Bipolar neutrosophic graphs with applications

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Abstract

In this research article, we present a novel frame work for handling bipolar neutrosophic information by combining the theory of bipolar neutrosophic sets with graphs. We introduce some operations on bipolar neutrosophic graphs. We describe the dominating and independent sets of bipolar neutrosophic graphs. We discuss an outranking approach for risk analysis and construction of minimum number of radio channels using bipolar neutrosophic sets and bipolar neutrosophic graphs.

Key-words: Applications of bipolar neutrosophic graphs, Union, Intersection, Join, Cartesian product, Direct product, Strong product, Domination number, Independent number.

1 Introduction

A fuzzy set [29] is an important mathematical structure to represent a collection of objects whose boundary is vague. Fuzzy models are becoming useful because of their aim in reducing the differences between the traditional numerical models used in engineering and sciences and the symbolic models used in expert systems.

In 1994, Zhang [31] introduced the notion of bipolar fuzzy sets and relations. Bipolar fuzzy sets are extension of fuzzy sets whose membership degree ranges $[-1, 1]$. The membership degree $(0, 1]$ indicates that the object satisfies a certain property whereas the membership degree $[-1, 0)$ indicates that the element satisfies the implicit counter property. Positive information represent what is considered to be possible and negative information represent what is granted to be impossible. Actually, a variety of decision making problems are based on two-sided bipolar judgements on a positive side and a negative side. Nowadays bipolr fuzzy sets are playing a substantial role in chemistry, economics, computer science, engineering, medicine and decision making problems. Samarandache [22] introduced the idea of neutrosophic probability, sets and logic. Some properties and applications of neutrosophic sets were further studied by Jaun-Jaun Peng et al. [19] in 2014. The other terminologies and applications of neutrosophic sets can be seen in [23, 27, 28, 9, 8, 12, 24]. In a neutrosophic set, the membership value is associated with truth, false and indeterminacy degrees but there is no restriction on their sum. Deli et al. [11] extended the ideas of bipolar fuzzy sets and neutrosophic sets to bipolar neutrosophic sets and studied its operations and applications in decision making problems.

Graph theory has numerous applications in science and engineering. However, in some cases, some aspects of graph theoretic concepts may be uncertain. In such cases, it is important to deal with uncertainty using the methods of fuzzy sets and logics. Based on Zadeh's fuzzy relations [30] Kaufmann [13] defined a fuzzy graph. The fuzzy relations between fuzzy sets were also considered by Rosenfeld [20] and he developed the structure of fuzzy graphs, obtaining analogs of several graph theoretical concepts. Later on, Bhattacharya [5] gave some remarks on fuzzy graphs, and some operations on fuzzy graphs were introduced by Mordeson and Peng [17]. The complement of a fuzzy graph was defined by Mordeson [16]. Bhutani and Rosenfeld introduced the concept of M-strong fuzzy graphs in [6] and studied some of their properties. The concept of strong arcs in fuzzy graphs was discussed in [7]. The theory of fuzzy graphs has extended widely by many researchers as it can be seen in [14, 15, 21]. The idea of domination was first arose in chessboard problem in 1862. Somasundaram amd Somasundaram [25] introduced domination and independent domination in fuzzy graphs. Nagoor Gani and Chandrasekaran [18] studied the notion of fuzzy domination and independent domination using strong arcs. Akram [1, 2] introduced bipolar fuzzy graphs and discuss its various properties which were further studied by Yang [26] in 2013. Akram et al. [3] studied regular bipolar fuzzy graphs. The theory of bipolar fuzzy graphs is extended to $m-$polar fuzzy graphs by Chen et al. [10] in 2014.

In this article, we propose the idea of bipolar neutrosophic graphs. We discuss some fundamental operations in bipolar neutrosphic graphs, regular and irregular bipolar neutrosophic graphs, domination number and independent number. We calculate the domination and independent numbers of bipolar neutrosphic union, intersection, join and products. At the end, some applications in bipolar neutrosphic graphs are given that support the ideas discussed in this article.

2 Preliminaries

Let X be a non-empty set. Let $\widetilde{X^2}$ denotes the collection of all $2-$elements subsets of X. A pair $G^* = (V, E)$, where $E \subseteq \widetilde{X^2}$ is called a *graph.* The cardinality of any subset $D \subseteq X$ is the number of vertices in D, it is denoted by $|D|$.

Definition 2.1. [29, 30] A *fuzzy subset* ν on a non-empty set X is a mapping $\nu : X \to [0, 1]$. A *fuzzy binary relation* on X is a fuzzy subset λ on $X \times X$. Fuzzy relation is a fuzzy binary relation given by the mapping $\lambda : X \times X \to [0, 1]$.

Definition 2.2. [13] A *fuzzy graph* of a graph $G^* = (X, E)$ is a pair $G = (\mu, \lambda)$, where μ and λ are fuzzy sets on X and $\widetilde{X}^2$ respectively, such that $\lambda(xy) \leq \min\{\mu(x), \mu(y)\}$ for all $xy \in E$. Note that $\lambda(xy) = 0$ for all $x, y \in \widetilde{X}^2 - E$.

Definition 2.3. [31] A bipolar fuzzy set on a non-empty set X is an object of the form $C = \{(x, \mu^p(x), \mu^n(x)) : x \in X\}$ where, $\mu^p : X \to [0, 1]$ and $\mu^n : X \to [-1, 0]$ are mappings.

The positive membership degree $\mu^p(x)$ denotes the truth or satisfaction degree of an element x to a certain property corresponding to bipolar fuzzy set C and $\mu^n(x)$ represents the satisfaction degree of element x to some counter property of bipolar fuzzy set C. If $\mu^n(x) \neq 0$ and $\mu^p(x) = 0$, it is the situation that x is not satisfying the property of C but satisfying the counter property to C. If $\mu^p(x) \neq 0$ and $\mu^n(x) = 0$, it is the case when x has only positive satisfaction for C. It is possible for x to be such that $\mu^p(x) \neq 0$ and $\mu^n(x) \neq 0$ when x satisfies the property of C as well as its counter property in some part of X.

Definition 2.4. [1] Let X be a nonempty set. A mapping $D = (\mu^p, \mu^n) : X \times X \to [0,1] \times [-1,0]$ is called a bipolar fuzzy relation on X such that $\mu^p(xy) \in [0,1]$ and $\mu^n(xy) \in [-1,0]$, for $x, y \in X$.

Definition 2.5. [1] A bipolar fuzzy graph on a crisp graph $G^* = (X, E)$ is a pair $G = (C, D)$ where $A = (\mu_C^p, \mu_C^n)$ is a bipolar fuzzy set on X and $D = (\mu_D^p, \mu_D^n)$ is a bipolar fuzzy relation in E such that

$$\mu_D^p(xy) \leq \mu_C^p(x) \wedge \mu_C^p(y) \text{ and } \mu_D^n(xy) \geq \mu_C^n(x) \vee \mu_C^n(y) \text{ for all } xy \in E.$$

Definition 2.6. [23] A neutrosophic set C on a non-empty set X is characterized by a truth membership function $t_C : X \to [0,1]$, indeterminacy membership function $I_C : X \to [0,1]$ and a falsity membership function $f_C : X \to [0,1]$. There is no restriction on the sum of $t_C(x)$, $I_C(x)$ and $f_C(x)$ for all $x \in X$.

Definition 2.7. [11] A bipolar neutrosophic set on a empty set X is an object of the form

$$C = \{(x, t_C^p(x), I_C^p(x), f_C^p(x), t_C^n(x), I_C^n(x), f_C^n(x)) : x \in X\}$$

where, $t_C^p, I_C^p, f_C^p : X \to [0,1]$ and $t_C^n, I_C^n, f_C^n : X \to [-1,0]$. The positive values $t_C^p(x), I_C^p(x), f_C^p(x)$ denote respectively the truth, indeterminacy and false membership degrees of an element $x \in X$ whereas $t_C^n(x), I_C^n(x), f_C^n(x)$ denote the implicit counter property of the truth, indeterminacy and false membership degrees of the element $x \in X$ corresponding to the bipolar neutrosophic set C.

3 Bipolar neutrosophic graphs

Definition 3.1. A bipolar neutrosophic relation on a non-empty set X is a bipolar neutrosophic subset of $X \times X$ of the form $D = \{(xy, t_D^p(xy), I_D^p(xy), f_D^p(xy), t_D^n(xy), I_D^n(xy), f_D^n(xy)) : xy \in E \subseteq X \times X\}$ where, $t_D^p, I_D^p, f_D^p, t_D^n, I_D^n, f_D^n$ are defined by the the mappings $t_D^p, I_D^p, f_D^p : X \times X \to [0,1]$ and $t_D^n, I_D^n, f_D^n : X \times X \to [-1,0]$.

