omega_2,E)} \tilde{\tilde{\sqcap}} \overline{(\Lambda_1,\Omega_1,E)^c}] \tilde{\tilde{\sqcup}} [\overline{(\Lambda_1,\Omega_1,E)} \tilde{\tilde{\sqcap}} \overline{(\Lambda_2,\Omega_2,E)} \tilde{\tilde{\sqcap}} \overline{(\Lambda_2,\Omega_2,E)^c}] = [(\overline{(\Lambda_1,\Omega_1,E)} \tilde{\tilde{\sqcap}} \overline{(\Lambda_1,\Omega_1,E)^c}) \tilde{\tilde{\sqcap}} \overline{(\Lambda_2,\Omega_2,E)}] \tilde{\tilde{\sqcup}} [\overline{(\Lambda_1,\Omega_1,E)} \tilde{\tilde{\sqcap}} (\overline{(\Lambda_2,\Omega_2,E)} \tilde{\tilde{\sqcap}} \overline{(\Lambda_2,\Omega_2,E)^c})] = [(\Lambda_1,\Omega_1,E)^b \tilde{\tilde{\sqcap}} \overline{(\Lambda_2,\Omega_2,E)}] \tilde{\tilde{\sqcup}} [\overline{(\Lambda_1,\Omega_1,E)} \tilde{\tilde{\sqcap}} (\Lambda_2,\Omega_2,E)^b] \tilde{\tilde{\sqsubseteq}} (\Lambda_1,\Omega_1,E)^b \tilde{\tilde{\sqcup}} (\Lambda_2,\Omega_2,E)^b$. □

Proposition 19. *Suppose that $(X,T_{\mathscr{BH}},E,\neg E)$ is a bipolar hypersoft space and (Λ,Ω,E) is a bipolar hypersoft set over X. Then:*

(1) $(\Lambda,\Omega,E)^o \tilde{\tilde{\sqcup}} (\Lambda,\Omega,E)^b \tilde{\tilde{\sqsubseteq}} \overline{(\Lambda,\Omega,E)}$.

(2) $(\Lambda,\Omega,E)^o \tilde{\tilde{\sqcup}} (\Lambda,\Omega,E)^e \tilde{\tilde{\sqcup}} (\Lambda,\Omega,E)^b \tilde{\tilde{\sqsubseteq}} (\Psi,\Phi,E)$.

Proof

(1) $(\Lambda,\Omega,E)^o \tilde{\tilde{\sqcup}} (\Lambda,\Omega,E)^b = (\Lambda,\Omega,E)^o \tilde{\tilde{\sqcup}} [\overline{(\Lambda,\Omega,E)} \tilde{\tilde{\sqcap}} \overline{(\Lambda,\Omega,E)^c}] = [(\Lambda,\Omega,E)^o \tilde{\tilde{\sqcup}} \overline{(\Lambda,\Omega,E)}] \tilde{\tilde{\sqcap}} [(\Lambda,\Omega,E)^o \tilde{\tilde{\sqcup}} \overline{(\Lambda,\Omega,E)^c}] = \overline{(\Lambda,\Omega,E)} \tilde{\tilde{\sqcap}} [(\Lambda,\Omega,E)^o \tilde{\tilde{\sqcup}} ((\Lambda,\Omega,E)^o)^c] \tilde{\tilde{\sqsubseteq}} \overline{(\Lambda,\Omega,E)} \tilde{\tilde{\sqcap}} (\Psi,\Phi,E) = \overline{(\Lambda,\Omega,E)}$.

(2) By Proposition 17 (3.), $(\Lambda,\Omega,E)^o \tilde{\tilde{\sqcup}} (\Lambda,\Omega,E)^o = ((\Lambda,\Omega,E)^b)^c$, then $(\Lambda,\Omega,E)^o \tilde{\tilde{\sqcup}} (\Lambda,\Omega,E)^e \tilde{\tilde{\sqcup}} (\Lambda,\Omega,E)^b = ((\Lambda,\Omega,E)^b)^c \tilde{\tilde{\sqcup}} ((\Lambda,\Omega,E)^b) \tilde{\tilde{\sqsubseteq}} (\Psi,\Phi,E)$. □

Remark 10. The quality of Proposition 19 (1.) and (2.) does not hold in general, whereas it is equal in hypersoft topology.

