

# A problem related to twin primes

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**Abstract** Let  $p_1$  and  $p_2$  are two primes with  $p_1 < p_2$  and  $p_2 - p_1 = 2$ , we call such pairs of primes are twin primes. About the elementary properties of twin primes, some authors had studied it, and obtained some interesting results. In reference [1], F. Smarandache asked us to prove that  $p$  and  $p + 2$  are primes if and only if

$$(p-1)! \left\{ \frac{1}{p} + \frac{2}{p+2} \right\} + \frac{1}{p} + \frac{1}{p+2}$$

is an integer. This result is called Smarandache Criterion for twin primes and has been proved in [6], [7] and [8]. In this paper, we use the elementary methods to study this problem, and prove that it is true.

**Keywords** Twin primes, pseudo-twin primes, congruence.

## §1. Introduction and results

Let  $p_1$  and  $p_2$  are primes, if  $p_1 < p_2$  and  $p_2 - p_1 = 2$ , we call such pairs of primes are twin primes. For example, 3 and 5, 5 and 7, 11 and 13, 17 and 19, 29 and 31,  $\dots$ , are all twin primes. About the elementary properties of twin primes, some authors had studied it, and obtained some interesting results, see reference [2], [3], [4] and [5].

Let  $p$  be a positive integer, in reference [9], Kenichiro Kashihara called that  $p$  and  $p + 2$  are pseudo-twin primes if and only if

$$\frac{(p-1)! + 1}{p} + \frac{(p+1)! + 1}{p+2} \text{ is an integer.}$$

Simultaneously, Florentin Smarandache also proposed the following two problems:

**Problem 1.** Let  $p$  be positive integer, prove  $p$  and  $p + 2$  are twin primes if and only if

$$(p-1)! \left\{ \frac{1}{p} + \frac{2}{p+2} \right\} + \frac{1}{p} + \frac{1}{p+2} \text{ is an integer.}$$

**Problem 2.** Are there pseudo-twin primes that are not classic twin primes?

About these two problems, it seems that none had studied them, at least we have not seen related paper before. The main purpose of this paper is using the elementary methods to study these two problems, and solved them completely. That is, we shall prove the following:

**Theorem 1.** let  $p$  be a positive integer, then  $p$  and  $p + 2$  are twin primes if and only if

$$(p-1)! \left\{ \frac{1}{p} + \frac{2}{p+2} \right\} + \frac{1}{p} + \frac{1}{p+2} \text{ is an integer.}$$

**Theorem 2.** Pseudo-twin primes must be classic twin primes except  $p = 1, p + 2 = 3$ .

## §2. Proof of the theorems

In this section, we shall complete the proof of the theorems directly. First we prove that if  $p$  and  $p + 2$  are twin primes, then

$$(p-1)! \left\{ \frac{1}{p} + \frac{2}{p+2} \right\} + \frac{1}{p} + \frac{1}{p+2}$$

is an integer. In fact from the Wilson's Theorem we know that for any prime  $p$ ,

$$(p-1)! \equiv -1 \pmod{p}.$$

So

$$p \mid (p-1)! + 1.$$

Therefore

$$\frac{(p-1)! + 1}{p} \quad \text{is an integer.} \quad (1)$$

Since  $p + 2$  be a prime, we also have  $(p+1)! + 1 \equiv 0 \pmod{p+2}$ , so

$$(p-1)! \cdot p \cdot (p+1) + 1 \equiv 0 \pmod{p+2}$$

or

$$2(p-1)! + 1 \equiv 0 \pmod{p+2}.$$

That is to say,

$$\frac{2(p-1)! + 1}{p+2} \quad \text{is an integer.} \quad (2)$$

Note that

$$(p-1)! \left\{ \frac{1}{p} + \frac{2}{p+2} \right\} + \frac{1}{p} + \frac{1}{p+2} = \frac{(p-1)! + 1}{p} + \frac{2(p-1)! + 1}{p+2},$$

From (1) and (2), we know that

$$(p-1)! \left\{ \frac{1}{p} + \frac{2}{p+2} \right\} + \frac{1}{p} + \frac{1}{p+2} \quad \text{is an integer.}$$

Now we prove that if

$$(p-1)! \left\{ \frac{1}{p} + \frac{2}{p+2} \right\} + \frac{1}{p} + \frac{1}{p+2} \quad (3)$$

is an integer, then  $p$  and  $p + 2$  must be primes.