Definition 3.2. A bipolar neutrosophic graph on a crisp graph $G^* = (X, E)$ is a pair $G = (C, D)$, where C is a bipolar neutrosophic set on X and D is a bipolar neutrosophic relation on X such that

$$\begin{aligned} &t_D^p(xy) \leq t_C^p(x) \wedge t_C^p(y), \quad &&I_D^p(xy) \leq I_C^p(x) \vee I_C^p(y), \quad &&f_D^p(xy) \leq f_C^p(x) \vee f_C^p(y), \\ &t_D^n(xy) \geq t_C^n(x) \vee t_C^n(y), \quad &&I_D^n(xy) \geq I_C^n(x) \wedge I_C^n(y), \quad &&f_D^n(xy) \geq t_C^n(x) \wedge t_C^n(y) \text{ for all } xy \in E. \end{aligned}$$

Note that $D(xy) = (0,0,0,0,0,0)$ for all $xy \in X \times X \setminus E$.

Example 3.1. Consider a graph $G^* = (X, E)$ such that $X = \{x, y, z\}$, $E = \{xy, yz, zx\}$. Let C be a bipolar neutrosophic set on X given in Table.1 and D be a bipolar neutrosophic relation of $E \subseteq X \times X$ given in Table.2. Routine calculations show that $G = (C, D)$ is a bipolar neutrosophic graph of $G^* = (X, E)$. The bipolar neutrosophic graph G is shown in Fig. 1.

Table 1	x	y	z
t^p_C	0.3	0.5	0.4
I^p_C	0.4	0.4	0.3
f^p_C	0.5	0.2	0.2
t^n_C	-0.6	-0.1	-0.5
I^n_C	-0.5	-0.8	-0.5
f^n_C	-0.2	-0.2	-0.5

Table 2	xy	yz	xz
t^p_D	0.3	0.3	0.3
I^p_D	0.4	0.4	0.4
f^p_D	0.5	0.2	0.5
t^n_D	-0.1	-0.1	-0.5
I^n_D	-0.8	-0.8	-0.5
f^n_D	-0.2	-0.5	-0.5

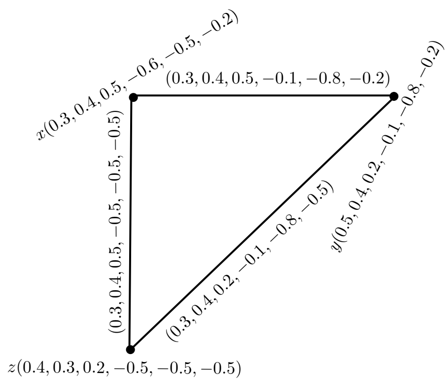


Figure 1: Bipolar neutrosophic graph G

Definition 3.3. The union of two bipolar neutrosophic graphs $G_1 = (C_1, D_1)$ and $G_2 = (C_2, D_2)$ is a pair $G_1 \cup G_2 = (C_1 \cup C_2, D_1 \cup D_2)$ where, $C_1 \cup C_2$ is a bipolar neutrosophic set on $X_1 \cup X_2$ and $D_1 \cup D_2$ is a bipolar neutrosophic set on $E_1 \cup E_2$ such that

$$t^p_{C_1\cup C_2}(x) = \begin{cases} t^p_{C_1}(x), & x \in X_1,\ x \notin X_2 \\ t^p_{C_2}(x), & x \notin X_1,\ x \in X_2 \\ t^p_{C_1}(x) \vee t^p_{C_2}(y), & x \in X_1 \cap X_2 \end{cases} \qquad I^p_{C_1\cup C_2}(x) = \begin{cases} I^p_{C_1}(x), & x \in X_1,\ x \notin X_2 \\ I^p_{C_2}(x), & x \notin X_1,\ x \in X_2 \\ I^p_{C_1}(x) \wedge I^p_{C_2}(y), & x \in X_1 \cap X_2 \end{cases}$$

$$f^p_{C_1\cup C_2}(x) = \begin{cases} f^p_{C_1}(x), & x \in X_1,\ x \notin X_2 \\ f^p_{C_2}(x), & x \notin X_1,\ x \in X_2 \\ f^p_{C_1}(x) \wedge f^p_{C_2}(y), & x \in X_1 \cap X_2 \end{cases} \qquad t^n_{C_1\cup C_2}(x) = \begin{cases} t^n_{C_1}(x), & x \in X_1,\ x \notin X_2 \\ t^n_{C_2}(x), & x \notin X_1,\ x \in X_2 \\ t^n_{C_1}(x) \wedge t^n_{C_2}(y), & x \in X_1 \cap X_2 \end{cases}$$

$$I^n_{C_1\cup C_2}(x) = \begin{cases} I^n_{C_1}(x), & x \in X_1,\ x \notin X_2 \\ I^n_{C_2}(x), & x \notin X_1,\ x \in X_2 \\ I^n_{C_1}(x) \vee I^n_{C_2}(y), & x \in X_1 \cap X_2 \end{cases} \qquad f^n_{C_1\cup C_2}(x) = \begin{cases} f^n_{C_1}(x), & x \in X_1,\ x \notin X_2 \\ f^n_{C_2}(x), & x \notin X_1,\ x \in X_2 \\ f^n_{C_1}(x) \vee f^n_{C_2}(y), & x \in X_1 \cap X_2 \end{cases}$$

and membership values of edges are

$$t^p_{D_1\cup D_2}(xy) = \begin{cases} t^p_{D_1}(xy), & xy \in E_1,\ xy \notin E_2 \\ t^p_{D_2}(xy), & xy \notin E_1,\ xy \in E_2 \\ t^p_{D_1}(xy) \vee t^p_{D_2}(xy), & xy \in E_1 \cap E_2 \end{cases}$$

$$I^p_{D_1\cup D_2}(xy) = \begin{cases} I^p_{D_1}(xy), & xy \in E_1,\ xy \notin E_2 \\ I^p_{D_2}(xy), & xy \notin E_1,\ xy \in E_2 \\ I^p_{D_1}(xy) \wedge I^p_{D_2}(xy), & xy \in E_1 \cap E_2 \end{cases}$$

$$f^p_{D_1\cup D_2}(xy)=\begin{cases} f^p_{D_1}(xy), & xy\in E_1,\ xy\notin E_2\\ f^p_{D_2}(xy), & xy\notin E_1,\ xy\in E_2\\ f^p_{D_1}(xy)\wedge f^p_{D_2}(xy), & xy\in E_1\cap E_2\end{cases}$$

$$t^n_{D_1\cup D_2}(xy)=\begin{cases} t^n_{D_1}(xy), & xy\in E_1,\ xy\notin E_2\\ t^n_{D_2}(xy), & xy\notin E_1,\ xy\in E_2\\ t^n_{D_1}(xy)\wedge t^n_{D_2}(xy), & xy\in E_1\cap E_2\end{cases}$$

$$I^n_{D_1\cup D_2}(xy)=\begin{cases} I^n_{D_1}(xy), & xy\in E_1,\ xy\notin E_2\\ I^n_{D_2}(xy), & xy\notin E_1,\ xy\in E_2\\ I^n_{D_1}(xy)\vee I^n_{D_2}(xy), & xy\in E_1\cap E_2\end{cases}$$

$$f^n_{D_1\cup D_2}(xy)=\begin{cases} f^n_{D_1}(xy), & xy\in E_1,\ xy\notin E_2\\ f^n_{D_2}(xy), & xy\notin E_1,\ xy\in E_2\\ f^n_{D_1}(xy)\vee f^n_{D_2}(xy), & xy\in E_1\cap E_2\end{cases}$$

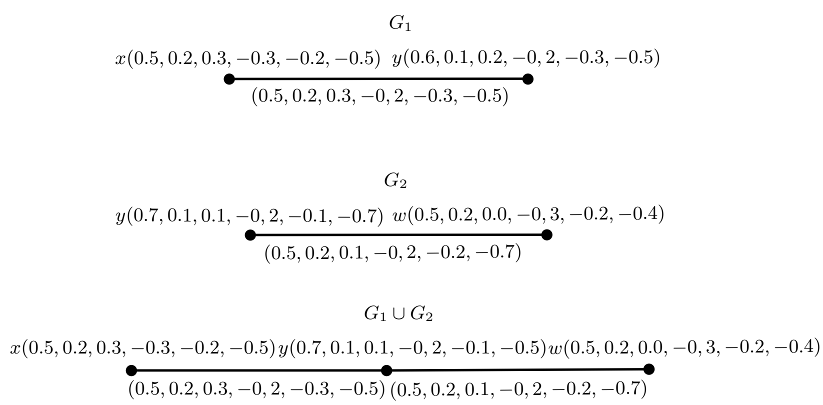


Figure 2: Union of two bipolar neutrosophic graphs

Example 3.2. The union of two bipolar neutrosophic graphs G_1 and G_2 is shown in Fig.2.

