ON TWO NEW ARITHMETIC FUNCTIONS AND THE K-POWER COMPLEMENT NUMBER SEQUENCES *

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Abstract The main purpose of this paper is to study the asymptotic property of the k-power complement numbers (where $k \geq 2$ is a fixed integer), and obtain some interesting asymptotic formulas.

Keywords: k-power complement number; Asymptotic formula; Arithmetic function.

§1. Introduction

Let $k \geq 2$ is a fixed integer, for each integer n, let $C(n)$ denotes the smallest integer such that $n \times C(n)$ is a perfect k-power, $C(n)$ is called k-power complement number of n. In problem 29 of reference [1], Professor F. Smarandache asked us to study the properties of the k-power complement number sequences. Let $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_s^{\alpha_s}$, we define two arithmetic function $D(n)$ and $I(n)$ similar to the derivative and integral function in mathematical analysis as follows:

$$D(n) = D\left(p_1^{\alpha_1}\right) D\left(p_2^{\alpha_2}\right) \cdots D\left(p_s^{\alpha_s}\right), \qquad D\left(p^{\alpha}\right) = \alpha p^{\alpha - 1}$$

and

$$I(n) = I\left(p_1^{\alpha_1}\right) I\left(p_2^{\alpha_2}\right) \cdots I\left(p_s^{\alpha_s}\right), \qquad I\left(p^{\alpha}\right) = \frac{1}{\alpha + 1} p^{\alpha + 1}.$$

In this paper, we use the analytic method to study the asymptotic properties of the functions $D(n)$ and $I(n)$ for the k-power complement number sequences, and obtain some interesting asymptotic formulas. That is, we shall prove the following conclusions:

*This work is supported by the N.S.F.(60472068) and the P.N.S.F.of P.R.China .

Theorem 1. For any real number $x \geq 1$, we have the asymptotic formula

$$\begin{aligned} &\sum_{n\leq x} \frac{1}{D(C(n))} \\ = \; & \frac{6(k-1)\zeta\left(\frac{k}{k-1}\right)\cdot x^{\frac{1}{k-1}}}{\pi^2} \prod_p \left(1 + \frac{p}{p+1}\sum_{i=1}^{k-2} \frac{1}{(k-i)p^{k-1-i}\cdot p^{\frac{i}{k-1}}}\right) \\ & + O\left(x^{\frac{2k-1}{2k(k-1)}+\varepsilon}\right), \end{aligned}$$

where ε denotes any fixed positive number.

Theorem 2. For any real number $x \geq 1$, we have the asymptotic formula

$$\begin{aligned} &\sum_{n\leq x} I(C(n))d(C(n)) \\ = \; & \frac{6\zeta(k(k+1))\cdot x^{k+1}}{(k+1)\pi^2} \prod_p \left(1 + \frac{p}{p+1}\left(\sum_{i=2}^{k} \frac{p^{k+1-i}}{p^{(k+1)i}} - \frac{1}{p^{k(k+1)}}\right)\right) \\ & + O\left(x^{k+\frac{1}{2}+\varepsilon}\right), \end{aligned}$$

where $d(n) = \sum_{d|n} 1$ is the divisor function.

§2. Proof of the theorems

In this section, we shall complete the proof of the theorems. Let

$$f(s) = \sum_{n=1}^{\infty} \frac{1}{D(C(n))n^s}.$$

Because $D(n)$ and $C(n)$ are all multiplicative function, so from the Euler product formula [2] and the definition of $D(n)$ and $C(n)$ we have

$$\begin{aligned} & f(s) \\ = \; & \prod_p \left(1 + \frac{1}{D(C(p))p^s} + \frac{1}{D(C(p^2))p^{2s}} + \cdots\right) \\ = \; & \prod_p \left(1 + \left(\sum_{i=1}^{k-1} \frac{1}{(k-i)p^{k-1-i}\cdot p^{is}} + \frac{1}{p^{ks}}\right)\left(1 + \frac{1}{p^{ks}} + \frac{1}{p^{2ks}} + \cdots\right)\right) \\ = \; & \zeta(ks) \prod_p \left(1 + \frac{1}{p^{(k-1)s}} + \sum_{i=1}^{k-2} \frac{1}{(k-i)p^{k-1-i}\cdot p^{is}}\right) \\ = \; & \frac{\zeta(ks)\zeta((k-1)s)}{\zeta((2k-2)s)} \prod_p \left(1 + \frac{p^{(k-1)s}}{p^{(k-1)s}+1}\sum_{i=1}^{k-2} \frac{1}{(k-i)p^{k-1-i}\cdot p^{is}}\right), \end{aligned}$$

