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A Class of Recursive Sets

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In this article one builds a class of recursive sets, one establishes properties of these sets and one proposes applications. This article widens some results of [1].

1) Definitions, properties.

One calls recursive sets the sets of elements which are built in a recursive manner: let T be a set of elements and f_i for i between 1 and s, of operations n_i, such that $f_i : T^{n_i} \to T$. Let's build by recurrence the set M included in T and such that:

(Def. 1) 1°) certain elements $a_1,...,a_n$ of T, belong to M.

2°) if $(\alpha_{i_1},...,\alpha_{i_{n_i}})$ belong to M, then $f_i(\alpha_{i_1},...,\alpha_{i_{n_i}})$ belong to M for all $i \in \{1,2,...,s\}$.

3°) each element of M is obtained by applying a number finite of times the rules 1° or 2°.

We will prove several proprieties of these sets M, which will result from the manner in which they were defined. The set M is the representative of a class of recursive sets because in the rules 1° and 2°, by particularizing the elements $a_1,...,a_n$ respectively $f_1,...,f_s$ one obtains different sets.

Remark 1 : To obtain an element of M, it is necessary to apply initially the rule 1.

(Def. 2) The elements of M are called elements M-recursive.

(Def. 3) One calls order of an element a of M the smallest natural $p \geq 1$ which has the propriety that a is obtained by applying p times the rule 1° or 2°.

One notes M_p the set which contains all the elements of order p of M. It is obvious that $M_1 = \{a_1,...,a_n\}$.

$$M_2 = \bigcup_{i=1}^{s} \left\{ \bigcup_{(\alpha_{i_1},...,\alpha_{i_{n_i}}) \in M_1^{n_i}} f_i(\alpha_{i_1},...,\alpha_{i_{n_i}}) \right\} \setminus M_1 .$$

One withdraws M_1 because it is possible that $f_j(a_{j_1},...,a_{j_{n_j}}) = a_i$ which belongs to M_1, and thus does not belong to M_2.

One proves that for $k \geq 1$ one has:

$$M_{k+1} = \bigcup_{i=1}^{s} \left\{ \bigcup_{(\alpha_{i_1},...,\alpha_{i_{n_i}}) \in \prod_k^{(i)}} f_i(\alpha_{i_1},...,\alpha_{i_{n_i}}) \right\} \setminus \bigcup_{h=1}^{k} M_h$$

where each

$\prod_k^{(i)} = \{ (\alpha_{i_1},...,\alpha_{i_{n_i}}) / \alpha_{i_j} \in M_{q_j} \quad j \in \{1,2,...,n_i\};\ 1 \leq q_j \leq k$ and at least an element $a_{i_{j_o}} \in M_k, 1 \leq j_o \leq n_i \}$.

The sets M_p, $p \in \mathbb{N}^*$, form a partition of the set M.

Theorem 1:

$$M = \bigcup_{p \in \mathbb{N}^*} M_p \text{, where } \mathbb{N}^* = \{1, 2, 3, ...\}.$$

Proof:

From the rule 1^o it results that $M_1 \subseteq M$.

One supposes that this propriety is true for values which are less than p. It results that $M_p \subseteq M$, because M_p is obtained by applying the rule 2^o to the elements of $\bigcup_{i=1}^{p-1} M_i$.

Thus $\bigcup_{p \in \mathbb{N}^*} M_p \subseteq M$. Reciprocally, one has the inclusion in the contrary sense in accordance with the rule 3^o.

Theorem 2: The set M is the smallest set, which has the properties 1^o and 2^o.

Proof:

Let R be the smallest set having properties 1^o and 2^o. One will prove that this set is unique.

Let's suppose that there exists another set R' having properties 1^o and 2^o, which is the smallest. Because R is the smallest set having these proprieties, and because R' has these properties also, it results that $R \subseteq R'$; of an analogue manner, we have $R' \subseteq R$: therefore $R = R'$.

It is evident that $M' \subseteq R$. One supposes that $M_i \subseteq R$ for $1 \leq i < p$. Then (rule 3^o), and taking in consideration the fact that each element of M_p is obtained by applying rule 2^o to certain elements of M_i, $1 \leq i < p$, it results that $M_p \subseteq R$. Therefore $\bigcup_p M_p \subseteq R$ $(p \in \mathbb{N}^*)$, thus $M \subseteq R$. And because R is unique, $M = R$.

Remark 2. The theorem 2 replaces the rule 3^o of the recursive definition of the set M by: " M is the smallest set that satisfies proprieties 1^o and 2^o".

Theorem 3: M is the intersection of all the sets of T which satisfy conditions 1^o and 2^o.