Proposition 3.1. *Let G_1 and G_2 be any two bipolar neutrosophic graphs then $G_1 \cup G_2$ is a bipolar neutrosophic graph.*

Definition 3.4. The intersection of two bipolar neutrosophic graphs $G_1 = (C_1, D_1)$ and $G_2 = (C_2, D_2)$ is a pair $G_1 \cap G_2 = (C_1 \cap C_2, D_1 \cap D_2)$ where, $C_1 \cap C_2$ is a bipolar neutrosophic set on $X_1 \cap X_2$ and $D_1 \cap D_2$ is a bipolar neutrosophic set on $E_1 \cap E_2$. The membership degrees are defined as

$$\begin{aligned} &t^p_{C_1\cap C_2}(x)=t^p_{C_1}(x)\wedge t^p_{C_2}(y) && I^p_{C_1\cap C_2}(x) &&= I^p_{C_1}(x)\vee I^p_{C_2}(y) && f^p_{C_1\cap C_2}(x)=f^p_{C_1}(x)\vee f^p_{C_2}(y)\\ &t^n_{C_1\cap C_2}(x)=t^n_{C_1}(x)\vee t^n_{C_2}(y) && I^n_{C_1\cap C_2}(x) &&= I^n_{C_1}(x)\wedge I^n_{C_2}(y) && f^n_{C_1\cap C_2}(x)=f^n_{C_1}(x)\wedge f^n_{C_2}(y)\\ &\text{for all}\quad x\in X_1\cap X_2. \end{aligned}$$

$$\begin{aligned} &t^p_{D_1\cap D_2}(xy)=t^p_{D_1}(xy)\wedge t^p_{D_2}(xy) && I^p_{D_1\cap D_2}(xy)= && I^p_{D_1}(xy)\vee I^p_{D_2}(xy) && f^p_{D_1\cap D_2}(xy)=f^p_{D_1}(xy)\vee f^p_{D_2}(xy)\\ &t^n_{D_1\cap D_2}(xy)=t^n_{D_1}(xy)\vee t^n_{D_2}(xy) && I^n_{D_1\cap D_2}(xy)= && I^n_{D_1}(xy)\wedge I^n_{D_2}(xy) && f^n_{D_1\cap D_2}(xy)=f^n_{D_1}(xy)\wedge f^n_{D_2}(xy),\\ &\text{for all}\quad xy\in E_1\cap E_2. \end{aligned}$$

Example 3.3. The intersection of two bipolar neutrosophic graphs G_1 and G_2 shown in Fig.2 is the vertex y with membership value $(0.6, 0.1, 0.2, -0.2, -0.3, -0.7)$.

Proposition 3.2. *The intersection of any two bipolar neutrosophic graphs is also a bipolar netrosophic graph.*

Definition 3.5. Let C_1 and C_2 be two bipolar neutrosophic subsets of the set of vertices X_1 and X_2 and D_1, D_2 be the bipolar neutrosophic relations on X_1 and X_2, respectively. The join of the bipolar neutrosophic graphs $G_1 = (C_1, D_1)$ and $G_2 = (C_2, D_2)$ is defined by the pair $G_1 + G_2 = (C_1 + C_2, D_1 + D_2)$ such that, $C_1 + C_2 = C_1 \cup C_2$ for all $x \in X_1 \cup X_2$ and

1. $D_1 + D_2 = D_1 \cup D_2$ for all $xy \in E_1 \cap E_2$,
2. Let $E^{'}$ be the set of all edges joining the vertices of G_1 and G_2 then for all $xy \in E^{'}$, where $x \in X_1$ and $y \in X_2$,

$$t^p_{D_1+D_2}(xy) = t^p_{C_1}(x) \vee t^p_{C_2}(y), \quad I^p_{D_1+D_2}(xy) = I^p_{C_1}(x) \wedge I^p_{C_2}(y), \quad f^p_{D_1+D_2}(xy) = f^p_{C_1}(x) \wedge f^p_{C_2}(y),$$
$$t^n_{D_1+D_2}(xy) = t^n_{C_1}(x) \wedge t^n_{C_2}(y), \quad I^n_{D_1+D_2}(xy) = I^n_{C_1}(x) \vee I^n_{C_2}(y), \quad f^n_{D_1+D_2}(xy) = f^n_{C_1}(x) \vee f^n_{C_2}(y).$$

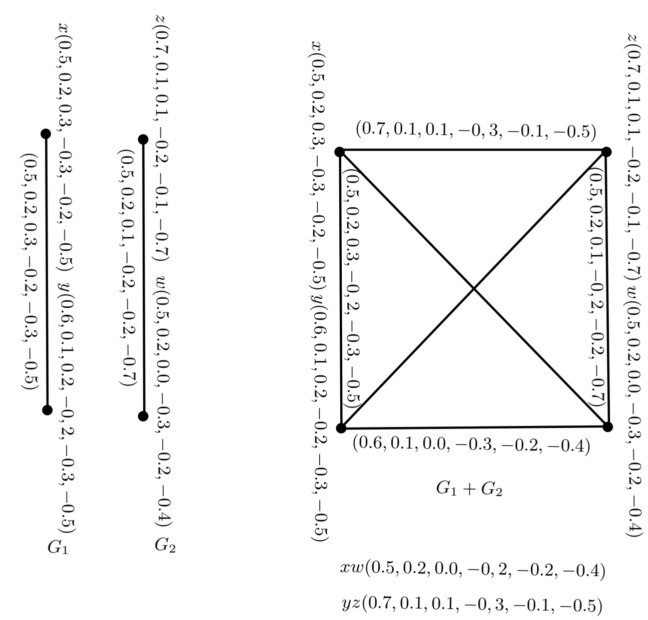


Figure 3: Join of G_1 and G_2.

Example 3.4. The join of two bipolar neutrosophic graphs G_1 and G_2 is shown in Fig.3.

Proposition 3.3. *Let G_1 and G_2 be two bipolar neutrosophic graphs then $G_1 + G_2$ is also a bipolar neutrosophic graph.*

Definition 3.6. Let C_1, C_2, D_1 and D_2 be the bipolar neutrosophic subsets of X_1, X_2, E_1 and E_2, respectively. We denote the cartesian product of G_1 and G_2 by the pair $G_1 \square G_2 = (C_1 \square C_2, D_1 \square D_2)$ and define as:

$$t^p_{C_1\Box C_2}(x)=t^p_{C_1}(x)\wedge t^p_{D_2}(x),\qquad I^p_{C_1\Box C_2}(x)=I^p_{C_1}(x)\vee I^p_{C_2}(x),\qquad f^p_{C_1\Box C_2}(x)=f^p_{C_1}(x)\vee f^p_{C_2}(x),$$
$$t^n_{C_1\Box D_2}(x)=t^n_{C_1}(x)\vee t^n_{C_2}(x),\qquad I^n_{C_1\Box C_2}(x)=I^n_{C_1}(x)\wedge I^n_{C_2}(x),\qquad f^n_{C_1\Box C_2}(x)=f^n_{C_1}(x)\wedge f^n_{C_2}(x).$$

for all $x\in X_1\times X_2$.

1. $t^p_{D_1\Box D_2}((x_1,x_2)(x_1,y_2))=t^p_{C_1}(x_1)\wedge t^p_{D_2}(x_2y_2)$, $t^n_{D_1\Box D_2}((x_1,x_2)(x_1,y_2))=t^p_{C_1}(x_1)\vee t^p_{D_2}(x_2y_2)$, for all $x_1\in X_1, x_2y_2\in E_2$,
2. $t^p_{D_1\Box D_2}((x_1,x_2)(y_1,x_2))=t^p_{D_1}(x_1y_1)\wedge t^p_{C_2}(x_2)$, $t^n_{D_1\Box D_2}((x_1,x_2)(y_1,x_2))=t^p_{D_1}(x_1y_1)\vee t^p_{C_2}(x_2)$, for all $x_1y_1\in E_1, x_2\in X_2$,
3. $I^p_{D_1\Box D_2}((x_1,x_2)(x_1,y_2))=I^p_{C_1}(x_1)\vee I^p_{D_2}(x_2y_2)$, $I^n_{D_1\Box D_2}((x_1,x_2)(x_1,y_2))=I^p_{C_1}(x_1)\wedge I^p_{D_2}(x_2y_2)$, for all $x_1\in X_1, x_2y_2\in E_2$,
4. $I^p_{D_1\Box D_2}((x_1,x_2)(y_1,x_2))=I^p_{D_1}(x_1y_1)\vee I^p_{C_2}(x_2)$, $I^n_{D_1\Box D_2}((x_1,x_2)(y_1,x_2))=I^p_{D_1}(x_1y_1)\wedge I^p_{C_2}(x_2)$, for all $x_1y_1\in E_1, x_2\in X_2$,
5. $f^p_{D_1\Box D_2}((x_1,x_2)(x_1,y_2))=f^p_{C_1}(x_1)\vee f^p_{D_2}(x_2y_2)$, $f^n_{D_1\Box D_2}((x_1,x_2)(x_1,y_2))=f^p_{C_1}(x_1)\wedge f^p_{D_2}(x_2y_2)$, for all $x_1\in X_1, x_2y_2\in E_2$,
6. $f^p_{D_1\Box D_2}((x_1,x_2)(y_1,x_2))=f^p_{D_1}(x_1y_1)\vee f^p_{C_2}(x_2)$, $f^n_{D_1\Box D_2}((x_1,x_2)(y_1,x_2))=f^p_{D_1}(x_1y_1)\wedge f^p_{C_2}(x_2)$, for all $x_1y_1\in E_1, x_2\in X_2$.

