SOME PROPERTIES OF SMARANDACHE FUNCTIONS OF THE TYPE I

by

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We consider the construction of Smarandache functions of the type I S_p ($p \in \mathbb{N}^*$, p prim) which are defined in [1] and [2] as follows:

$$S_n : \mathbb{N}^* \longrightarrow \mathbb{N}^* \quad ; \quad S_1(k) = 1 \quad ; \quad S_n(k) = \max_{1 \le j \le r} \{ S_{p_j}(i_j . k) \}$$

$$\text{for} \qquad n = p_1^{i_1} \, p_2^{i_2} \ldots p_r^{i_r}$$

In this paper there are presented some properties of these functions. We shall study the monotonicity of each function S_n and also the monotonicity of some subsequences of the sequence $(S_n)_{n \in \mathbb{N}^*}$.

1. Proposition. The function S_n is monotonous increasing for every positiv integer n.

Proof. The function S_1 is abviously monotonous increasing. Let $k_1 < k_2$ where $k_1, k_2 \in \mathbb{N}^*$. Supposing that n is a prime number and taking accont that $(S_n(k_2))! =$ multiple $n^{k_1} =$ multiple n^{k_2}, it results that $S_n(k_1) \le S_n(k_2)$, therefore S_n is monotonous increasing. Let

$$S_n(k_1) = \max_{1\le j\le k} \{ S_{p_j}(i_j \cdot k_1)\} = S_{p_m}(i_m \cdot k_1)$$

$$S_n(k_2) = \max_{1\le j\le r} \{ S_{p_j}(i_j \cdot k_2)\} = S_{p_t}(i_t \cdot k_2)$$

Because $S_{p_m}(i_m \cdot k_1) \le S_{p_m}(i_m \cdot k_2) \le S_{p_t}(i_t \cdot k_2)$

it results that $S_n(k_1) \le S_n(k_2)$ so S_n is monotonous increasing.

2. Proposition. The sequence of functions $(S_{p^i})_{i\in\mathbb{N}^*}$ is monotonous increasing, for every prime number p.

Proof. For any two nombers $i_1, i_2 \in \mathbb{N}^*$, $i_1 < i_2$ and for any $n \in \mathbb{N}^*$ we have :

$S_{p^{i_1}}(n) = S_p(i_1 \cdot n) \le S_p(i_2 \cdot n) = S_{p^{i_2}}(n)$ therefore $S_{p^{i_1}} \le S_{p^{i_2}}$.

Hence the sequence $\{ S_{p^i} \}_{i\in\mathbb{N}^*}$ is monotonous increasing for every prime number p.

3. Proposition. Let p and q two given prime numbers. If $p<q$ then

$$S_p(k) < S_q(k) \quad , \quad k \in \mathbb{N}^*$$

Proof. Let the sequence of coefficients (see [2]) $a_1^{(p)}, a_2^{(p)}, \ldots, a_s^{(p)}, \ldots$

Every $k \in \mathbb{N}^*$ can be uniquely written as

$$k = t_1 a_s^{(p)} + t_2 a_{s-1}^{(p)} + \ . \ . \ . \ + t_s a_1^{(p)} \qquad (1)$$

where $0 \le t_i \le p-1$, for $i = \overline{1,s-1}$, and $0 \le t_s \le p$.

The procedure of passing from k to $k+1$ in formule (1) is :

(i) t_s is increasing with a unity.

(ii) if t_s can not increase with a unity, then t_{s-1} is increasing with a unity and $t_s = 0$

(iii) if neithe t_s, nor t_{s-1} are not increasing with a unity then t_{s-2} is increasing with a unity and $t_s = t_{s-1} = 0$

The procedure is continued in the same way until we obtain the expresion of $k+1$.

Denoting $\Delta_k(S_p) = S_p(k+1) - S_p(k)$ the leap of the function S_p when we pass from k to $k+1$ corresponding to the procedure described above. We find that

- in the case (i) $\Delta_k(S_p) = p$
- in the case (ii) $\Delta_k(S_p) = 0$
- in the case (iii) $\Delta_k(S_p) = 0$

. . . .

It is abviously seen that: $S_p(n) = \sum_{k=1}^{n} \Delta_k(n_p) + S_p(1)$.

