

ABOUT BERNOULLI'S NUMBERS

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Abstract.

Many methods to compute the sum of the first n natural numbers of the same powers (see [4]) are well known. In this article we present a simple proof of the method from [3].

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Introduction.

The Bernoulli's numbers are defined by

$$(1) \quad B_n = \frac{-1}{n+1} \left(C_{n+1}^0 B_0 + C_{n+1}^1 B_1 + \dots + C_{n+1}^{n-1} B_{n-1} \right)$$

where $B_0 = 1$. It is known that $B_{n+1} = 0$ if $n \geq 1$. By calculation we find that:

$$(2) \quad B_1 = -\frac{1}{2}, B_2 = \frac{1}{6}, B_4 = -\frac{1}{30}, B_6 = \frac{1}{42}, B_8 = -\frac{1}{30}, B_{10} = \frac{5}{66},$$
$$B_{12} = -\frac{691}{2730}, B_{14} = \frac{7}{6}, B_{16} = -\frac{3617}{510}, B_{18} = \frac{43867}{798}, B_{20} = -\frac{174611}{330},$$
$$B_{22} = \frac{854513}{138}, B_{24} = -\frac{236364091}{2730}, \text{ etc.}$$

Let $S_n^k = 1^k + 2^k + \dots + n^k$ the sum of the first n natural numbers which have the same power.

Theorem.

$$(3) \quad S_n^k = \frac{1}{k+1} \left(n^{k+1} + \frac{1}{2} C_{k+1}^1 n^k + C_{k+1}^2 B_2 n^{k-1} + \dots + C_{k+1}^k B_n n \right)$$

Proof: (1) can be written as:

$$(4) \quad \sum_{i=0}^n C_{n+1}^i B_i = 0, \quad n \geq 1.$$

$$\text{If } P(x) = \sum_{i=0}^k C_{k+1}^i B_i x^{k+1-i},$$

then

$$P(n+1) - P(n) = \sum_{i=0}^k C_{k+1}^i B_i \left((n+1)^{k+1-i} - n^{k+1-i} \right) = \sum_{i=0}^k C_{k+1}^i B_i \left(\sum_{j=1}^{k+1-i} C_{k+1-i}^j n^{k+1-i-j} \right).$$

Let A_t be the coefficients of n^{k-1} , where $t \in \{0, 1, \dots, k\}$.

$$A_t = \sum_{i=0}^t C_{k+1}^i C_{k+1-i}^{i+t+1} B_i = C_{k+1}^{i+1} \left(\sum_{i=0}^t C_{i+1}^i B_i \right).$$

If $n \geq 1$, then $A_t = 0$, only $A_0 = C_{k+1}^1$.

Because of these $P(n+1) - P(n) = C_{k+1}^1 n^k$. Using this

$$\sum_{i=0}^{n-1} i^k = \frac{1}{k+1} \sum_{i=0}^{n-1} (P(i+1) - P(i)) = \frac{1}{k+1} P(n),$$

because $P(0) = 0$. Then $S_n^k = \frac{1}{k+1} P(n) + n^k$. From here one obtains (3).

Note. From the previous result we can also find the formula

$$S_n^k = \frac{1}{k+1} P(n+1).$$

Using this, we find the following equalities:

$$S_n^0 = n, S_n^1 = \frac{1}{2} n(n+1), S_n^2 = \frac{1}{6} n(n+1)(2n+1), S_n^3 = \frac{1}{4} n^2 (n+1)^2,$$

$$S_n^4 = \frac{1}{30} n(n+1)(2n+1)(3n^2 + 3n - 1), S_n^5 = \frac{1}{12} n^2 (n+1)^2 (2n^2 + 2n - 1),$$

$$S_n^6 = \frac{1}{42} n(n+1)(2n+1)(3n^4 + 6n^3 - 3n + 1),$$

$$S_n^7 = \frac{1}{24} n^2 (n+1)^2 (3n^4 + 6n^3 - n^2 - 4n + 2),$$

$$S_n^8 = \frac{1}{90} n(n+1)(2n+1)(5n^6 + 15n^5 + 5n^4 - 15n^3 - n^2 + 9n - 3),$$

$$S_n^9 = \frac{1}{20} (2n^{10} + 10n^9 + 15n^8 - 14n^6 + 10n^4 - 3n^2),$$

$$S_n^{10} = \frac{1}{66} (6n^{11} + 33n^{10} + 55n^9 - 66n^7 + 66n^5 - 33n^3 + 5n),$$

$$S_n^{11} = \frac{1}{24} (2n^{12} + 12n^{11} + 22n^{10} - 33n^8 + 44n^6 - 33n^4 + 10n^2),$$

$$S_n^{12} = \frac{1}{2730} (210n^{13} + 1365n^{12} + 3630n^{11} - 4935n^9 + 115n^8 + 9640n^7 + 1960n^6 - 5899n^5 + 35n^4 + 4550n^3 + 1382n^2 - 691n), \text{ etc.}$$

Problems:

- 1) Using the mathematical induction on the base of (1), we prove that $B_{2n+1} = 0$, if $n \geq 1$.
- 2) Prove that S_n^k is divisible by $n(n+1)$.
- 3) Prove that S_n^{2k+1} is divisible by $n^2(n+1)^2$.
- 4) Determine those natural numbers n, k for which S_n^{2k} is divisible $n(n+1)(2n+1)$.
- 5) Detach in parts the sums $S_n^9, S_n^{10}, S_n^{11}, S_n^{12}$.
- 6) Using (2), (3), compute the sums $S_n^{13}, \dots, S_n^{21}$.

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