where $\zeta(s)$ is Riemann zeta function. Obviously, we have the inequality

$$|\frac{1}{D(C(n))}| \leq 1, \qquad \left|\sum_{n=1}^{\infty} \frac{1}{D(C(n))n^{\sigma}}\right| < \frac{1}{\sigma - \frac{1}{k-1}},$$

where $\sigma > \frac{1}{k-1}$ is the real part of s. So by Perron formula [3]

$$\begin{aligned} & \sum_{n \leq x} \frac{a(n)}{n^{s_0}} \\ = & \frac{1}{2i\pi} \int_{b-iT}^{b+iT} f(s+s_0) \frac{x^s}{s} ds + O\left(\frac{x^b B(b+\sigma_0)}{T}\right) \\ & +O\left(x^{1-\sigma_0} H(2x) \min(1, \frac{\log x}{T})\right) + O\left(x^{-\sigma_0} H(N) \min(1, \frac{x}{||x||})\right), \end{aligned}$$

where N is the nearest integer to x, $\|x\| = |x - N|$. Taking $s_0 = 0$, $b = 1 + \frac{1}{k-1}$, $T = x^{1+\frac{1}{2k(k-1)}}$, $H(x) = 1$, $B(\sigma) = \frac{1}{\sigma - \frac{1}{k-1}}$, we have

$$\begin{aligned} & \sum_{n \leq x} \frac{1}{D(C(n))} \\ = & \frac{1}{2i\pi} \int_{1+\frac{1}{k-1}-iT}^{1+\frac{1}{k-1}+iT} \frac{\zeta(ks)\zeta((k-1)s)}{\zeta((2k-2)s)} R(s) \frac{x^s}{s} ds \\ & +O\left(x^{\frac{2k-1}{2k(k-1)}+\varepsilon}\right), \end{aligned}$$

where

$$R(s) = \prod_p \left(1 + \frac{p^{(k-1)s}}{p^{(k-1)s}+1} \sum_{i=1}^{k-2} \frac{1}{(k-i)p^{k-1-i} \cdot p^{is}}\right).$$

To calculate the main term

$$\frac{1}{2i\pi} \int_{1+\frac{1}{k-1}-iT}^{1+\frac{1}{k-1}+iT} \frac{\zeta(ks)\zeta((k-1)s)x^s}{\zeta((2k-2)s)s} R(s) ds,$$

we move the integral line from $s = 1 + \frac{1}{k-1} \pm iT$ to $s = \frac{1}{k} + \frac{1}{2k(k-1)} \pm iT$. This time, the function

$$f_1(s) = \frac{\zeta(ks)\zeta((k-1)s)}{\zeta((2k-2)s)} R(s) \frac{x^s}{s}$$

have a simple pole point at $s = \frac{1}{k-1}$ with residue $\frac{(k-1)\zeta(\frac{k}{k-1}) \cdot x^{\frac{1}{k-1}}}{\zeta(2)} R(\frac{1}{k-1})$. So we have

$$\frac{1}{2i\pi}\left(\int_{1+\frac{1}{k-1}-iT}^{1+\frac{1}{k-1}+iT} + \int_{1+\frac{1}{k-1}+iT}^{\frac{1}{k}+\frac{1}{2k(k-1)}+iT} + \int_{\frac{1}{k}+\frac{1}{2k(k-1)}+iT}^{\frac{1}{k}+\frac{1}{2k(k-1)}-iT} + \int_{\frac{1}{k}+\frac{1}{2k(k-1)}-iT}^{1+\frac{1}{k-1}-iT}\right)$$

$$\begin{aligned}&\frac{\zeta(ks)\zeta((k-1)s)x^s}{\zeta((2k-2)s)s}R(s)ds\\ =\ &\frac{(k-1)\zeta\left(\frac{k}{k-1}\right)\cdot x^{\frac{1}{k-1}}}{\zeta(2)}\prod_p\left(1+\frac{p}{p+1}\sum_{i=1}^{k-2}\frac{1}{(k-i)p^{k-1-i}\cdot p^{\frac{i}{k-1}}}\right).\end{aligned}$$