In fact if this conclusion is not true, then there must be three cases:

- (a)  $p$  and  $p + 2$  both are not primes;
- (b)  $p$  is a prime,  $p + 2$  is not a prime;

(c)  $p$  is not a prime,  $p + 2$  is a prime.

If (a) is true, then there at least exist two pair integers  $a$  and  $b$ ,  $c$  and  $d$  with  $p = a \cdot b$ ,  $p + 2 = c \cdot d$ . Obviously,  $a < p$ ,  $b < p$ ,  $c < p + 2$ ,  $d < p + 2$ . If  $p = 4$  and  $p + 2 = 6$ , then (3) is not an integer. So we can assume that  $p > 4$ , this time  $a|(p - 1)!$  and  $b|(p - 1)!$ , so  $p = ab|(p - 1)!$  (if  $a = b$ , then  $2a|(p - 1)!$ , so we also have  $p|(p - 1)!$ ). Therefore,

$$\frac{(p - 1)!}{p} \quad \text{and} \quad \frac{2(p + 1)!}{p + 2}$$

both are integers. But

$$\frac{1}{p} + \frac{1}{p + 2} \quad \text{is not integer.}$$

So

$$(p - 1)! \left\{ \frac{1}{p} + \frac{2}{p + 2} \right\} + \frac{1}{p} + \frac{1}{p + 2} \quad \text{is not an integer.}$$

If (b) is true, then

$$\frac{(p - 1)! + 1}{p} \quad \text{is an integer,}$$

and

$$\frac{2(p + 1)!}{p + 2} \quad \text{is an integer,}$$

but

$$\frac{1}{p + 2} \quad \text{is not an integer.}$$

So

$$(p - 1)! \left\{ \frac{1}{p} + \frac{2}{p + 2} \right\} + \frac{1}{p} + \frac{1}{p + 2} = \frac{(p - 1)! + 1}{p} + \frac{2(p + 1)!}{p + 2} + \frac{1}{p + 2} \quad \text{is not integer.}$$

If (c) is true, then

$$\frac{(p - 1)!}{p} \quad \text{is an integer,}$$

and

$$\frac{2(p + 1)! + 1}{p + 2} \quad \text{is an integer,}$$

but

$$\frac{1}{p} \quad \text{is not an integer.}$$

So

$$(p - 1)! \left\{ \frac{1}{p} + \frac{2}{p + 2} \right\} + \frac{1}{p} + \frac{1}{p + 2} = \frac{(p - 1)!}{p} + \frac{2(p + 1)! + 1}{p + 2} + \frac{1}{p} \quad \text{is not integer.}$$

This completes the proof of Theorem 1.

Now we prove Theorem 2. Note that the identity

$$\begin{aligned} & \frac{(p - 1)! + 1}{p} + \frac{(p + 1)! + 1}{p + 2} = \frac{(p - 1)!}{p} + \frac{(p + 1)! + 1}{p + 2} + \frac{1}{p} \\ = & \frac{(p - 1)! + 1}{p} + \frac{(p + 1)!}{p + 2} + \frac{1}{p + 2} = \frac{(p - 1)!}{p} + \frac{(p + 1)!}{p + 2} + \frac{1}{p} + \frac{1}{p + 2}. \end{aligned}$$

$\frac{1}{p+2}$  is not an integer.

If  $p+2$  is a prime and  $p > 1$  is not a prime, then in the formula

$$\frac{(p-1)!}{p} + \frac{(p+1)!+1}{p+2} + \frac{1}{p},$$

$\frac{1}{p}$  is not an integer.

If  $p$  and  $p+2$  both are not primes with  $p > 1$ , then in the formula

$$\frac{(p-1)!}{p} + \frac{(p+1)!}{p+2} + \frac{1}{p} + \frac{1}{p+2},$$

$\frac{1}{p} + \frac{1}{p+2}$  is not an integer.

This completes the proof of Theorem 2.

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