Proof:

Let T_{12} be the family of all sets of T satisfying the conditions 1^o and 2^o. We note $I = \bigcap_{A \in T_{12}} A$.

I has the properties 1^o and 2^o because:

1) For all $i \in \{1, 2, ..., n\}$, $a_i \in I$, because $a_i \in A$ for all A of T_{12}.

2) If $\alpha_{i_1}, ..., \alpha_{i_{n_i}} \in I$, it results that $\alpha_{i_1}, ..., \alpha_{i_{n_i}}$ belong to A that is A of T_{12}. Therefore,

$\forall i \in \{1,2,...,s\}$, $f_i(\alpha_{i_1},...,\alpha_{i_{n_i}}) \in A$ which is A of T_{12}, therefore $f_i(\alpha_{i_1},...,\alpha_{i_{n_i}}) \in I$ for all i from $\{1,2,...,s\}$.

From theorem 2 it results that $M \subseteq I$.

Because M satisfies the conditions 1° and 2°, it results that $M \in T_{12}$, from which $I \subseteq M$. Therefore $M = I$

(Def. 4) A set $A \subseteq I$ is called closed for the operation f_{i_0} if and only if for all $\alpha_{i_0 1},...,\alpha_{i_0 n_{i_0}}$ of A, one has $f_{i_0}(\alpha_{i_0 1},...,\alpha_{i_0 n_{i_0}})$ belong to A.

(Def. 5) A set $A \subseteq T$ is called M-recursively closed if and only if:

1) $\{a_1,...,a_n\} \subseteq A$.

2) A is closed in respect to operations $f_1,...,f_s$.

With these definitions, the precedent theorems become:

Theorem 2': The set M is the smallest M - recursively closed set.

Theorem 3': M is the intersection of all M - recursively closed sets.

(Def. 6) The system of elements $\langle \alpha_1,...,\alpha_m \rangle$, $m \geq 1$ and $\alpha_i \in T$ for $i \in \{1,2,...,m\}$, constitute a M-recursive description for the element α, if $\alpha_m = \alpha$ and that each α_i ($i \in \{1,2,...,m\}$) satisfies at least one of the proprieties:

1) $\alpha_i \in \{a_1,...,a_n\}$.

2) α_i is obtained starting with the elements which precede it in the system by applying the functions f_j, $1 \leq j \leq s$ defined by property 2° of (Def. 1).

(Def. 7) The number m of this system is called the length of the M-recursive description for the element α.

Remark 3: If the element α admits a M-recursive description, then it admits an infinity of such descriptions.

Indeed, if $\langle \alpha_1,...,\alpha_m \rangle$ is a M-recursive description of α then $\left\langle \underbrace{a_1,...,a_1}_{h\ times}, \alpha_1,...,\alpha_m \right\rangle$ is also a M-recursive description for α, h being able to take all values from $\mathbb{N}$.

Theorem 4: The set M is identical with the set of all elements of T which admit a M-recursive description.

Proof: Let D be the set of all elements, which admit a M-recursive description. We will prove by recurrence that $M_p \subseteq D$ for all p of $\mathbb{N}^*$.

For $p = 1$ we have: $M_1 = \{a_1,...,a_n\}$, and the a_j, $1 \leq j \leq n$, having as M-recursive description: $< a_j >$. Thus $M_1 \subseteq D$. Let's suppose that the property is true for the values smaller than p. M_p is obtained by applying the rule 2° to the elements of

$\bigcup_{i=1}^{p-1} M_i$; $\alpha \in M_p$ implies that $\alpha \in f_j(\alpha_{i_1},...,\alpha_{i_{n_i}})$ and $\alpha_{i_j} \in M_{h_j}$ for $h_j < p$ and $1 \le j \le n_i$.

But a_{i_j}, $1 \le j \le n_i$, admits M-recursive descriptions according to the hypothesis of recurrence, let's have $\langle \beta_{j1},...,\beta_{js_j} \rangle$. Then $\langle \beta_{11},...,\beta_{1s_1},\beta_{21},...,\beta_{2s_2},...,\beta_{n_i1},...,\beta_{n_is_{n_i}},\alpha \rangle$ constitute a M-recursive description for the element α. Therefore if α belongs to D, then $M_p \subseteq D$ which is $M = \bigcup_{p \in \mathbb{N}^*} M_p \subseteq D$.

