**Article**

# An Economic Analogy with Maxwell Equations in Fractional Space

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## Abstract

Hyman Minsky pioneered the idea of the financial instability hypothesis to explain how swings between robustness and fragility in financial markets generate business cycles in the economic system. Therefore, in his model business cycles and instability are endogenous. The problem now is how to put his idea of financial instability into a working model which can be tested with empirical data. Such a Minskyan model is quite rare, though some economists have proposed have tried to achieve that. For example, Toichiro Asada suggested generalized Lotka-Volterra nonlinear systems of equations as a model for Minskyan cycles.
Now I will touch a different field in applied mathematics. In the meantime, there are other papers discussing fractional Maxwell equations. Therefore, in this paper I am going to derive an Economic model as an analogy with Maxwell equations in fractional space. This is not entirely new, because it is based on Sanjay Dasari's paper. First, I will review Maxwell equations in fractional space, and then Sanjay Dasari's paper.

**Key Words**: fractional space, Hyman Minsky, economic model, symmetric Maxwell equations.

## Introduction

In contrast to the claim of mainstream (Neoclassical) teaching that the capitalist economy is inherently stable, Hyman Minsky pioneered the idea of the financial instability hypothesis to explain how swings between robustness and fragility in financial markets generate business cycles in the economic system. Therefore, in his model business cycles and instability are endogenous.[4][5] The problem now is how to put his idea of financial instability into a working model which can be tested with empirical data. Such a Minskyan model is quite rare, though some economists have proposed have tried to achieve that. For example, Toichiro Asada suggested a generalized Lotka-Volterra nonlinear systems of equations as a model for Minskyan cycles [12].

Now I will touch a different field in applied mathematics. There have been much interests to study different physical phenomenon in fractional dimensional space during the last few decades. It is also important to mention that the experimental measurement of the dimension of real world is $3 \pm 10^{-6}$, not exactly 3 [1]. In a recent paper, M. Zubair et al. described a novel approach for fractional space generalization of the differential electromagnetic equations [1]. A new form of vector differential operator Del, and its related differential operators, is formulated in fractional space. Using these modified vector differential operators, the classical Maxwell equations have

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been worked out for fractal media. In the meantime, there are other papers discussing fractional Maxwell equations [2-3].

Therefore, in this paper I am going to derive an Economic model as an analogy with Maxwell equations in fractional space. This is not entirely new, because it is based on Sanjay Dasari's paper [7]. First, I will review Maxwell equations in fractional space, and then Sanjay Dasari's paper.

## Review of previous result - Maxwell equations in fractional space

I will not re-derive Maxwell equations here. For a good reference on *Classical Electrodynamics*, see for exampleJulian Schwinger et al.'s book [9]. Penrose also discusses Maxwell equations shortly in his book:*The Road to Reality*[10]. Maxwell equations in SI unit can be written as follows [8]:

$$\nabla \cdot \varepsilon E = \rho , \tag{i}$$

$$\nabla \cdot \mu H = 0 , \tag{ii}$$

$$\nabla \times E = -\mu \frac{\partial H}{\partial t} , \tag{iii}$$

$$\nabla \times H = \varepsilon \frac{\partial E}{\partial t} + j , \tag{iv}$$

Now it seems interesting to note that Zubair et al. were able to write a differential form of Maxwell equations in far-field region in the fractional space in *div* and *curl* notation as follows [1]:

$$div_D D = \rho_v , \tag{1}$$

$$div_D B = 0 , \tag{2}$$

$$curl_D E = -\frac{\partial B}{\partial t} , \tag{3}$$

$$curl_D H = J + \frac{\partial D}{\partial t} , \tag{4}$$

and the continuity equation in fractional space as:

$$div_D J = -\frac{\partial \rho_v}{\partial t} , \tag{5}$$

where $div_D$ and $curl_D$ are defined as follows [1]:

$$div_D F = \nabla_D \cdot F = \frac{\partial F_x}{\partial x} + \frac{1}{2}\frac{(\alpha_1 - 1)F_x}{x} + \frac{\partial F_y}{\partial y} + \frac{1}{2}\frac{(\alpha_2 - 1)F_y}{y} + \frac{\partial F_z}{\partial z} + \frac{1}{2}\frac{(\alpha_3 - 1)F_z}{z}, \tag{6}$$

$$curl_D F = \nabla_D \times F = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} + \frac{1}{2}\frac{\alpha_1 - 1}{x} & \frac{\partial}{\partial y} + \frac{1}{2}\frac{\alpha_2 - 1}{y} & \frac{\partial}{\partial z} + \frac{1}{2}\frac{\alpha_3 - 1}{z} \\ F_x & F_y & F_z \end{vmatrix}, \tag{7}$$

where parameters ($0 < \alpha_1 \le 1$, $0 < \alpha_2 \le 1$ and $0 < \alpha_3 \le 1$) are used to describe the measure distribution of space where each one is acting independently on a single coordinate and the total dimension of the system is $D = \alpha_1 + \alpha_2 + \alpha_3$. [1]

