Two asymptotic formulae on the $k + 1$-power free numbers

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Abstract The main purpose of this paper is to study the distributive properties of $k + 1$-power free numbers, and give two interesting asymptotic formulae.

Keywords $k + 1$-power free numbers; Asymptotic formula.

§1. Introduction

A natural number n is called a $k + 1$-power free number if it can not be divided by any p^{k+1}, where p is a prime number. One can obtain all $k + 1$-power free numbers by the following method:

From the set of natural numbers (except 0 and 1)

-take off all multiples of 2^{k+1}(i.e. 2^{k+1}, 2^{k+2}, $\cdots$).

-take off all multiples of 3^{k+1}.

-take off all multiples of 5^{k+1}.

$\cdots$ and so on (take off all multiples of all $k + 1$-power primes).

In reference [1], Professor F. Smarandache asked us to study the properties of the $k + 1$-power free numbers sequence. Yet we still know very little about it.

Now we define two new number-theoretic functions $U(n)$ and $V(n)$ as following,

$$U(1) = 1, \quad U(n) = \prod_{p|n} p,$$

$$V(1) = 1, \quad V(n) = V\left(p_1^{\alpha_1}\right) \cdots U\left(p_r^{\alpha_r}\right) = (p^{\alpha_1} - 1) \cdots (p^{\alpha_r} - 1),$$

where n is any natural number with the form $n = p_1^{\alpha_1} \cdots p_r^{\alpha_r}$. Obviously they are both multiplicative functions. In this paper, we shall use the analytic method to study the distribution properties of this sequence, and obtain two interesting asymptotic formulae. That is, we have the following two theorems:

Theorem 1. *Let $\mathcal{A}$ denote the set of all $k + 1$-power free numbers, then for any real number $x \geq 1$, we have the asymptotic formula*

$$\sum_{\substack{n \in \mathcal{A} \\ n \leq x}} U(n) = \frac{3x^2}{\pi^2} \prod_p \left(1 + \frac{p^{2k-2} - 1}{p^{2k+1} + p^{2k} - p^{2k-1} - p^{2k-2}}\right) + O\left(x^{\frac{3}{2}+\varepsilon}\right),$$

where ε denotes any fixed positive number and $\prod\limits_{p}$ denotes the product of all the prime numbers.

Theorem 2. *For any real number $x \geq 1$, we have the asymptotic formula*

$$\sum_{\substack{n \in \mathcal{A} \\ n \leq x}} V(n) = \frac{x^2}{2} \prod_p \left(1 - \frac{1}{p^{k+1}} - \frac{p^{2k+1} + p^{2k} - p - 1}{p^{2k+3} + p^{2k+1}}\right) + O\left(x^{\frac{3}{2}+\varepsilon}\right).$$

§2. Proof of Theorems

In this section, we shall complete the proof of Theorems. First we prove Theorem 1, let

$$f(s) = 1 + \sum_{\substack{n \in \mathcal{A} \\ n \leq x}} \frac{U(n)}{n^s}.$$

From the Euler product formula [2] and the definition of $U(n)$, we may have

$$
\begin{aligned}
f(s) &= \prod_p \left(1 + \frac{U(p)}{p^s} + \frac{U(p^2)}{p^{2s}} + \cdots + \frac{U(p^k)}{p^{ks}}\right) \\
&= \prod_p \left(1 + \frac{1}{p^{s-1}} + \frac{1}{p^{2s-1}} + \cdots + \frac{1}{p^{ks-1}}\right) \\
&= \prod_p \left(1 + \frac{1}{p^{s-1}} + \frac{p^{(k-1)s} - 1}{p^{2s-1}(p^{(k-1)s} - p^{(k-2)s})}\right) \\
&= \frac{\zeta(s-1)}{\zeta(2(s-1))} \prod_p \left(1 + \frac{p^{(k-1)s} - 1}{(p^{2s-1} + p^s)(p^{(k-1)s} - p^{(k-2)s})}\right),
\end{aligned}
$$

where $\zeta(s)$ is the Riemann-zeta function. Obviously, we have the following two inequalities

$$|U(n)| \leq n, \qquad \left|\sum_{n=1}^{\infty} \frac{U(n)}{n^\sigma}\right| < \frac{1}{\sigma - 2},$$

where $\sigma > 2$ is the real part of s. So by Perron formula [3]

$$
\begin{aligned}
\sum_{n \leq x} \frac{U(n)}{n^{s_0}} &= \frac{1}{2i\pi} \int_{b-iT}^{b+iT} f(s + s_0) \frac{x^s}{s} ds + O\left(\frac{x^b B(b + \sigma_0)}{T}\right) \\
&+ O\left(x^{1-\sigma_0} H(2x) \min(1, \frac{\log x}{T})\right) + O\left(x^{-\sigma_0} H(N) \min(1, \frac{x}{||x||})\right),
\end{aligned}
$$

where N is the nearest integer to x, $||x|| = |x - N|$. Taking $s_0 = 0$, $b = 3$, $T = x^{\frac{3}{2}}$, $H(x) = x$, $B(\sigma) = \frac{1}{\sigma - 2}$, we have

$$\sum_{n \leq x} U(n) = \frac{1}{2i\pi} \int_{3-iT}^{3+iT} \frac{\zeta(s-1)}{\zeta(2(s-1))} R(s) \frac{x^s}{s} ds + O(x^{\frac{3}{2}+\varepsilon}),$$

where

$$R(s) = \prod_p \left(1 + \frac{p^{2k-2} - 1}{p^{2k+1} + p^{2k} - p^{2k-1} - p^{2k-2}}\right).$$