We can easy get the estimate

$$\begin{aligned}&\left|\frac{1}{2\pi i}\left(\int_{1+\frac{1}{k-1}+iT}^{\frac{1}{k}+\frac{1}{2k(k-1)}+iT}+\int_{\frac{1}{k}+\frac{1}{2k(k-1)}-iT}^{1+\frac{1}{k-1}-iT}\right)\frac{\zeta(ks)\zeta((k-1)s)x^s}{\zeta((2k-2)s)s}R(s)ds\right|\\ \ll\ &\int_{\frac{1}{k}+\frac{1}{2k(k-1)}}^{1+\frac{1}{k-1}}\left|\frac{\zeta(k(\sigma+iT))\zeta((k-1)(\sigma+iT))}{\zeta((2k-2)(\sigma+iT))}R(s)\frac{x^{1+\frac{1}{k-1}}}{T}\right|d\sigma\\ \ll\ &\frac{x^{1+\frac{1}{k-1}}}{T}=x^{\frac{1}{k}+\frac{1}{2k(k-1)}}\end{aligned}$$

and

$$\begin{aligned}&\left|\frac{1}{2\pi i}\int_{\frac{1}{k}+\frac{1}{2k(k-1)}+iT}^{\frac{1}{k}+\frac{1}{2k(k-1)}-iT}\frac{\zeta(ks)\zeta((k-1)s)x^s}{\zeta((2k-2)s)s}R(s)ds\right|\\ \ll\ &\int_0^T\left|\frac{\zeta(1+\frac{1}{2(k-1)}+ikt)\zeta(\frac{2k-1}{2k}+i(k-1)t)}{\zeta(\frac{2k-1}{k}+i(2k-2)t)}\frac{x^{\frac{1}{k}+\frac{1}{2k(k-1)}}}{t}\right|dt\\ \ll\ &x^{\frac{1}{k}+\frac{1}{2k(k-1)}+\varepsilon}.\end{aligned}$$

Note that $\zeta(2)=\frac{\pi^2}{6}$, we have

$$\begin{aligned}\sum_{n\le x}\frac{1}{D(C(n))}\ =\ &\frac{6(k-1)\zeta(\frac{k}{k-1})\cdot x^{\frac{1}{k-1}}}{\pi^2}\prod_p\left(1+\frac{p}{p+1}\sum_{i=1}^{k-2}\frac{1}{(k-i)p^{k-1-i}\cdot p^{\frac{i}{k-1}}}\right)\\ &+O\left(x^{\frac{2k-1}{2k(k-1)}+\varepsilon}\right).\end{aligned}$$

This completes the proof of Theorem 1.

Let

$$g(s)=\sum_{n=1}^{\infty}I(C(n))d(C(n))$$

from the definition of $I(n)$ and $C(n)$, we can also have

$$\begin{aligned}g(s)\ &=\ \prod_p\left(1+\frac{I(C(p))d(C(p))}{p^s}+\frac{I(C(p^2))d(C(p^2))}{p^{2s}}+\cdots\right)\\ &=\ \prod_p\left(1+\left(\frac{p^k}{p^s}+\frac{p^{k-1}}{p^{2s}}+\cdots+\frac{p}{p^{ks}}\right)\left(1+\frac{1}{p^{ks}}+\frac{1}{p^{2ks}}+\cdots\right)\right)\end{aligned}$$

$$
\begin{aligned}
&= \zeta(ks)\prod_p\left(1-\frac{1}{p^{ks}}+\frac{p^k}{p^s}+\frac{p^{k-1}}{p^{2s}}+\cdots+\frac{p}{p^{ks}}\right)\\
&= \frac{\zeta(ks)\zeta(s-k)}{\zeta(2s-2k)}\prod_p\left(1+\frac{p^{s-k}}{p^{s-k}+1}\left(\sum_{i=2}^{k}\frac{p^{k+1-i}}{p^{is}}-\frac{1}{p^{ks}}\right)\right).
\end{aligned}
$$

Now by Perron formula [3] and the method of proving Theorem 1, we can also obtain Theorem 2.

Reference

[1] F. Smarndache, Only Problems, Not Solution, Xiquan Publishing House, Chicago, 1993, pp. 26.

[2] Tom M. Apostol, Introduction to Analytic Number Theory, Springer-Verlag, New York, 1976.

[3] Pan Chengdong and Pan Chengbiao, Foundation of Analytic Number Theory, Science Press, Beijing, 1997, pp. 98.