SOLVING PROBLEMS BY USING A FUNCTION IN THE NUMBER THEORY

Let $n \geq 1$, $h \geq 1$, and $a \geq 2$ be integers. For which values of a and n is $(n + h)!$ a multiple of a^n ?
(A generalization of the problem $n^0 = 1270$, Mathematics Magazine, Vol. 60, No. 3, June 1987, p. 179, proposed by Roger B. Eggleton, The University of Newcastle, Australia.)

Solution

(For h = 1 the problem $n^0 = 1270$ is obtained.)

§1. Introduction

We have constructed a function η (see [1]) having the following properties:

(a) For each non-null integer n, $\eta(n)!$ is a multiple of n;

(b) $\eta(n)$ is the smallest natural number with the property (a).

It is easy to prove:

Lemma 1. $(\forall)$ $k, p \in N^*$, $p \neq 1$, k is uniquely written in the form:

$$k = t_1 a_{n_1}^{(p)} + \dots + t_\ell a_{n_\ell}^{(p)},$$

where $a_{n_i}^{(p)} = (p^{n_i} - 1) \,/\, (p - 1)$, $i = 1, 2, \dots, \ell$,

$n_1 > n_2 > \dots > n_\ell > 0$ and $1 \le t_j \le p - 1$, $j = 1, 2, \dots, \ell - 1$, $1 \le t_\ell \le p$, $n_i, t_i \in N$, $i = 1, 2, \dots, \ell$, $\ell \in N^*$.

We have constructed the function η_p, p prime > 0, $\eta_p : N^* \to N^*$, thus:

$$(\forall)\ n \in N^*,\ \eta_p\left(a_n^{(p)}\right) = p^n, \text{ and}$$

$$\eta_p\left(t_1 a_{n_1}^{(p)} + \dots + t_\ell a_{n_\ell}^{(p)}\right) =$$

$$= t_1 \eta_p\left(a_{n_1}^{(p)}\right) + \dots + t_\ell \eta_p\left(a_{n_\ell}^{(p)}\right).$$

Of course:

<u>Lemma 2</u>.

(a) $(\forall)\ k \in N^*$, $\eta_p(k)! = \dot{M}p^k$.

(b) $\eta_p(k)$ is the smallest number with the property (a). Now, we construct another function:

$\eta : Z\setminus\{0\} \to N$ defined as follows:

$$\begin{cases} \eta(\pm 1) = 0, \\ (\forall)\ n = \epsilon\, p_1^{\alpha_1} \dots p_s^{\alpha_s} \text{ with } \epsilon = \pm 1,\ p_i \text{ prime and} \\ p_i \neq p_j \text{ for } i \neq j,\ \text{all } \alpha_i \in N^*,\ \eta(n) = \\ = \max\limits_{1 \le i \le s} \{\eta_{p_i}(\alpha_i)\}. \end{cases}$$

It is not difficult to prove η has the demanded properties of §1.

§2. Now, let $a = p_1^{\alpha_1} \dots p_s^{\alpha_s}$, with all $\alpha_i \in N^*$ and all p_i distinct primes. By the previous theory we have:

$$\eta(a) = \max_{1 \le i \le s} \{n_{p_i}(\alpha_i)\} = \eta_p(\alpha) \text{ (by notation)}.$$

Hence $\quad \eta(a) = \eta(p^\alpha),\ \eta(p^\alpha)! = Mp^\alpha.$

We know:

$$(t_1 p^{n_1} + \dots + t_\ell p^{n_\ell})! = Mp^{t_1 \frac{p^{n_1}-1}{p-1} + \dots + t_\ell \frac{p^{n_\ell}-1}{p-1}}.$$

We put:

$$t_1 p^{n_1} + \dots + t_\ell p^{n_\ell} = n + h$$

and $t_1 \frac{p^{n_1}-1}{p-1} + \ldots + t_\ell \frac{p^{n_\ell}-1}{p-1} = \alpha\, n.$

Whence

$$\frac{1}{\alpha}\left[\frac{p^{n_1}-1}{p-1} + \ldots + t_\ell \frac{p^{n_\ell}-1}{p-1} \right] \geq t_1 p^{n_1} + \ldots + t_\ell p^{n_\ell} - h$$

or

$$(1) \quad \alpha\,(p - 1)\,h \geq (\alpha\,p - \alpha - 1)\,[t_1 p^{n_1} + \ldots + t_\ell p^{n_\ell}] +$$

$$+ (t_1 + \ldots + t_\ell).$$

On this condition we take $n_0 = t_1 p^{n_1} + \ldots + t_\ell p^{n_\ell} - h$

(see Lemma 1), hence $n = \begin{cases} n_0, & n_0 > 0; \\ 1, & n_0 \leq 0 \end{cases}$.