## An Economic Model with Maxwell equations

Sanjay Dasari works out an economic model as an analogy to Maxwell equations. He begins with symmetric electrodynamics of Cabibo and Ferrari [13][14], as follows [7]:

$$\nabla \cdot \varepsilon_0 \vec{E} = \rho_e, \tag{8}$$

$$\nabla \cdot \vec{B} = \mu_0 \rho_m, \tag{9}$$

$$\nabla \times E = -\frac{\partial \vec{B}}{\partial t} - \mu_0 \vec{j}_m, \tag{10}$$

$$\nabla \times \vec{B} = \mu_0 \varepsilon_0 \frac{\partial \vec{E}}{\partial t} + \mu_0 \vec{j}_m, \tag{11}$$

According to Singleton [13], we can write E and B in terms of four vector potentials as follows:

$$\vec{E} = -\nabla \phi_e - \frac{\partial \vec{A}}{\partial t} - \nabla \times \vec{C}, \tag{12}$$

$$\vec{B} = -\nabla \phi_m - \frac{\partial \vec{C}}{\partial t} + \nabla \times \vec{A}. \tag{13}$$

Then Dasari denotes the main economic variables as follows:

- competition flow as →c
- profit flow as →P
- money flow as →M
- money density, money per unit volume, as n
- Ambition of a person as, A→m

• Price index desirable by a consumer as Pic

• Price index desirable by a supplier as Pis

• Choice flow of a consumer (supplier) as → Chc,s

• Economic power flow as →Ep

• Economic activity as Ea

• inverse of basic strength-scale of currency, at least for macro economy, as
s0

• basic technical knowhow+political power, at least for macro economy, as
k0

• human infrastructure as h

• capital density as ρ K

• capital flow as →K

## An Economic Model with Maxwell equations in Fractional Space

Then it is possible to write down four equations of symmetric Maxwell equations in Fractional Space, by virtue of Zubair et al.'s definitions (5)-(7), as follows:

$$div_D \varepsilon_0 \vec{E} = \rho_e , \tag{14}$$

$$div_D \vec{B} = \mu_0 \rho_m , \tag{15}$$

$$curl_D E = -\frac{\partial \vec{B}}{\partial t} - \mu_0 \vec{j}_m , \tag{16}$$

$$curl_D \vec{B} = \mu_0 \varepsilon_0 \frac{\partial \vec{E}}{\partial t} + \mu_0 \vec{j}_m , \tag{17}$$

The next step is using the above comparison economic variables to write down an economic model with symmetric Maxwell equations in Fractional Space, as follows:

$$div_D s_0 \vec{c} = -n , \tag{18}$$

$$div_D \vec{P} = k_0 \rho_k , \tag{19}$$

$$curl_D \vec{c} = -\frac{\partial \vec{P}}{\partial t} - k_0 \vec{K} , \tag{20}$$

$$curl_D \vec{P} = k_0 s_0 \frac{\partial \vec{c}}{\partial t} + k_0 \vec{K} , \tag{21}$$

In the above equations (18)-(21) I keep Dasari's notations remain intact [7], the only change here is the use of curl and div of fractional differentiation according to Zubair *et al*. [1]

Beyond possible advantages as mentioned by Dasari such as deriving Black-Scholes equation, I guess one clear advantage of the above expression is to explain endogenous cycles as postulated by Hyman Minsky. Moreover, these fractional differential equations will make it possible to reflect the daily fluctuations of market price. Therefore, I hope the above equations will be found useful both by economists and also by econophysicists.

## Fractional Helmholtz equation and solution of classical wave equation in fractional space

It is worth noting here that Zubair also wrote Helmholtz equations in fractional space for E and H field as a consequence of his Maxwell equations in fractional space, as follows [1]:

$$\nabla_D^2 E - \mu\varepsilon \frac{\partial^2 E}{\partial t^2} = 0\,, \tag{22}$$

$$\nabla_D^2 H - \mu\varepsilon \frac{\partial^2 H}{\partial t^2} = 0\,. \tag{23}$$

In another paper, Zubair, Mughal & Naqvi give a solution of this kind of Helmholtz equation in fractional space [11]. The Laplacian operator in D-dimensional fractional space is defined as follows:

$$\nabla_D^2 = \frac{\partial^2}{\partial x^2} + \frac{\alpha_1 - 1}{x}\frac{\partial}{\partial x} + \frac{\partial^2}{\partial y^2} + \frac{\alpha_2 - 1}{y}\frac{\partial}{\partial y} + \frac{\partial^2}{\partial z^2} + \frac{\alpha_3 - 1}{z}\frac{\partial}{\partial z}\,. \tag{24}$$

Then they derive a solution of equation (31) with the help of Bessel equation [11].