Reciprocally, let x belong to D. It admits a M-recursive description $\langle b_1,...,b_t \rangle$ with $b_t = x$. It results by recurrence by the length of the M-recursive description of the element x, that $x \in M$. For $t = 1$ we have $\langle b_1 \rangle$, $b_1 = x$ and $b_1 \in \{a_1,...,a_n\} \subseteq M$. One supposes that all elements y of D which admit a M-recursive description of a length inferior to t belong to M. Let $x \in D$ be described by a system of length t: $\langle b_1,...,b_t \rangle$, $b_t = x$. Then $x \in \{a_1,...,a_n\} \subseteq M$, where x is obtained by applying the rule 2° to the elements which precede it in the system: $b_1,...,b_{t-1}$. But these elements admit the M-recursive descriptions of length which is smaller that t: $\langle b_1 \rangle, \langle b_1, b_2 \rangle, ..., \langle b_1,...,b_{t-1} \rangle$. According to the hypothesis of the recurrence, $b_1,...,b_{t-1}$ belong to M. Therefore b_t belongs also to M. It results that $M \equiv D$.

Theorem 5: Let $b_1,...,b_q$ be elements of T, which are obtained from the elements $a_1,...,a_n$ by applying a finite number of times the operations [illegible]. Then M can be defined recursively in the following mode:

1) Certain elements $a_1,...,a_n,b_1,...,b_q$ of T belong to M.
2) M is closed for the applications f_i, with $i \in \{1,2,...,s\}$.
3) Each element of M is obtained by applying a finite number of times the rules (1) or (2) which precede.

Proof: evident. Because $b_1,...,b_q$ belong to T, and are obtained starting with the elements $a_1,...,a_n$ of M by applying a finite number of times the operations f_i, it results that $b_1,...,b_q$ belong to M.

Theorem 6: Let's have g_j, $1 \le j \le r$, of the operations n_j, where $g_j : T^{n_j} \to T$ such that M to be closed in rapport to these operations. Then M can be recursively defined in the following manner:

1) Certain elements $a_1,...,a_n$ de T belong to M.
2) M is closed for the operations f_i, $i \in \{1,2,...,s\}$ and g_j, $j \in \{1,2,...,r\}$.
3) Each element of M is obtained by applying a finite number of times the precedent rules.

Proof is simple: Because M is closed for the operations g_j (with $j \in \{1,2,...,r\}$), one has, that for any $\alpha_{j1},...,\alpha_{jn_j}$ from M, $g_j(\alpha_{j1},...,\alpha_{jn_j}) \in M$ for all $j \in \{1,2,...,r\}$.

From the theorems 5 and 6 it results:

Theorem 7: The set M can be recursively defined in the following manner:

1) Certain elements $a_1,...,a_n,b_1,...,b_q$ of T belong to M.

2) M is closed for the operations f_i ($i \in \{1,2,...,s\}$) and for the operations g_j ($j \in \{1,2,...,r\}$) previously defined.

3) Each element of M is defined by applying a finite number of times the previous 2 rules.

(Def. 8) The operation f_i conserves the property P iff for any elements $\alpha_{i1},...,\alpha_{in_i}$ having the property P, $f_i(\alpha_{i1},...,\alpha_{in_i})$ has the property P.

Theorem 8: If $a_1,...,a_n$ have the property P, and if the functions $f_1,...,f_s$ preserve this property, then all elements of M have the property P.

Poof:

$M = \bigcup_{p \in \mathbb{N}^*} M_p$. The elements of M_1 have the property P.

Let's suppose that the elements of M_i for $i < p$ have the property P. Then the elements of M_p also have this property because M_p is obtained by applying the operations $f_1, f_2,..., f_s$ to the elements of: $\bigcup_{i=1} M_i$, elements which have the property P. Therefore, for any p of $\mathbb{N}$, the elements of M_p have the property P.

Thus all elements of M have it.

Corollary 1: Let's have the property P: "x can be represented in the form $F(x)$".

If $a_1,...,a_n$ can be represented in the form $F(a_1),...,$ respectively $F(a_n)$, and if $f_1,...,f_s$ maintains the property P, then all elements α of M can be represented in the form $F(\alpha)$.

Remark. One can find more other equivalent def. of M.

2) APPLICATIONS, EXAMPLES.

In applications, certain general notions like: M - recursive element, M-recursive description, M - recursive closed set will be replaced by the attributes which characterize the set M. For example in the theory of recursive functions, one finds notions like: recursive primitive functions, primitive recursive description, primitively recursive closed sets. In this case " M " has been replaced by the attribute "primitive" which characterizes this class of functions, but it can be replaced by the attributes "general", "partial".

By particularizing the rules 1^o and 2^o of the def. 1, one obtains several interesting sets:

Example 1: (see [2], pp. 120-122, problem 7.97).

