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About Some Progressions

In Florentin Smarandache: "Collected Papers", vol. I (second edition).
Ann Arbor (USA): InfoLearnQuest, 2007.

In this article one builds sets which have the following property: for any division in two subsets, at least one of these subsets contains at least three elements in arithmetic (or geometrical) progression.

Lemma 1. The set of natural numbers cannot be divided in two subsets not containing either one or the other 3 numbers in arithmetic progression.

Let us suppose the opposite, and have M_1 and M_2 two subsets. Let $k \in M_1$:

a) If $k+1 \in M_1$, then $k-1$ and $k+2$ belong to M_2, if not we can build an arithmetic progression in M_1. For the same reason, since $k-1$ and $k+2$ belong to M_2, then $k-4$ and $k+5$ are in M_1. Thus $k+1$ and $k+5$ are in M_1 thus $k+3$ is in M_2; $k-4$ and k are in M_1 thus $k+4$ is in M_1; we have obtained that M_2 contains $k+2$, $k+3$ and $k+4$, which is in contradiction with the hypothesis.

b) If $k+1 \in M_1$ then $k+1 \in M_2$. We analyze the element $k-1$. If $k-1 \in M_1$, we are in the case a) where two consecutive elements belong to the same set. If $k-1 \in M_2$, then, because $k-1$ and $k+1$ belong to M_2, it results that $k-3$ and $k+3 \in M_2$, then $\in M_1$. But we obtained the arithmetic progression $k-3$, k, $k+3$ in M_1, contradiction.

Lemma 2. If one puts aside a finite number of terms of the natural integer set, the set obtained still satisfies the property of the lemma 1.

In the lemma 1, the choice of k was arbitrary, and for each k one obtains at least in one of the sets M_1 or M_2 a triplet of elements in arithmetic progression: thus at least one of these two sets contains an infinity of such triplets.

If one takes a finite number of natural numbers, it takes also a finite number of triplets in arithmetic progression. But at least one of the sets M_1 or M_2 will contain an infinite number of triplets in arithmetic progression.

Lemma 3. If $i_1,...,i_s$ are natural numbers in arithmetic progression, and $a_1,a_2,...$ is an arithmetic progression (respectively geometric), then $a_{i_1},....,a_{i_s}$ is also an arithmetic progression (respectively geometric).

Proof:

For every j we have: $2i_j = i_{j-1} + i_{j+1}$

a) If $a_1,a_2,...$ is an arithmetic progression of ratio r:

$$2a_{i_j} = 2(a_1 + (i_j - 1)r) = (a_1 + (i_{j-1} - 1)r) + (a_1 + (i_{j+1} - 1)r) = a_{i_{j-1}} + a_{i_{j+1}}$$

b) If $a_1,a_2,...$ is a geometric progression of ratio r:

$$\left(a_{i_j}\right)^2 = \left(a \cdot r^{i_j - 1}\right)^2 = a^2 \cdot r^{2i_j - 2} = \left(a \cdot r^{i_{j-1}-1}\right)\cdot\left(a \cdot r^{i_{j+1}-1}\right) = a_{i_{j-1}} + a_{i_{j+1}}$$

Theorem 1.

It does not matter the way in which one partitions the set of the terms of an arithmetic progression (respectively geometric) in subsets: in at least one of these subsets there will be at least 3 terms in arithmetic progression (respectively geometric).

Proof:

According to lemma 3, it is enough to study the division of the set of the indices of the terms of the progression in 2 subsets, and to analyze the existence (or not) of at least 3 indices in arithmetic progression in one of these subsets.
But the set of the indices of the terms of the progression is the set of the natural numbers, and we proved in lemma 1 that it cannot be division in 2 subsets without having at least 3 numbers in arithmetic progression in one of these subsets: the theorem is proved.

Theorem 2.

A set M, which contains an arithmetic progression (respectively geometric) infinite, not constant, preserves the property of the theorem 1.

Indeed, this directly results from the fact that any partition of M implies the partition of the terms of the progression.

Application: Whatever the way in which one partitions the set $A=\left\{1^m, 2^m, 3^m, \ldots\right\}$, $(m \in \mathbb{R})$ in subsets, at least one of these subsets contains 3 terms in geometric progression.

(Generalization of the problem 0:255 from "Gazeta Matematică", Bucharest, no. 10/1981, p. 400).

The solution naturally results from theorem 2, if it is noticed that A contains the geometric progression $a_n=(2^m)^n$, $(n \in \mathbb{N}^*)$.

Moreover one can prove that in at least one of the subsets there is an infinity of triplets in geometric progression, because A contains an infinity of different geometric progressions: $a_n^{(p)}=(p^m)^n$ with p prime and $n \in \mathbb{N}^*$, to which one can apply the theorems 1 and 2.