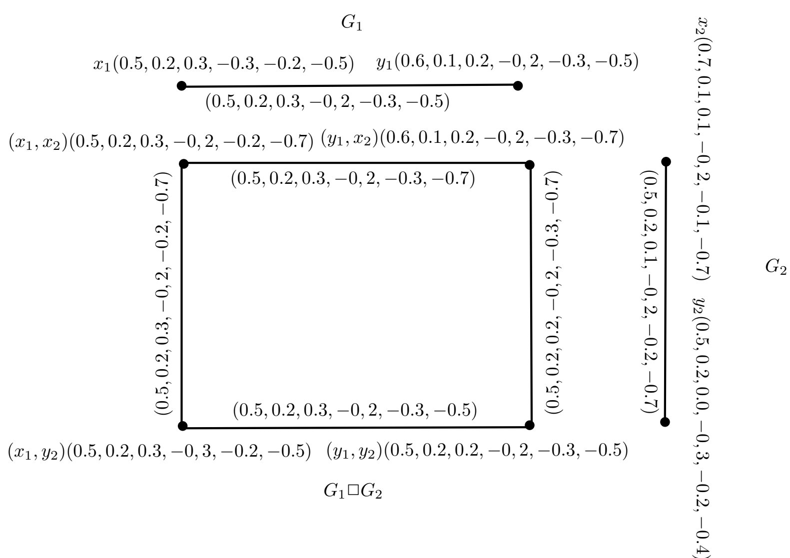


Figure 4: Cartesian product $G_1\Box G_2$

Example 3.5. The Cartesian product of two bipolar neutrosophic graphs G_1 and G_2 is shown in Fig.4.

Proposition 3.4. *Let G_1 and G_2 be two bipolar neutrosophic graphs then $G_1\Box G_2$ is also a bipolar neutrosophic graph.*

Definition 3.7. Let C_1, C_2, D_1 and D_2 be the bipolar neutrosophic subsets of X_1, X_2, E_1 and E_2, respectively. We denote the direct product of $G_1 = (C_1, D_1)$ and $G_2 = (C_2, D_2)$ by the pair $G_1 \times G_2 = (C_1 \times C_2, D_1 \times D_2)$ and define the membership degrees as

$$\begin{aligned} t^p_{C_1\times C_2}(x) &= t^p_{C_1}(x) \wedge t^p_{D_2}(x), & I^p_{C_1\times C_2}(x) &= I^p_{C_1}(x) \vee I^p_{C_2}(x), & f^p_{C_1\times C_2}(x) &= f^p_{C_1}(x) \vee f^p_{C_2}(x), \\ t^n_{C_1\times C_2}(x) &= t^n_{C_1}(x) \vee t^n_{C_2}(x), & I^n_{C_1\times C_2}(x) &= I^n_{C_1}(x) \wedge I^n_{C_2}(x), & f^n_{C_1\times C_2}(x) &= f^n_{C_1}(x) \wedge f^n_{C_2}(x), \end{aligned}$$

for all $\quad x \in X_1 \times X_2$.

1. $t^p_{D_1\times D_2}((x_1, x_2)(y_1, y_2)) = t^p_{D_1}(x_1y_1) \wedge t^p_{D_2}(x_2y_2)$, $\quad t^n_{D_1\times D_2}((x_1, x_2)(y_1, y_2)) = t^p_{D_1}(x_1y_1) \vee t^p_{D_2}(x_2y_2)$, for all $x_1y_1 \in E_1, x_2y_2 \in E_2$,

2. $I^p_{D_1\times D_2}((x_1, x_2)(y_1, y_2)) = I^p_{D_1}(x_1y_1) \vee I^p_{D_2}(x_2y_2)$, $\quad I^n_{D_1\times D_2}((x_1, x_2)(y_1, y_2)) = I^p_{D_1}(x_1y_1) \wedge I^p_{D_2}(x_2y_2)$, for all $x_1y_1 \in E_1, x_2y_2 \in E_2$,

3. $f^p_{D_1\times D_2}((x_1, x_2)(y_1, y_2)) = f^p_{D_1}(x_1y_1) \vee f^p_{D_2}(x_2y_2)$, $\quad f^n_{D_1\times D_2}((x_1, x_2)(y_1, y_2)) = f^p_{D_1}(x_1y_1) \wedge f^p_{D_2}(x_2y_2)$, for all $x_1y_1 \in E_1, x_2y_2 \in E_2$.

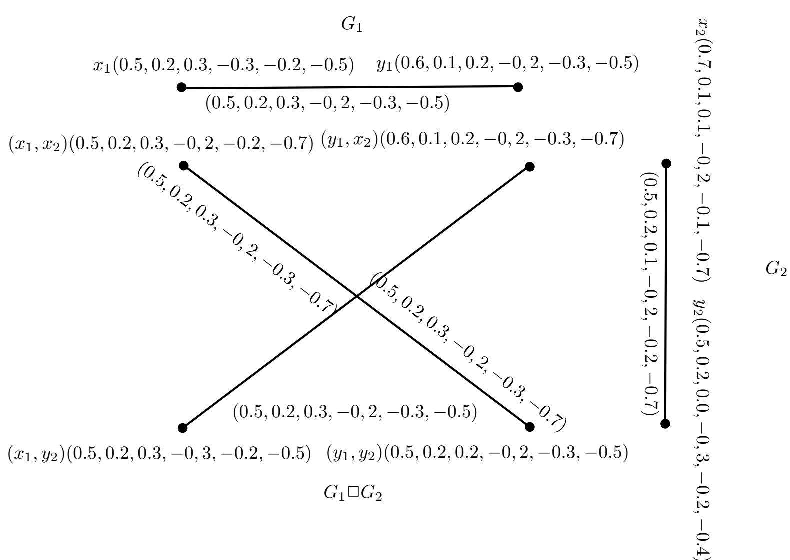


Figure 5: Direct product $G_1 \times G_2$

Example 3.6. The direct product of two bipolar neutrosophic G_1 and G_2 graphs is shown in Figure. 5

Proposition 3.5. *Let G_1 and G_2 be two bipolar neutrosophic graphs then $G_1 \times G_2$ is also a bipolar neutrosophic graph.*

Definition 3.8. Let C_1, C_2, D_1 and D_2 be the bipolar neutrosophic subsets of X_1, X_2, E_1 and E_2, respectively. We denote the strong product of G_1 and G_2 by the pair $G_1 \boxtimes G_2 = (C_1 \boxtimes C_2, D_1 \boxtimes D_2)$ and define as:

$$\begin{aligned} t^p_{C_1\boxtimes C_2}(x) &= t^p_{C_1}(x) \wedge t^p_{D_2}(x), & I^p_{C_1\boxtimes C_2}(x) &= I^p_{C_1}(x) \vee I^p_{C_2}(x), & f^p_{C_1\boxtimes C_2}(x) &= f^p_{C_1}(x) \vee f^p_{C_2}(x), \\ t^n_{C_1\boxtimes D_2}(x) &= t^n_{C_1}(x) \vee t^n_{C_2}(x), & I^n_{C_1\boxtimes C_2}(x) &= I^n_{C_1}(x) \wedge I^n_{C_2}(x), & f^n_{C_1\boxtimes C_2}(x) &= f^n_{C_1}(x) \wedge f^n_{C_2}(x), \end{aligned}$$

for all $\quad x \in X_1 \times X_2$.

1. $t^p_{D_1 \boxtimes D_2}((x_1,x_2)(x_1,y_2)) = t^p_{C_1}(x_1) \wedge t^p_{D_2}(x_2y_2), \quad t^n_{D_1 \boxtimes D_2}((x_1,x_2)(x_1,y_2)) = t^p_{C_1}(x_1) \vee t^p_{D_2}(x_2y_2),$ for all $x_1 \in X_1, x_2y_2 \in E_2$,

2. $t^p_{D_1 \boxtimes D_2}((x_1,x_2)(y_1,x_2)) = t^p_{D_1}(x_1y_1) \wedge t^p_{C_2}(x_2), \quad t^n_{D_1 \boxtimes D_2}((x_1,x_2)(y_1,x_2)) = t^p_{D_1}(x_1y_1) \vee t^p_{C_2}(x_2),$ for all $x_1y_1 \in E_1, x_2 \in X_2$,

3. $t^p_{D_1 \boxtimes D_2}((x_1,x_2)(y_1,y_2)) = t^p_{D_1}(x_1y_1) \wedge t^p_{D_2}(x_2y_2), \quad t^n_{D_1 \boxtimes D_2}((x_1,x_2)(y_1,y_2)) = t^p_{D_1}(x_1y_1) \vee t^p_{D_2}(x_2y_2),$ for all $x_1y_1 \in E_1, x_2y_2 \in E_2$,

4. $I^p_{D_1 \boxtimes D_2}((x_1,x_2)(x_1,y_2)) = I^p_{C_1}(x_1) \vee I^p_{D_2}(x_2y_2), \quad I^n_{D_1 \boxtimes D_2}((x_1,x_2)(x_1,y_2)) = I^p_{C_1}(x_1) \wedge I^p_{D_2}(x_2y_2),$ for all $x_1 \in X_1, x_2y_2 \in E_2$,