To estimate the main term

$$\frac{1}{2i\pi} \int_{3-iT}^{3+iT} \frac{\zeta(s-1)}{\zeta(2(s-1))} R(s) \frac{x^s}{s} ds,$$

we move the integral line from $s = 3 \pm iT$ to $s = \frac{3}{2} \pm iT$. This time, the function

$$f(s) = \frac{\zeta(s-1)x^s}{\zeta(2(s-1))s} R(s)$$

has a simple pole point at $s = 2$ with residue $\frac{x^2}{2\zeta(2)} R(2)$. So we have

$$\frac{1}{2i\pi} \left(\int_{3-iT}^{3+iT} + \int_{3+iT}^{\frac{3}{2}+iT} + \int_{\frac{3}{2}+iT}^{\frac{3}{2}-iT} + \int_{\frac{3}{2}-iT}^{3-iT} \right) \frac{\zeta(s-1)x^s}{\zeta(2(s-1))s} R(s) ds$$

$$= \frac{x^2}{2\zeta(2)} \prod_p \left(1 + \frac{p^{2k-2} - 1}{p^{2k+1} + p^{2k} - p^{2k-1} - p^{2k-2}} \right).$$

We can easily get the estimates

$$\left| \frac{1}{2\pi i} \left(\int_{3+iT}^{\frac{3}{2}+iT} + \int_{\frac{3}{2}-iT}^{3-iT} \right) \frac{\zeta(s-1)x^s}{\zeta(2(s-1))s} R(s) ds \right|$$

$$\ll \int_{\frac{3}{2}}^{3} \left| \frac{\zeta(\sigma-1+iT)}{\zeta(2(\sigma-1+iT))} R(s) \frac{x^3}{T} \right| d\sigma \ll \frac{x^3}{T} = x^{\frac{3}{2}}$$

and

$$\left| \frac{1}{2\pi i} \int_{\frac{3}{2}+iT}^{\frac{3}{2}-iT} \frac{\zeta(s-1)x^s}{\zeta(2(s-2))s} R(s) ds \right| \ll \int_0^T \left| \frac{\zeta(1/2+it)}{\zeta(1+2it)} \frac{x^{\frac{3}{2}}}{t} \right| dt \ll x^{\frac{3}{2}+\varepsilon}.$$

Note the fact that $\zeta(2) = \frac{\pi^2}{6}$, then from the above we can obtain

$$\sum_{\substack{n \in \mathcal{A} \\ n \leq x}} U(n) = \frac{3x^2}{\pi^2} \prod_p \left(1 + \frac{p^{2k-2} - 1}{p^{2k+1} + p^{2k} - p^{2k-1} - p^{2k-2}} \right) + O\left(x^{\frac{3}{2}+\varepsilon} \right).$$

This completes the proof of Theorem 1.

Now we come to prove Theorem 2. Let

$$g(s) = 1 + \sum_{\substack{n \in \mathcal{A} \\ n \leq x}} \frac{V(n)}{n^s}.$$

From the Euler product formula [2] and the definition of $V(n)$, we also have

$$\begin{aligned}
g(s) &= \prod_p \left(1 + \frac{V(p)}{p^s} + \frac{V(p^2)}{p^{2s}} + \cdots + \frac{V(p^k)}{p^{ks}} \right) \\
&= \prod_p \left(1 + \frac{p-1}{p^s} + \frac{p^2-1}{p^{2s}} + \cdots + \frac{p^k-1}{p^{ks}} \right) \\
&= \prod_p \left(\frac{1 - \frac{1}{p^{(k+1)(s-1)}}}{1 - \frac{1}{p^{s-1}}} - \frac{1 - \frac{1}{p^{ks}}}{p^s - 1} \right) \\
&= \zeta(s-1) \prod_p \left(1 - \frac{1}{p^{(k+1)(s-1)}} - \frac{(p^{ks} - 1)(p^{s-1} + 1)}{(p^{ks} - p^{(k-1)s})p^{2s-1}} \right).
\end{aligned}$$

Now applying Perron formula [3], and the method of proving Theorem 1, we can also obtain the result of Theorem 2.

This completes the proof of Theorems.

References

[1]F.Smarandache, Only problems, Not solutions, Chicago, Xiquan Publ. House, 1993.

[2] Tom M.Apostol, Introduction to analytic number theory, New York, Springer-Verlag, 1976.

[3] Pan Chengdong and Pan Chengbiao, Foundation of analytic number theory, Beijing, Science Press, 1997.