Example 10. Consider $T_{\mathscr{BH}}$ and (Λ,Ω,E) given in Example 9. We found that

$$(\Lambda,\Omega,E)^o = \{((\omega_1,\omega_3,\omega_4),\phi,\{\chi_2,\chi_3\}),((\omega_2,\omega_3,\omega_4),\{\chi_1\},\{\chi_3\})\},$$
$$(\Lambda,\Omega,E)^b = \{((\omega_1,\omega_3,\omega_4),\phi,\{\chi_3\}),((\omega_2,\omega_3,\omega_4),\phi,\{\chi_1,\chi_3\})\}. \tag{26}$$

Now,

$$\overline{(\Lambda,\Omega,E)} = (\Lambda_1,\Omega_1,E)^c = \{((\omega_1,\omega_3,\omega_4),\{\chi_1\},\{\chi_3\}),((\omega_2,\omega_3,\omega_4),\{\chi_1,\chi_2\},\{\chi_3\})\},$$
$$(\Lambda,\Omega,E)^e = \{((\omega_1,\omega_3,\omega_4),\{\chi_3\},\{\chi_1\}),((\omega_2,\omega_3,\omega_4),\{\chi_3\},\{\chi_1,\chi_2\})\}, \tag{27}$$

Then,

$$(\Lambda,\Omega,E)^o \tilde{\tilde{\sqcup}} (\Lambda,\Omega,E)^b = \{((\omega_1,\omega_3,\omega_4),\phi,\{\chi_3\}), \cdot ((\omega_2,\omega_3,\omega_4),\{\chi_1\},\{\chi_3\})\} \neq \overline{(\Lambda,\Omega,E)}, \tag{28}$$

Also,

$$(\Lambda,\Omega,E)^o \tilde{\tilde{\sqcup}} (\Lambda,\Omega,E)^e \tilde{\tilde{\sqcup}} (\Lambda,\Omega,E)^b = \{((\omega_1,\omega_3,\omega_4),\{\chi_3\},\phi) ((\omega_2,\omega_3,\omega_4),\{\chi_1,\chi_3\},\phi)\} \neq (\Psi,\Phi,E). \tag{29}$$

Proposition 20. *Suppose that $(X,T_{\mathscr{BH}},E,\neg E)$ is a bipolar hypersoft space and (Λ,Ω,E) is a bipolar hypersoft set over X. If (Λ,Ω,E) is a bipolar hypersoft open set, then (Λ,Ω,E) and $(\Lambda,\Omega,E)^b$ are disjoint bipolar hypersoft sets, that is, $(\Lambda,\Omega,E) \tilde{\tilde{\sqcap}} (\Lambda,\Omega,E)^b = (\Phi,\Omega,E)$.*

proof. Let (Λ,Ω,E) be a bipolar hypersoft open set. By Proposition 17 (3.), $(\Lambda,\Omega,E)^o \tilde{\tilde{\sqsubseteq}} ((\Lambda,\Omega,E)^b)^c$. But $(\Lambda,\Omega,E)^o = (\Lambda,\Omega,E)$ since (Λ,Ω,E) is a bipolar hypersoft open set. Hence, $(\Lambda,\Omega,E) \tilde{\tilde{\sqsubseteq}} ((\Lambda,\Omega,E)^b)^c$. This implies that (Λ,Ω,E) and $(\Lambda,\Omega,E)^b$ are disjoint bipolar hypersoft sets, that is, $(\Lambda,\Omega,E) \tilde{\tilde{\sqcap}} (\Lambda,\Omega,E)^b = (\Phi,\Omega,E)$. □

Remark 11. The next example illustrates that the opposite of Proposition 20 does not hold in general, whereas it is true in hypersoft topology.

Example 11. Consider $T_{\mathscr{BH}}$, (Λ,Ω,E), and $(\Lambda,\Omega,E)^b$ given in Example 9. It is easy to see that $(\Lambda,\Omega,E) \tilde{\tilde{\sqcap}} (\Lambda,\Omega,E)^b = (\Phi,\Omega,E)$. Hence, (Λ,Ω,E) and $(\Lambda,\Omega,E)^b$ are disjoint bipolar hypersoft sets, but (Λ,Ω,E) is not a bipolar hypersoft open set.

Proposition 21. *Suppose that $(X,T_{\mathscr{BH}},E,\neg E)$ is a bipolar hypersoft space and (Λ,Ω,E) is a bipolar hypersoft set over X. If (Λ,Ω,E) is a bipolar hypersoft closed set, then $(\Lambda,\Omega,E)^b \tilde{\tilde{\sqsubseteq}} (\Lambda,\Omega,E)$.*

proof. Suppose that (Λ,Ω,E) is a bipolar hypersoft closed set. By Proposition 5 (1.), $(\Lambda,\Omega,E)^b \tilde{\tilde{\sqsubseteq}} \overline{(\Lambda,\Omega,E)}$. Since (Λ,Ω,E) is a bipolar hypersoft closed set, then $\overline{(\Lambda,\Omega,E)} = (\Lambda,\Omega,E)$. This implies that $(\Lambda,\Omega,E)^b \tilde{\tilde{\sqsubseteq}} (\Lambda,\Omega,E)$. □

Remark 12. The converse of Proposition 21 is false in general, while it is true in hypersoft topology.