Consider giving $a \neq 2$, we have a finite number of n. There are an infinite number of n if and only if $\alpha p - \alpha - 1 = 0$, i.e., $\alpha = 1$ and $p = 2$, i.e., $a = 2$.

§3. Particular Case

If $h = 1$ and $a \neq 2$, because

$$t_1 p^{n_1} + \ldots + t_\ell p^{n_\ell} \geq p^{n_\ell} > 1$$

and $t_1 + \ldots + t_\ell \geq 1$, it follows from (1) that:

(1') $(\alpha p - \alpha) > (\alpha p - \alpha - 1) \cdot 1 + 1 = \alpha p - \alpha$,

which is impossible. If $h = 1$ and $a = 2$ then $\alpha = 1$, $p = 2$, or

(1") $1 \geq t_1 + \ldots + t_\ell$,

hence $\ell = 1$, $t_1 = 1$ whence $n = t_1 p^{n_1} + \ldots + t_\ell p^{n_\ell} - h =$

$= 2^{n_1} - 1$, $n_1 \in N^*$ (the solution to problem 1270).

Example 1. Let $h = 16$ and $a = 3^4 \cdot 5^2$. Find all n such that

$$(n + 16)! = M\ 2025^n.$$

Solution

$\eta(2025) = \max\{\eta_3(4), \eta_5(2)\} = \max\{9, 10\} = 10 =$
$= \eta_5(2) = \eta(5^2)$. Whence $\alpha = 2$, $p = 5$. From (1) we have:

$$128 \geq 7[t_1 5^{n_1} + \ldots + t_\ell 5^{n_\ell}] + t_1 + \ldots + t_\ell.$$

Because $5^4 > 128$ and $7[t_1 5^{n_1} + \ldots + t_\ell 5^{n_\ell}] < 128$ we find $\ell = 1$,

$$128 \geq 7\ t_1\ 5^{n_1} + t_1\ ,$$

whence $n_1 \leq 1$, i.e., $n_1 = 1$, and $t_1 = 1, 2, 3$. Then $n_0 =$ $= t_1\ 5 - 16 < 0$, hence we take $n = 1$.

Example 2

$(n + 7)! = \mathcal{M}\ 3^n$ when $n = 1, 2, 3, 4, 5$.

$(n + 7)! = \mathcal{M}\ 5^n$ when $n = 1$.

$(n + 7)! = \mathcal{M}\ 7^n$ when $n = 1$.

But $(n + 7)! \neq \mathcal{M}\ p^n$, for p prime > 7, $(\forall)\ n \in N^*$.

$(n + 7)! = \mathcal{M}\ 2^n$ when

$$n_0 = t_1\ 2^{n_1} + \ldots + t_\ell\ 2^{n_\ell} - 7,$$

$$t_1, \ldots, t_{\ell-1} = 1,$$

$$1 \leq t_\ell \leq 2,\ t_1 + \ldots + t_\ell \leq 7$$

and
$$n = \begin{cases} n_0, & n_0 > 0; \\ 1, & n_0 \leq 0. \end{cases}$$

etc.

Exercise for Readers

If $n \in N^*$, $a \in N^*\setminus\{1\}$, find all values of a and n such that:

$(n + 7)!$ be a multiple of a^n.

Some Unsolved Problems (see [2])

Solve the diophantine equations:

(1) $\eta(x) \cdot \eta(y) = \eta(x+y)$.

(2) $\eta(x) = y!$ (A solution: $x = 9$, $y = 3$).

(3) Conjecture: the equation $\eta(x) = \eta(x+1)$ has no solution.

References

[1] Florentin Smarandache, "A Function in the Number Theory," Analele Univ. Timisoara, Fasc. 1, Vol. XVIII, pp. 79-88, 1980, MR: 83c: 10008.

[2] Idem, Un Infinity of Unsolved Problems Concerning a Function in Number Theory, International Congress of Mathematicians, Univ. of Berkeley, CA, August 3-11, 1986.

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[A comment about this generalization was published in "Mathematics Magazine", Vol. 61, No. 3, June 1988, p. 202: "Smarandache considered the general problem of finding positive integers n, a, and k, so that (n + k)! should be a multiple of a^n. Also, for positive integers p and k, with p prime, he found a formula for determining the smallest integer f(k) with the property that (f(k))! is a multiple of p^k."]