Solution of a problem by J. Rodriguez

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Problem: "Find the largest strictly increasing series of integer numbers for which the Smarandache Function is strictly decreasing".

My intention is to prove that there exists series of arbitrary finite length with the properties described above.

To begin with, define $p_1 = 2$, $p_2 = 3$, $p_3 = 5$ and more generally, $p_n =$ the nth prime. Now we have the following lemma:

Lemma: $p_k < p_{k+1} < 2p_k$ for all $k \in \mathbb{N}$. (Δ)

Proof: A theorem conjectured by Bertrand Russell and proven by Tchebychef states that for all natural numbers $n \geq 2$, there exists a prime p such that $n < p < 2n$. Using this theorem for $n = p_k$, we get $p_k < p < 2p_k$ $(\star)$ for at least one prime p. The smallest prime $> p_k$ is p_{k+1}, so $p \geq p_{k+1}$. But then it is obvious that $(\star)$ is satisfied by $p = p_{k+1}$. Hence $p_k < p_{k+1} < 2p_k$. $\square$

This lemma plays an important role in the proof of the following theorem:

Theorem: Let n be a natural number ≥ 2 and define the series $\{x_k\}_{k=0}^{n-1}$ of length n by $x_k = 2^k p_{2n-k}$ for $k \in \{0, \ldots, n-1\}$. Then $x_k < x_{k+1}$ and $S(x_k) > S(x_{k+1})$ for all $k \in \{0, \ldots, n-2\}$. (Ω)

Proof: For $k \in \{0, \ldots, n-2\}$ we have the following equivalences: $x_k < x_{k+1} \Leftrightarrow 2^k p_{2n-k} < 2^{k+1} p_{2n-k-1} \Leftrightarrow p_{2n-k} < 2p_{2n-k-1}$ according to Lemma (Δ).

Futhermore $p_{2n-k} \geq p_{2n-(n-1)} = p_{n+1} \geq p_3 = 5 > 2$, so $(p_{2n-k}, 2) = 1$ for all $k \in \{0, \ldots, n-1\}$. Hence $S(x_k) = S(2^k p_{2n-k}) = \max\{S(2^k), S(p_{2n-k})\} = \max\{S(2^k), p_{2n-k}\}$. Consequently $p_{2n-k} \leq S(x_k) \leq \max\{2k, p_{2n-k}\}$ $(*)$ since $S(2^k) \leq 2k$.

Moreover we know that $p_{k+1} - p_k \geq 2$ for all $k \geq 2$ because both p_k and p_{k+1} are odd integers. This inequality gives us the following result:

$$\sum_{k=2}^{n-1}(p_{k+1} - p_k) = p_n - p_2 = p_n - 3 \geq \sum_{k=2}^{n-1} 2 = 2(n-2),$$

so $p_n \geq 2n - 1$ for all $n \geq 3$. In other words, $p_{n+1} \geq 2n + 1 > 2(n-1)$ for $n \geq 2$, i.e. $p_{2n-k} > 2k$ for $k = n - 1$. The fact that p_{2n-k} increases and $2k$ decreases as k decreases from $n-1$ to 0 implies that $p_{2n-k} > 2k$ for all $k \in \{0, \ldots, n-1\}$. From this last inequality and $(*)$ it follows that $S(x_k) = p_{2n-k}$. This formula brings us to the conclusion: $S(x_k) = p_{2n-k} > p_{2n-k-1} = S(x_{k+1})$ for all $k \in \{0, \ldots, n-2\}$. $\square$

Example: For $n = 10$ Theorem (Ω) generates the following series:

k	0	1	2	3	4	5	6	7	8	9
x_k	71	134	244	472	848	1504	2752	5248	9472	15872
$S(x_k)$	71	67	61	59	53	47	43	41	37	31