Similarly, the equations (22)-(23) also imply that the proposed economic model based on symmetric Maxwell equations will have endogenous cycles just as postulated by Minsky. This is a remarkable advantage of this model.

## Concluding remarks

In contrast to the claim of mainstream (Neoclassical) teaching that the capitalist economy is inherently stable, Hyman Minsky pioneered the idea of the financial instability hypothesis to explain how swings between robustness and fragility in financial markets generate business cycles in the economic system. Therefore, in his model business cycles and instability are endogenous. The problem now is how to put his idea of financial instability into a working model which can be tested with empirical data. Such a Minskyan model is quite rare, though some economists have proposed have tried to achieve that. For example, Toichiro Asada suggested generalized Lotka-Volterra nonlinear systems of equations as a model for Minskyan cycles.

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## Appendix: **The Lotka-Volterra System does not always oscillate**

### Introduction

In recent years, there are proposals suggesting that the Lotka-Volterra system can be used as a model of financial instability in order to prove Hyman Minsky's financial instability hypothesis, see for instance [1][2][3][4]. While such a proposal looks interesting, nonetheless one should be careful on built-in assumptions in the model. For instance Lotka-Volterra's original equations are known to be unrealistic, so if this model shows cyclical behavior, it is based on unrealistic assumptions. It can be shown that more realistic model does not show cyclical behavior. In turn, our conclusion is that the Lotka-Volterra system is inadequate as a model of financial instability.

### The original Lotka-Volterra equations

The Lotka–Volterra system arises in mathematical biology and models the growth of animal species [5]. Consider two species where $Y_1(T)$ denotes the number of predators and $Y_2(T)$ denotes the number of prey. A particular case of the Lotka–Volterra differential system is

$$\dot{Y}_1 = Y_1(Y_2 - 1), \tag{1}$$

$$\dot{Y}_2 = Y_2(2Y_1 - 1). \tag{2}$$

where the dot denotes differentiation with respect to time T.

The Lotka–Volterra system (1)-(2) has an invariant H, which is constant for all T [8]:

$$H(Y_1, Y_2) = 2\ln Y_1 - Y_1 + \ln Y_2 - Y_2 \tag{3}$$

The level curves of the invariant (3) are closed so that the solution is periodic. It is desirable that the numerical solution of (1)-(2) is also periodic, but this is not always the case. Note that (1)-(2) is a Poisson system [8]:

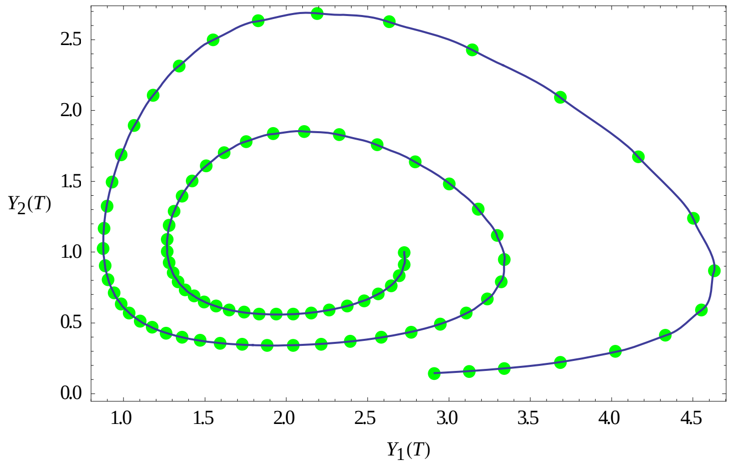
$$\dot{Y} = B(Y)\nabla H(Y) = \begin{pmatrix} 0 & Y_1Y_2 \\ Y_1Y_2 & 0 \end{pmatrix} \begin{pmatrix} \dfrac{2}{Y_1} - 1 \\ \dfrac{1}{Y_2} - 1 \end{pmatrix}, \qquad (4)$$

where H(Y) is defined in (3).