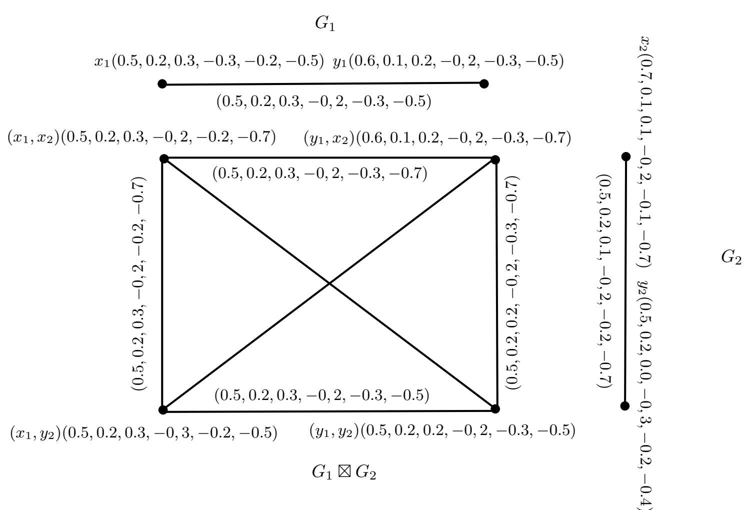
5. $I^p_{D_1 \boxtimes D_2}((x_1,x_2)(y_1,x_2)) = I^p_{D_1}(x_1y_1) \vee I^p_{C_2}(x_2), \quad I^n_{D_1 \boxtimes D_2}((x_1,x_2)(y_1,x_2)) = I^p_{D_1}(x_1y_1) \wedge I^p_{C_2}(x_2),$ for all $x_1y_1 \in E_1, x_2 \in X_2$,

6. $I^p_{D_1 \boxtimes D_2}((x_1,x_2)(y_1,y_2)) = I^p_{D_1}(x_1y_1) \vee I^p_{D_2}(x_2y_2), \quad I^n_{D_1 \boxtimes D_2}((x_1,x_2)(y_1,y_2)) = I^p_{D_1}(x_1y_1) \wedge I^p_{D_2}(x_2y_2),$ for all $x_1y_1 \in E_1, x_2y_2 \in E_2$,

7. $f^p_{D_1 \boxtimes D_2}((x_1,x_2)(x_1,y_2)) = f^p_{C_1}(x_1) \vee f^p_{D_2}(x_2y_2), \quad f^n_{D_1 \boxtimes D_2}((x_1,x_2)(x_1,y_2)) = f^p_{C_1}(x_1) \wedge f^p_{D_2}(x_2y_2),$ for all $x_1 \in X_1, x_2y_2 \in E_2$,

8. $f^p_{D_1 \boxtimes D_2}((x_1,x_2)(y_1,x_2)) = f^p_{D_1}(x_1y_1) \vee f^p_{C_2}(x_2), \quad f^n_{D_1 \boxtimes D_2}((x_1,x_2)(y_1,x_2)) = f^p_{D_1}(x_1y_1) \wedge f^p_{C_2}(x_2),$ for all $x_1y_1 \in E_1, x_2 \in X_2$,

9. $f^p_{D_1 \boxtimes D_2}((x_1,x_2)(y_1,y_2)) = f^p_{D_1}(x_1y_1) \vee f^p_{D_2}(x_2y_2), \quad f^n_{D_1 \boxtimes D_2}((x_1,x_2)(y_1,y_2)) = f^p_{D_1}(x_1y_1) \wedge f^p_{D_2}(x_2y_2),$ for all $x_1y_1 \in E_1, x_2y_2 \in E_2$.



$(x_1, x_2)(y_1, y_2)(0.5, 0.2, 0.3, -0.2, -0.3, -0.7)$

$(y_1, x_2)(x_1, y_2)(0.5, 0.2, 0.3, -0.2, -0.3, -0.7)$

Figure 6: Strong product of G_1 and G_2

Example 3.7. The strong product of two bipolar neutrosophic G_1 and graphs G_2 is shown in Fig.6

Proposition 3.6. *The strong product of any two bipolar neutrosophic graphs is a bipolar neutrosophic graph.*

Definition 3.9. The *complement* of a bipolar neutrosophic graph $G = (C, D)$ is defined as a pair $G^c = (C^c, D^c)$ such that, for all $x \in X$ and $xy \in E$,

$$t^p_{C^c}(x) = t^p_C(x), \quad I^p_{C^c}(x) = I^p_C(x), \quad f^p_{C^c}(x) = f^p_C(x), \quad t^n_{C^c}(x) = t^n_C(x), \quad I^n_{C^c}(x) = I^n_C(x), \quad f^p_{C^c}(x) = f^p_C(x).$$

$$\begin{aligned} t^p_{D^c}(xy) &= t^p_C(x) \wedge t^p_C(y) - t^p_D(xy), \quad I^p_{D^c}(xy) = I^p_C(x) \vee I^p_C(y) - I^p_D(xy), \quad f^p_{D^c}(xy) = f^p_C(x) \vee f^p_C(y) - f^p_D(xy), \\ t^n_{D^c}(xy) &= t^n_C(x) \vee t^n_C(y) - t^n_D(xy), \quad I^n_{D^c}(xy) = I^n_C(x) \wedge I^n_C(y) - I^n_D(xy), \quad f^n_{D^c}(xy) = f^n_C(x) \wedge f^n_C(y) - f^n_D(xy). \end{aligned}$$

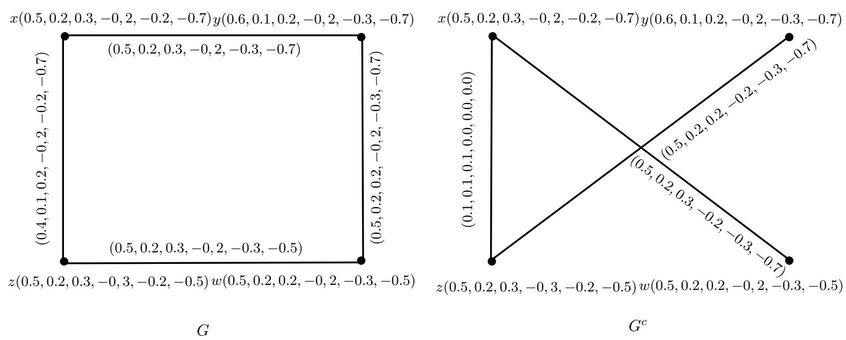


Figure 7: Complement of G

Example 3.8. An example of complement of a bipolar neutrosophic G is shown in Fig.7

Remark 3.1. A bipolar neutrosophic graph G is said to be *self complementary* if $G = G^c$.

Definition 3.10. A bipolar neutrosophic graph $G = (C, D)$ is known as *strong bipolar neutrosophic graph* if

$$\begin{aligned} t^p_{D^c}(xy) &= t^p_C(x) \wedge t^p_C(y), \quad I^p_{D^c}(xy) = I^p_C(x) \vee I^p_C(y), \quad f^p_{D^c}(xy) = f^p_C(x) \vee f^p_C(y), \\ t^n_{D^c}(xy) &= t^n_C(x) \vee t^n_C(y), \quad I^n_{D^c}(xy) = I^n_C(x) \wedge I^n_C(y), \quad f^n_{D^c}(xy) = f^n_C(x) \wedge f^n_C(y), \quad \text{for all} \quad xy \in E. \end{aligned}$$

Theorem 3.1. *Let G_1 and G_2 be strong bipolar neutrosophic graphs then $G_1 + G_2$, $G_1 \Box G_2$, $G_1 \times G_2$ and $G_1 \boxtimes G_2$ are strong bipolar neutrosophic graphs.*

Theorem 3.2. *If $G_1 \Box G_2$, $G_1 \times G_2$ and $G_1 \boxtimes G_2$ are strong bipolar neutrosophic graphs then G_1 and G_2 are also strong.*

Definition 3.11. A bipolar neutrosophic graph $G = (C, D)$ is known as complete bipolar neutrosophic graph if

$$\begin{aligned} t^p_{D^c}(xy) &= t^p_C(x) \wedge t^p_C(y), \quad I^p_{D^c}(xy) = I^p_C(x) \vee I^p_C(y), \quad f^p_{D^c}(xy) = f^p_C(x) \vee f^p_C(y), \\ t^n_{D^c}(xy) &= t^n_C(x) \vee t^n_C(y), \quad I^n_{D^c}(xy) = I^n_C(x) \wedge I^n_C(y), \quad f^n_{D^c}(xy) = f^n_C(x) \wedge f^n_C(y), \quad \text{for all} \quad x, y \in X. \end{aligned}$$

Theorem 3.3. *Let G be a self complementary bipolar neutrosophic graph then*

$$\sum_{x\neq y} t_D^p(xy) = \frac{1}{2}\sum_{x\neq y} t_C^p(x)\wedge t_C^p(y), \quad \sum_{x\neq y} I_D^p(xy) = \frac{1}{2}\sum_{x\neq y} I_C^p(x)\vee I_C^p(y), \quad \sum_{x\neq y} f_D^p(xy) = \frac{1}{2}\sum_{x\neq y} f_C^p(x)\vee f_C^p(y),$$
$$\sum_{x\neq y} t_D^n(xy) = \frac{1}{2}\sum_{x\neq y} t_C^n(x)\wedge t_C^n(y), \quad \sum_{x\neq y} I_D^n(xy) = \frac{1}{2}\sum_{x\neq y} I_C^n(x)\vee I_C^n(y), \quad \sum_{x\neq y} f_D^n(xy) = \frac{1}{2}\sum_{x\neq y} f_C^n(x)\vee f_C^n(y).$$

Theorem 3.4. *Let $G = (C, D)$ be a bipolar neutrosophic graph such that for all $x, y \in X$,*

$$t_{D^c}^p(xy) = \frac{1}{2}(t_C^p(x)\wedge t_C^p(y)), \quad I_{D^c}^p(xy) = \frac{1}{2}(I_C^p(x)\vee I_C^p(y)), \quad f_{D^c}^p(xy) = \frac{1}{2}(f_C^p(x)\vee f_C^p(y)),$$
$$t_{D^c}^n(xy) = \frac{1}{2}(t_C^n(x)\vee t_C^n(y)), \quad I_{D^c}^n(xy) = \frac{1}{2}(I_C^n(x)\wedge I_C^n(y)), \quad f_{D^c}^n(xy) = \frac{1}{2}(f_C^n(x)\wedge f_C^n(y)).$$

Then G is self complementary bipolar neutrosophic graph.