Analogously we write $S_q(n) = \sum_{k=1}^{n} \Delta_k(n_q) + S_q(1)$

Taking into account that $S_p(1) = p < q = S_q(1)$ and using the procedure of passing from k to $k+1$ we deduce that the number of leaps with zero value of S_p is greater then the number of leaps with zero value of S_q, respectively the number of leaps with value p of S_p is less then the number of leaps of S_q with value

q it result that

$$\sum_{k=1}^{n} \Delta_k(S_p) + S_p(1) < \sum_{k=1}^{n} \Delta_k(S_q) + S_q(1) \qquad (2)$$

Hence $S_p(n) < S_q(n)$, $n \in \mathbb{N}^*$.

As an example we give a table with S_2 and S_3 for $0<n<21$

k	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
the leap		2	0	2	2	0	0	2	2	0	2	2	0	0	0	2	0	2	2	2
$S_2(k)$	2	4	4	6	8	8	8	10	12	12	14	16	16	16	16	18	18	20	22	24
the leap		3	3	0	3	3	3	0	3	3	3	0	0	3	3	3	0	3	3	3
$S_3(k)$	3	6	9	9	12	15	18	18	21	24	27	27	27	30	33	36	36	39	42	45

Hence $S_2(k) < S_3(k)$ for $k = 1,2,\ldots,20$.

4. Remark. For any monotonous increasing sequence of prime numbers $p_1 < p_2 < \ldots < p_n < \ldots$ it results that

$$S_1 < S_{p_1} < S_{p_2} < \ldots < S_{p_n} < \ldots$$

If $n = p_1^{i} \; p_2^{i} \ldots p_t^{i}$ and $p_1 < p_2 < \ldots < p_t$ then

$$S_n(k) = \max_{1 \le j \le t} \{ S_{p_j^i}(k) \} = S_{p_t^i}(k) = S_{p_t}(ik)$$

5. Proposition. If p and q are prime numbers and $p.i < q$ then $S_{p^i} < S_q$.

Proof. Because $p.i < q$ it results

$$S_{p^i}(1) \le p.i < q = S_q(1) \qquad (3)$$

and $S_{p^i}(k) = S_p(ik) \le i \, S_p(k)$.

From (3) passing from k to $k+1$, we deduce

$$\Delta_k(S_{p^i}) \le i \, \Delta_k(S_p) \qquad (4)$$

Taking into account the proposition 3. from (4) it results that when we pass from k to $k+1$ we obtain

$$\Delta_k(S_{p^i}) \le i\,\Delta_k(S_p) \le i.p < q \text{ and } i\sum_{k=1}^{n} \Delta_k(S_p) \le \sum_{k=1}^{n} \Delta_k(S_q) \quad (5)$$

Because we have

$$S_{p^i}(n) = S_{p^i}(1) + \sum_{k=1}^{n} \Delta_k(S_{p^i}) \le S_{p^i}(1) + i \sum_{k=1}^{n} \Delta_k(S_p)$$

and

$$S_q(n) = S_q(1) + \sum_{k=1}^{n} \Delta_k(S_q)$$

from (3) and (5) it results $S_{p^i}(n) \le S_q(n)$, $n \in \mathbb{N}^*$

6. Proposition. If p is a prime number then $S_n < S_p$ for every $n < p$.

Proof. If n is a prime number from $n < p$, using the proposition 3 it results $S_n(k) < S_p(k)$ for $k \in \mathbb{N}^*$. If n is a composed, that is $n = p_1^{i_1} \ldots p_t^{i_t}$ then $S_n(k) = \max_{1\le j\le t} \{ S_{p_j^{i_j}}(k)\} = S_{p_r^{i_r}}(k)$.

Because $n < p$ it results $p_r^{i_r} < p$ and using the proposition 5 and knowing that $i_r p_r \le p_r^{i_r} < p$ it results that $S_{p_r^{i_r}}(k) \le S_p(k)$ therefore for $k \in \mathbb{N}^*$ $S_n(k) < S_p(k)$.

References

[1] Balacenoiu I , Smarandache Numerical Functions in Smarandache Function Journal nr. 4 / 1994.

[2] Smarandache F., A function in the number theory. " An. Univ. Timisoara" vol XVIII , fasc 1, pp 79-88.