The solution of Poisson system (4) can be found using Mathematica 9 as follows [8]:

```
Needs["DifferentialEquations`NDSolveProblems`"];
Needs["DifferentialEquations`NDSolveUtilities`"];
Needs["DifferentialEquations`InterpolatingFunctionAnatomy`"];
system = GetNDSolveProblem["LotkaVolterra"];
invts = system["Invariants"];
time = system["TimeData"];
vars = system["DependentVariables"];
step = 3/25;
LotkaVolterraPlot[sol_,vars_,time_,opts___?OptionQ]:=
  Module[{commonopts, data,data1,data2,ifuns,lplot,pplot},
   ifuns=First[vars/.sol];
   data1=Part[ifuns,1,0]["ValuesOnGrid"];
   data2=Part[ifuns,2,0]["ValuesOnGrid"];
   data=Transpose[{data1,data2}];

commonopts=Sequence[Axes→False,Frame→True,FrameLabel→Join[Map[Traditio
nalForm,vars],{None,None}],RotateLabel→False];

lplot=ListPlot[data,Evaluate[FilterRules[{opts},Options[ListPlot]]],Pl
otStyle→{PointSize[0.02],RGBColor[0,1,0]},Evaluate[commonopts]];

pplot=ParametricPlot[Evaluate[ifuns],time,Evaluate[FilterRules[{opts},
Options[ParametricPlot]]],Evaluate[commonopts]];
   Show[lplot,pplot]
   ];
fesol = NDSolve[system,Method→"ExplicitEuler",StartingStepSize→step];

LotkaVolterraPlot[fesol, vars, time]
```

Graphical plot is shown below:




Graphic 1. Plot of solution of Lotka-Volterra system

## A Logistic version of Lotka-Volterra system

The classic Lotka-Volterra assumptions are very unrealistic. Specially the first one, where it is assumed that the prey population would increase indefinitely in absence of predators. One of the possible solutions is to assume logistic behavior in the prey and predator population [6]:

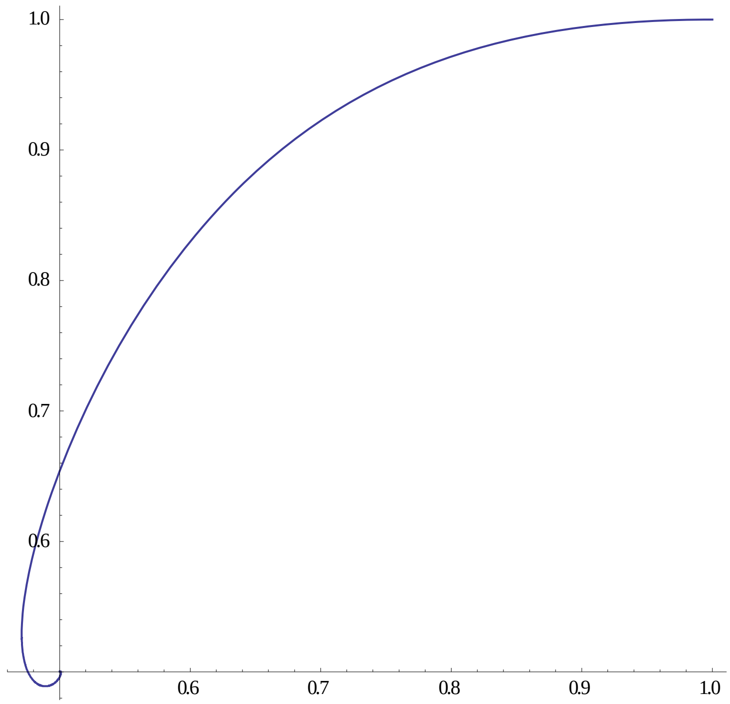
$$\dot{Y}_1 = Y_1(1-Y_1) - Y_1Y_2, \tag{5}$$

$$\dot{Y}_2 = Y_2(1-\frac{Y_2}{Y_1}). \tag{6}$$

Plot of the solution of equation (5) and (6) can be found by using Mathematica 9 too:

```
sol=NDSolve[{x'[t]==x[t]*(1-x[t])-x[t]*y[t],y'[t]==y[t]*(1-
y[t]/x[t]),x[0]==y[0]==1},{x,y},{t,10}]
ParametricPlot[Evaluate[{x[t],y[t]}/.sol],{t,0,10},PlotRange->All]
```

Output:
$\{\{x \to \text{InterpolatingFunction}[\{\{0.,10.\}\}, \text{"<>"}], y \to \text{InterpolatingFunction}[\{\{0.,10.\}\}, \text{"<>"}]\}\}$




Graphic 2. Plot of solution of Logistic version of Lotka-Volterra system [4]

## Concluding remarks

It is clear from graphic #2 above, that an improved version of the Lotka-Volterra system shows no cyclical behavior. Therefore unless it can be shown otherwise, it is unlikely that TLV system can be used as a model for financial instability, in contrast to some recent claims. A better model of financial instability and endogenous business cycles is needed. One possible solution is to use the Generalized Lotka-Volterra models, but they are beyond the scope of this paper.[7]

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