Example 12. Consider $T_{\mathscr{BH}}$, (Λ,Ω,E), and $(\Lambda,\Omega,E)^b$ given in Example 9. It is easy to see that $(\Lambda,\Omega,E)^b \tilde{\tilde{\sqsubseteq}} (\Lambda,\Omega,E)$ but (Λ,Ω,E) is not a bipolar hypersoft closed set.

Proposition 22. *Suppose that (Λ,Ω,E) is a bipolar hypersoft set of a bipolar hypersoft space over X. If (Λ,Ω,E) is a bipolar hypersoft open set and a bipolar hypersoft closed set, then $(\Lambda,\Omega,E)^b = (\Phi,\Omega,E)$*

proof. Suppose that (Λ,Ω,E) is a bipolar hypersoft open set and a bipolar hypersoft closed set. Then $(\Lambda,\Omega,E)^b = \overline{(\Lambda,\Omega,E)} \tilde{\tilde{\sqcap}} \overline{(\Lambda,\Omega,E)^c} = \overline{(\Lambda,\Omega,E)} \tilde{\tilde{\sqcap}} ((\Lambda,\Omega,E)^o)^c = (\Lambda,\Omega,E) \tilde{\tilde{\sqcap}} (\Lambda,\Omega,E)^c = (\Phi,\Omega,E)$. □

Remark 13. The next example shows that the opposite of Proposition 22 is incorrect in general, while it is hold in hypersoft topology.

Example 13. Consider $T_{\mathscr{BH}}$, (Λ,Ω,E), and $(\Lambda,\Omega,E)^b$ given in Example 9. We saw that $(\Lambda,\Omega,E)^b = (\Phi,\Omega,E)$, but (Λ,Ω,E) is neither a bipolar hypersoft open set nor a bipolar hypersoft closed set.

Proposition 23. *Suppose that* $(X, T_{\mathscr{B}\mathscr{H}}, E, \neg E)$ *is a bipolar hypersoft space and* (Λ, Ω, E) *is a bipolar hypersoft set over* X. *Then:*

(1) $(\Lambda, \Omega, E)^o \tilde{\tilde{\sqcap}} (\Lambda, \Omega, E)^b = (\Phi, \Omega, E)$.

(2) $(\Lambda, \Omega, E)^e \tilde{\tilde{\sqcap}} (\Lambda, \Omega, E)^b = (\Phi, \Omega, E)$.

proof

(1) $(\Lambda, \Omega, E)^o \tilde{\tilde{\sqcap}} (\Lambda, \Omega, E)^b = (\Lambda, \Omega, E)^o \tilde{\tilde{\sqcap}} [\overline{(\Lambda, \Omega, E)} \tilde{\tilde{\sqcap}} \overline{(\Lambda, \Omega, E)^c}] = [(\Lambda, \Omega, E)^o \tilde{\tilde{\sqcap}} \overline{(\Lambda, \Omega, E)}] \tilde{\tilde{\sqcap}} \overline{(\Lambda, \Omega, E)^c} = (\Lambda, \Omega, E)^o \tilde{\tilde{\sqcap}} \overline{(\Lambda, \Omega, E)^c} = (\Lambda, \Omega, E)^o \tilde{\tilde{\sqcap}} ((\Lambda, \Omega, E)^o)^c = (\Phi, \Omega, E)$.

(2) $(\Lambda, \Omega, E)^e \tilde{\tilde{\sqcap}} (\Lambda, \Omega, E)^b = ((\Lambda, \Omega, E)^c)^o \tilde{\tilde{\sqcap}} [\overline{(\Lambda, \Omega, E)} \tilde{\tilde{\sqcap}} \overline{(\Lambda, \Omega, E)^c}] = (\overline{(\Lambda, \Omega, E)})^c \tilde{\tilde{\sqcap}} [\overline{(\Lambda, \Omega, E)} \tilde{\tilde{\sqcap}} (\overline{(\Lambda, \Omega, E)})^c] = [(\overline{(\Lambda, \Omega, E)})^c \tilde{\tilde{\sqcap}} \overline{(\Lambda, \Omega, E)}] \tilde{\tilde{\sqcap}} \overline{(\Lambda, \Omega, E)^c}] = (\Phi, \Omega, E) \tilde{\tilde{\sqcap}} (\overline{(\Lambda, \Omega, E)})^c = (\Phi, \Omega, E)$. □

5. Conclusions

We have introduced bipolar hypersoft topological spaces over the collection of bipolar hypersoft sets. The notions of bipolar hypersoft neighborhood, bipolar hypersoft subspace, and bipolar hypersoft limit points are introduced, and their elementary characteristics are investigated. In addition, we have defined bipolar hypersoft interior, bipolar hypersoft closure, bipolar hypersoft exterior, and bipolar hypersoft boundary, and the relations between them are studied. In the end, it is necessary to establish more topological structures such as compactness, connectedness, and separation axioms for the practical applications.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that there are no conflicts of interest.

Authors' Contributions

The published version of the manuscript has been read and approved by all authors.

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