Proof. Let $G^c = (C^c, D^c)$ be the complement of bipolar neutrosophic graph $G = (C, D)$, then by definition. 3.9,

$$\begin{aligned}
t_{D^c}^p(xy) &= t_C^p(x)\wedge t_C^p(y) - t_D^p(xy)\\
t_{D^c}^p(xy) &= t_C^p(x)\wedge t_C^p(y) - \frac{1}{2}(t_C^p(x)\wedge t_C^p(y))\\
t_{D^c}^p(xy) &= \frac{1}{2}(t_C^p(x)\wedge t_C^p(y))\\
t_{D^c}^p(xy) &= t_D^p(xy)
\end{aligned}$$

$$\begin{aligned}
t_{D^c}^n(xy) &= t_C^n(x)\vee t_C^n(y) - t_D^n(xy)\\
t_{D^c}^n(xy) &= t_C^n(x)\vee t_C^n(y) - \frac{1}{2}(t_C^n(x)\vee t_C^n(y))\\
t_{D^c}^n(xy) &= \frac{1}{2}(t_C^n(x)\vee t_C^n(y))\\
t_{D^c}^n(xy) &= t_D^n(xy)
\end{aligned}$$

Similarly, it can be proved that $I_{D^c}^p(xy) = I_D^p(xy)$, $I_{D^c}^n(xy) = I_D^n(xy)$, $f_{D^c}^p(xy) = f_D^p(xy)$ and $f_{D^c}^n(xy) = f_D^n(xy)$. Hence, G is self complementary. □

Definition 3.12. The *degree* of a vertex x in a bipolar neutrosophic graph is denoted by $\deg(x)$ and defined by the $6-$tuple as,

$$\begin{aligned}
deg(x) &= (deg_t^p(x),\ deg_I^p(x),\ deg_f^p(x),\ deg_t^n(x),\ deg_I^n(x),\ deg_f^n(x)),\\
&= (\sum_{xy\in E} t_D^p(xy), \sum_{xy\in E} I_D^p(xy), \sum_{xy\in E} f_D^p(xy), \sum_{xy\in E} t_D^n(xy), \sum_{xy\in E} I_D^n(xy), \sum_{xy\in E} f_D^n(xy)).
\end{aligned}$$

The term degree is also referred as *neighborhood degree*.

Definition 3.13. The closed neighborhood degree of a vertex x in a bipolar neutrosophic graph is denoted by $\deg[x]$ and defined as,

$$\begin{aligned}
deg[x] &= (deg_t^p[x],\ deg_I^p[x],\ deg_f^p[x],\ deg_t^n[x],\ deg_I^n[x],\ deg_f^n[x],\\
&= (deg_t^p(x) + t_C^p(x),\ deg_I^p(x) + I_C^p(x),\ deg_f^p(x) + f_C^p(x),\ deg_t^n(x) + t_C^n(x),\ deg_I^n(x) + t_C^n(x),\\
&\qquad deg_f^n(x) + f_C^p(x)).
\end{aligned}$$

Definition 3.14. A bipolar neutrosophic graph G is known as a *regular* bipolar neutrosophic graph if all vertices of G have same degree.

Definition 3.15. A bipolar neutrosophic graph G is known as a *totally regular* bipolar neutrosophic graph if all vertices of G have same closed neighborhood degree.

Theorem 3.5. *A complete bipolar neutrosophic graph is totally regular.*

Theorem 3.6. *Let $G = (C, D)$ be a bipolar neutrosohic graph then $C = (t^p, I^p, f^p, t^n, I^n, f^n)$ is a constant function if and only if the following statements are equivalent:*
(1) *G is a regular bipolar neutrosophic graph,*
(2) *G is totally regular bipolar neutrosophic graph.*

Proof. Assume that C is a constant function and for all $x \in X$,

$$t_C^p(x) = k_t,\ I_C^p(x) = k_I,\ f_C^p(x) = k_f,\ t_C^n(x) = k_t^{'},\ I_C^n(x) = k_I^{'},\ f_C^n(x) = k_f^{'}$$

where, $k_t,\ k_I,\ k_f,\ k_t^{'},\ k_I^{'},\ k_f^{'}$ are constants.
(1) $\Rightarrow$ (2) Suppose that G is a regular bipolar neutrosophic graph and $\deg(x) = (p_t,\ p_I,\ p_f,\ n_t,\ n_I,\ n_f)$ for all $x \in X$.
Now consider,
$\deg[x] = (\deg_t^p(x) + t_C^p(x), \deg_I^p(x) + I_C^p(x), \deg_f^p(x) + f_C^p(x), \deg_t^n(x) + t_C^n(x), \deg_I^n(x) + t_C^n(x), \deg_f^n(x) + f_C^p(x)) = (p_t + k_t, p_I + k_I, p_f + k_f, n_t + k_t^{'}, n_I + k_I^{'}, n_f + k_f^{'})$ for all $x \in X$.
Hence G is totally regular bipolar neutrosophic graph.
(2) $\Rightarrow$ (1) Suppose that G is totally regular bipolar neutrosophic graph and for all $x \in X$ $deg[x] = (p_t^{'},\ p_I^{'},\ p_f^{'},\ n_t^{'},\ n_I^{'},\ n_f^{'})$.

$$(\deg_t^p(x) + k_t, \deg_I^p(x) + k_I, \deg_f^p(x) + k_f, \deg_t^n(x) + k_t^{'}, \deg_I^n(x) + k_I^{'}, \deg_f^n(x) + k_f^{'}) = (p_t^{'},\ p_I^{'},\ p_f^{'},\ n_t^{'},\ n_I^{'},\ n_f^{'}),$$
$$\deg_t^p(x), \deg_I^p(x), \deg_f^p(x), \deg_t^n(x), \deg_I^n(x), \deg_f^n(x)) + (k_t, k_I, k_f, k_t^{'}, k_I^{'}, k_f^{'}) = (p_t^{'},\ p_I^{'},\ p_f^{'},\ n_t^{'},\ n_I^{'},\ n_f^{'}),$$
$$(\deg_t^p(x), \deg_I^p(x), \deg_f^p(x), \deg_t^n(x), \deg_I^n(x), \deg_f^n(x)) = (p_t^{'} - k_t,\ p_I^{'} - k_I,\ p_f^{'} - k_f,\ n_t^{'} - k_t^{'},\ n_I^{'} - k_I^{'},\ n_f^{'} - k_f^{'}),$$

for all $x \in X$. Thus G is a regular bipolar neutrosophic graph.
Conversely, assume that the conditions are equivalent. Let $\deg(x) = (c_t,\ c_I,\ c_f,\ d_t,\ d_I,\ d_f)$ and $\deg[x] = (c_t^{'},\ c_I^{'},\ c_f^{'},\ d_t^{'},\ d_I^{'},\ d_f^{'})$.
Since by definition of closed neighborhood degree for all $x \in X$,
$\deg[x] = \deg(x) + (t_C^p(x), I_C^p(x), f_C^p(x), t_C^n(x), I_C^n(x), f_C^p(x))$,
$\Rightarrow\ (t_C^p(x), I_C^p(x), f_C^p(x), t_C^n(x), I_C^n(x), f_C^p(x)) = \deg[x] - \deg(x)$,
$\Rightarrow\ (t_C^p(x), I_C^p(x), f_C^p(x), t_C^n(x), I_C^n(x), f_C^p(x)) = (c_t^{'} - c_t, c_I^{'} - c_I, c_f^{'} - c_f, d_t^{'} - d_t, d_I^{'} - d_I, d_f^{'} - d_f)$,
for all $x \in X$. Hence $C = (c_t^{'} - c_t, c_I^{'} - c_I, c_f^{'} - c_f, d_t^{'} - d_t, d_I^{'} - d_I, d_f^{'} - d_f)$, a constant function which completes the proof. □

Definition 3.16. A bipolar neutrosophic graph G is said to be *irregular* if at least two vertices have distinct degrees. If all vertices do not have same closed neighborhood degrees then G is known as *totally irregular* bipolar neutrosophic graph.