Example 2: The set of terms of a sequence defined by a recurring relation constitutes a recursive set.

Let's consider the sequence: $a_{n+k} = f(a_n, a_{n+1}, ..., a_{n+k-1})$ for all n of $\mathbb{N}^*$, with $a_i = a_i^0$, $1 \le i \le k$. One will recursively construct the set $A = \{a_m\}_{m \in \mathbb{N}^*}$ and one will define in the same time the position of an element in the set A:

1°) $a_1^0, ..., a_k^0$ belong to A, and each a_i^0 ($1 \le i \le k$) occupies the position i in the set A;

2°) if $a_n, a_{n+1}, ..., a_{n+k-1}$ belong to A, and each a_j for $n \le j \le n+k-1$ occupies the position j in the set A, then $f(a_n, a_{n+1}, ..., a_{n+k-1})$ belongs to A and occupies the position $n+k$ in the set A.

3°) each element of B is obtained by applying a finite number of times the rules 1° or 2°.

Example 3: Let $G = \{e, a^1, a^2, ..., a^p\}$ be a cyclic group generated by the element a. Then $(G, \bullet)$ can be recursively defined in the following manner:

1°) a belongs to G.

2°) if b and c belong to G then $b \bullet c$ belongs to G.

3°) each element of G is obtained by applying a finite number of times the rules 1 or 2.

Example 4: Each finite set $ML = \{x_1, x_2, ..., x_n\}$ can be recursively defined (with $ML \subseteq T$):

1°) The elements $x_1, x_2, ..., x_n$ of T belong to ML.

2°) If a belongs to ML, then $f(a)$ belongs to ML, where $f: T \to T$ such that $f(x) = x$;

3°) Each element of ML is obtained by applying a finite number of times the rules 1° or 2°.

Example 5: Let L be a vectorial space on the commutative corps K and $\{x_1, ..., x_m\}$ be a base of L. Then L, can be recursively defined in the following manner:

1°) $x_1, ..., x_m$ belong to L;

2°) if x, y belong to L and if a belongs to K, then $x \perp y$ y belong to L and $a * x$ belongs to L;

3°) each element of L is recursively obtained by applying a finite number of times the rules 1° or 2°.

(The operators $\perp$ and $*$ are respectively the internal and external operators of the vectorial space L).

Example 6: Let X be an A-module, and $M \subset X$ $(M \ne \emptyset)$, with $M = \{x_i\}_{i \in I}$. The sub-module generated by M is:

$$\langle M \rangle = \left\{ x \in X / x = a_1 x_1 + ... + a_n x_n, \;\; a_i \in A, \;\; x_i \in M, \;\; i \in \{1, ..., n\} \right\}$$

can be recursively defined in the following way:

1°) for all i of $\{1, 2, ..., n\}$, $\{1, 2, ..., n\} \bullet x_i \in \langle M \rangle$;

2°) if x and y belong to $\langle M \rangle$ and a belongs to A, then $x + y$ belongs to $\langle M \rangle$, and ax also;

3°) each element of $\langle M \rangle$ is obtained by applying a finite number of times the rules 1° or 2°.

In accordance to the paragraph 1 of this article, $\langle M \rangle$ is the smallest sub-set of X that verifies the conditions 1° and 2°, that is $\langle M \rangle$ is the smallest sub-module of X that includes M. $\langle M \rangle$ is also the intersection of all the subsets of X that verify the conditions 1° and 2°, that is $\langle M \rangle$ is the intersection of all sub-modules of X that contain M. One also directly refines some classic results from algebra.

One can also talk about sub-groups or ideal generated by a set: one also obtains some important applications in algebra.

Example 7: One also obtains like an application the theory of formal languages, because, like it was mentioned, each regular language (linear at right) is a regular set and reciprocally. But a regular set on an alphabet $\Sigma = \{a_1, ..., a_n\}$ can be recursively defined in the following way:

1°) $\emptyset, \{\varepsilon\}, \{a_1\}, ..., \{a_n\}$ belong to R.

2°) if P and Q belong to R, then $P \cup Q$, PQ, and P^* belong to R, with $P \cup Q = \{x / x \in P \text{ or } x \in Q\}$; $PQ = \{xy / x \in P \text{ and } y \in Q\}$, and $P^* = \bigcup_{n=0}^{\infty} P^n$ with $P^n = \underbrace{P \cdot P \cdots P}_{n \text{ times}}$ and, by convention, $P^0 = \{\varepsilon\}$.

3°) Nothing else belongs to R other that those which are obtained by using 1° or 2°.

From which many properties of this class of languages with applications to the programming languages will result.

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