Theorem 3.7. *Let $G = (C, D)$ be a bipolar neutrosophic graph and $C = (t^p_C, I^p_C, f^p_C, t^n_C, I^n_C, f^n_C)$ be a constant function then G is an irregular bipolar neutrophic graph if and only if G is a totally irregular bipolar neutrophic graph.*

Proof. Assume that G is an irregular bipolar neutrosophic graph then at least two vertices of G have distinct degrees. Let x and y be two vertices such that $\deg(x) = (r_1, r_2, r_3, s_1, s_2, s_3)$ and $\deg(y) = (r_1^{'}, r_2^{'}, r_3^{'}, s_1^{'}, s_2^{'}, s_3^{'})$ where, $r_i \neq r_i^{'}$, for some $i = 1, 2, 3$.
Since, C is a constant function let $C = (k_1, k_2, k_3, l_1, l_2, l_3)$. Therefore,

$$\begin{aligned} \deg[x] &= \deg(x) + (k_1,\ k_2,\ k_3,\ l_1,\ l_2,\ l_3) \\ \deg[x] &= (r_1 + k_1, r_2 + k_2, r_3 + k_3, s_1 + l_1, s_2 + l_2, s_3 + l_3) \\ \text{and} \quad \deg[y] &= (r_1^{'} + k_1, r_2^{'} + k_2, r_3^{'} + k_3, s_1^{'} + l_1, s_2^{'} + l_2, s_3^{'} + l_3). \end{aligned}$$

Clearly $r_i + k_i \neq r_i^{'} + k_i$, for some $i = 1, 2, 3$ therefore x and y have distinct closed neighborhood degrees. Hence G is a totally irregular bipolar neutrosophic graph.
The converse part is similar. □

4 Domination in bipolar neutrosophic graph

Definition 4.1. Let $G = (C, D)$ be a bipolar neutrosophic graph and x, y are two vertices in G then we say that x *dominates* y if

$$\begin{aligned} &t^p_D(xy) = t^p_C(x) \wedge t^p_C(y), && I^p_D(xy) = I^p_C(x) \vee I^p_C(y), && f^p_D(xy) = f^p_C(x) \vee f^p_C(y), \\ &t^n_D(xy) = t^n_C(x) \vee t^n_C(y), && I^n_D(xy) = I^n_C(x) \wedge I^n_C(y), && f^n_D(xy) = f^n_C(x) \wedge f^n_C(y). \end{aligned}$$

A subset $D^{'} \subseteq X$ is called a *dominating set* if for each $y \in X \setminus D^{'}$ there exists $x \in D^{'}$ such that x dominates y. A dominating set $D^{'}$ is said to be minimal if for any $x \in D^{'}$, $D^{'} \setminus \{x\}$ is not a dominating set. The minimum cardinality among all minimal dominating sets is called a *domination number* of G, denoted by $\lambda(G)$.

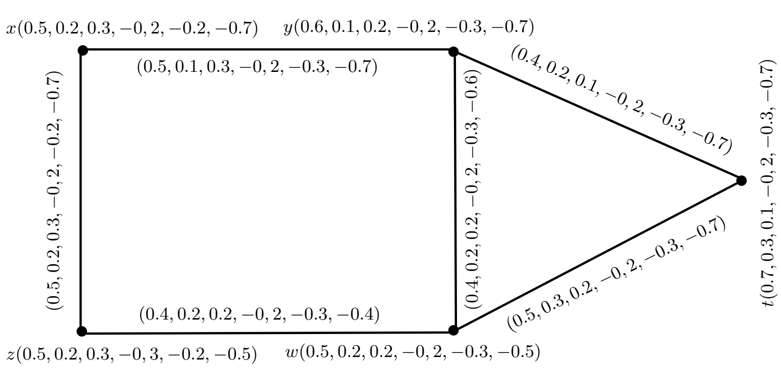


Figure 8: Bipolar neutrosophic graph G.

Example 4.1. Consider a bipolar neutrosophic graph as shown in Fig.8. The set $\{x, w\}$ is a minimal dominating set and $\lambda(G) = 2$

Theorem 4.1. *Let G_1 and G_2 be two bipolar neutrosophic graphs with $D_1^{'}$ and $D_2^{'}$ as dominating sets then $\lambda(G_1 \cup G_2) = \lambda(G_1) + \lambda(G_2) - |D_1^{'} \cap D_2^{'}|$.*

Proof. Since $D_1^{'}$ and $D_2^{'}$ are dominating sets of G_1 and G_2, $D_1^{'} \cup D_2^{'}$ is a dominating set of $G_1 \cup G_2$. Therefore, $\lambda(G_1 \cup G_2) \leq |D_1^{'} \cup D_2^{'}|$. It only remains to show that $D_1^{'} \cup D_2^{'}$ is the minimum dominating set. On contrary, assume that $D^{'} = D_1^{'} \cup D_2^{'} \setminus \{x\}$ is a minimum dominating set of $G_1 \cup G_2$. There are two cases,
Case 1. If $x \in D_1^{'}$ and $x \notin D_2^{'}$, then $D_1^{'} \setminus \{x\}$ is not a dominating set of G_1 which implies that $D_1^{'} \cup D_2^{'} \setminus \{x\} = D^{'}$ is not a dominating set of $G_1 \cup G_2$. A contradiction, hence $D_1^{'} \cup D_2^{'}$ is a minimum dominating set and

$$\begin{aligned} & \lambda(G_1 \cup G_2) = |D_1^{'} \cup D_2^{'}|, \\ \Rightarrow \quad & \lambda(G_1 \cup G_2) = \lambda(G_1) + \lambda(G_2) - |D_1^{'} \cap D_2^{'}|. \end{aligned}$$

Case 2. If $x \in D_2^{'}$ and $x \notin D_1^{'}$, same contradiction can be obtained. □

Theorem 4.2. *Let G_1 and G_2 be two bipolar neutrosophic graphs with $X_1 \cap X_2 \neq \emptyset$ then,*

$$\lambda(G_1 + G_2) = \min\{\lambda(G_1), \lambda(G_1), 2\}.$$

Proof. Let $x_1 \in X_1$ and $x_2 \in X_2$, sine of $G_1 + G_2$ is a bipolar neutrosophic graph, we have

$$\begin{aligned} t^p_{D_1+D_2}(x_1x_2) &= t^p_{C_1+C_2}(x_1) \wedge t^p_{C_1+C_2}(x_2), & t^n_{D_1+D_2}(x_1x_2) &= t^n_{C_1+C_2}(x_1) \vee t^n_{C_1+C_2}(x_2) \\ I^p_{D_1+D_2}(x_1x_2) &= I^p_{C_1+C_2}(x_1) \vee I^p_{C_1+C_2}(x_2), & I^n_{D_1+D_2}(x_1x_2) &= I^n_{C_1+C_2}(x_1) \wedge I^n_{C_1+C_2}(x_2) \\ f^p_{D_1+D_2}(x_1x_2) &= f^p_{C_1+C_2}(x_1) \vee f^p_{C_1+C_2}(x_2), & f^n_{D_1+D_2}(x_1x_2) &= f^n_{C_1+C_2}(x_1) \wedge f^n_{C_1+C_2}(x_2). \end{aligned}$$

Hence any vertex of G_1 dominates all vertices of G_2 and similarly any vertex of G_2 dominates all vertices of G_1. So, $\{x_1, x_2\}$ is a dominating set of $G_1 + G_2$. Let D be a minimum dominating set of $G_1 + G_2$, then D is one of the following forms:

1. $D = D_1$ where, $\lambda(G_1) = |D_1|$,
2. $D = D_2$ where, $\lambda(G_2) = |D_2|$,
3. $D = \{x_1, x_2\}$ where, $x_1 \in V_1$ and $x_2 \in V_2$. $\{x_1\}$ and $\{x_2\}$ are not dominating sets of G_1 or G_2, respectively.

Hence,

$$\lambda(G_1 + G_2) = \min\{\lambda(G_1), \lambda(G_1), 2\}.$$

□

Theorem 4.3. *Let $G_1 = (C_1, D_1)$ and $G_2 = (C_2, D_2)$ be two bipolar neutrosophic graphs. If for $x_1 \in V_1$, $C_1(x_1) > \mathbf{0}$ where, $\mathbf{0} = (0, 0, 0, 0, 0, 0)$, and x_2 dominates y_2 in G_2 then (x_1, y_1) dominates (x_1, y_2) in $G_1 \square G_2$.*

Proof. Since x_2 dominates y_2 therefore,

$$\begin{aligned}
t^p_{D_2}(x_2y_2) &= t^p_{C_2}(x_2) \wedge t^p_{C_2}(y_2), \quad I^p_{D_2}(x_2y_2) = I^p_{C_2}(x_2) \vee I^p_{C_2}(y_2), \quad f^p_{D_2}(x_2y_2) = f^p_{C_2}(x_2) \vee f^p_{C_2}(y_2),\\
t^n_{D_2}(x_2y_2) &= t^n_{C_2}(x_2) \vee t^n_{C_2}(y_2), \quad I^n_{D_2}(x_2y_2) = I^n_{C_2}(x_2) \wedge I^n_{C_2}(y_2), \quad f^n_{D_2}(x_2y_2) = f^n_{C_2}(x_2) \wedge f^n_{C_2}(y_2).
\end{aligned}$$

For $x_1 \in X_1$, take $(x_1, y_2) \in X_1 \times X_2$. By definition 3.6,

$$\begin{aligned}
t^p_{D_1\Box D_2}((x_1,x_2)(x_1,y_2)) &= t^p_{C_1}(x_1) \wedge t^p_{D_2}(x_2y_2),\\
&= t^p_{C_1}(x_1) \wedge \{t^p_{C_2}(x_2) \wedge t^p_{C_2}(y_2)\},\\
&= \{t^p_{C_1}(x_1) \wedge t^p_{C_2}(x_2)\} \wedge \{t^p_{C_1}(x_1) \wedge t^p_{C_2}(y_2)\},\\
&= t^p_{C_1\Box C_2}(x_1,x_2) \wedge t^p_{C_1\Box C_2}(x_1,y_2).
\end{aligned}$$

$$\begin{aligned}
t^n_{D_1\Box D_2}((x_1,x_2)(x_1,y_2)) &= t^n_{C_1}(x_1) \vee t^n_{D_2}(x_2y_2),\\
&= t^n_{C_1}(x_1) \vee \{t^n_{C_2}(x_2) \vee t^n_{C_2}(y_2)\},\\
&= \{t^n_{C_1}(x_1) \vee t^n_{C_2}(x_2)\} \vee \{t^n_{C_1}(x_1) \vee t^n_{C_2}(y_2)\},\\
&= t^n_{C_1\Box C_2}(x_1,x_2) \vee t^n_{C_1\Box C_2}(x_1,y_2).
\end{aligned}$$

Similarly, it can be proved that

$$\begin{aligned}
I^p_{D_1\Box D_2}((x_1,x_2)(x_1,y_2)) &= I^p_{C_1\Box C_2}(x_1,x_2) \vee I^p_{C_1\Box C_2}(x_1,y_2),\\
I^n_{D_1\Box D_2}((x_1,x_2)(x_1,y_2)) &= I^n_{C_1\Box C_2}(x_1,x_2) \wedge I^n_{C_1\Box C_2}(x_1,y_2),\\
f^p_{D_1\Box D_2}((x_1,x_2)(x_1,y_2)) &= f^p_{C_1\Box C_2}(x_1,x_2) \vee f^p_{C_1\Box C_2}(x_1,y_2),\\
f^n_{D_1\Box D_2}((x_1,x_2)(x_1,y_2)) &= f^n_{C_1\Box C_2}(x_1,x_2) \wedge f^n_{C_1\Box C_2}(x_1,y_2).
\end{aligned}$$

Hence (x_1, x_2) dominates (x_1, y_2) and the proof is complete. □

Proposition 4.1. *Let G_1 and G_2 be two bipolar neutrosophic graphs. If for $y_2 \in X_2$, $C_2(y_2) > \boldsymbol{0}$ where, $\boldsymbol{0} = (0, 0, 0, 0, 0, 0)$, and x_1 dominates y_1 in G_1 then (x_1, y_2) dominates (y_1, y_2) in $G_1\Box G_2$.*

Theorem 4.4. *Let $D_1^{'}$ and $D_2^{'}$ be the minimal dominating sets of $G_1 = (C_1, D_1)$ and $G_2 = (C_2, D_2)$, respectively. Then $D_1^{'} \times X_2$ and $X_1 \times D_2^{'}$ are dominating sets of $G_1\Box G_2$ and*

$$\lambda(G_1\Box G_2) \leq |D_1^{'} \times X_2| \wedge |X_1 \times D_2^{'}|. \tag{4.1}$$

Proof. To prove inequality 4.1, we need to show that $D_1^{'} \times X_2$ and $X_1 \times D_2^{'}$ are dominating sets of $G_1\Box G_2$. Let $(y_1, y_2) \notin D_1^{'} \times X_2$ then, $y_1 \notin D_1^{'}$. Since $D_1^{'}$ is a dominating set of G_1, there exists $x_1 \in D_1^{'}$ that dominates y_1. By theorem 4.1, (x_1, y_2) dominates (y_1, y_2) in $G_1\Box G_2$. Since (y_1, y_2) was taken to be arbitrary therefore, $D_1^{'} \times X_2$ is a dominating set of $G_1\Box G_2$. Similarly, $X_1 \times D_2^{'}$ is a dominating set if $G_1\Box G_2$. Hence the proof. □

Theorem 4.5. *Let $D_1^{'}$ and $D_2^{'}$ be the dominating sets of $G_1 = (C_1, D_1)$ and $G_2 = (C_2, D_2)$, respectively. Then $D_1^{'} \times D_2^{'}$ is a dominating set of the direct product $G_1 \times G_2$ and*

$$\lambda(G_1 \times G_2) = |D_1^{'} \times D_2^{'}|. \tag{4.2}$$

Proof. Let $(y_1, y_2) \in X_1 \times X_2 \setminus D_1' \times D_2'$ then $y_1 \in X_1 \setminus D_1'$ and $y_2 \in X_2 \setminus D_2'$. Since, D_1' and D_2' are dominating sets there exist $x_1 \in D_1'$ and $x_2 \in D_2'$ such that x_1 dominates y_1 and x_2 dominates y_2. Consider,

$$\begin{aligned} t^p_{D_1 \times D_2}((x_1, x_2)(y_1, y_2)) &= t^p_{D_1}(x_1y_1) \wedge t^p_{D_2}(x_2y_2), \\ &= \{t^p_{C_1}(x_1) \wedge t^p_{C_1}(y_1)\} \wedge \{t^p_{C_2}(x_2) \wedge t^p_{C_2}(y_2)\}, \\ &= \{t^p_{C_1}(x_1) \wedge t^p_{C_2}(x_2)\} \wedge \{t^p_{C_1}(y_1) \wedge t^p_{C_2}(y_2)\}, \\ &= t^p_{C_1 \times C_2}(x_1, x_2) \wedge t^p_{C_1 \times C_2}(y_1, y_2). \end{aligned}$$

It shows that (x_1, x_2) dominates (y_1, y_2). Since (x_1, x_2) was taken to be arbitrary therefore, every element of $X_1 \times X_2 \setminus D_1' \times D_2'$ is dominated by some element of $D_1' \times D_2'$. It only remains to show that $D_1' \times D_2'$ is a minimal dominating set. On contrary assume that $|D'| = D_1' \times D_2' \setminus \{(z_1, z_2)\}$ is a minimal dominating set of $G_1 \times G_2$ such that $|D'| < |D_1' \times D_2'|$. Let $(z_1, z_2) \in D_1' \times D_2'$ such that $(z_1, z_2) \notin D'$ i.e., $z_1 \in D_1'$ and $z_2 \in D_2'$ then there exist $z_1' \in X_1 \setminus D_1'$ and $z_2' \in X_2 \setminus D_2'$ which are only dominated by z_1 and z_2, respectively. Hence no element other than (z_1, z_2) dominates (z_1', z_2') so $(z_1, z_2) \in D'$. A contradiction, thus $\lambda(G_1 \times G_2) = |D_1 \times D_2|$. □

Corollary 4.1. *Let G_1 and G_2 be two bipolar neutrosophic graphs. If x_1 dominates y_1 in G_1 and x_2 dominates y_2 in G_2 then (x_1, y_1) dominates (x_2, y_2) in $G_1 \times G_2$.*

Definition 4.2. Two vertices x and y in a bipolar neutrosophic graph are said to be *independent* if

$$\begin{aligned} &t^p_D(xy) < t^p_C(x) \wedge t^p_C(y), \qquad I^p_D(xy) < I^p_C(x) \vee I^p_C(y), \qquad f^p_D(xy) < f^p_C(x) \vee f^p_C(y), \\ &t^n_D(xy) > t^n_C(x) \vee t^n_C(y), \qquad I^n_D(xy) > I^n_C(x) \wedge I^n_C(y), \qquad f^n_D(xy) > f^n_C(x) \wedge f^n_C(y). \end{aligned} \tag{4.3}$$

A subset N of X is said to *bipolar neutrosophic independent set* if for all $x, y \in N$ equations 4.3 are satisfied. A bipolar neutrosophic independent set is said to be *maximal* if for every $z \in X \setminus N$, $N \cup \{z\}$ is not a bipolar neutrosophic independent set. The maximal cardinality among all maximal independent sets is called *bipolar neutrosophic independent number.* It is denoted by $\alpha(G)$.

Theorem 4.6. *Let G_1 and G_2 be two bipolar neutrosophic graphs of the graphs $G_1^* = (X_1, E_1)$ and $G_2^* = (X_2, E_2)$ such that $X_1 \cap X_2 = \emptyset$ then $\alpha(G_1 \cup G_2) = \alpha(G_1) + \alpha(G_2)$.*

Proof. Let N_1 and N_2 be maximal bipolar neutrosophic independent sets of G_1 and G_2. Since $N_1 \cap N_2 = \emptyset$ therefore, $N_1 \cup N_2$ is a maximal independent set of $G_1 \cup G_2$. Hence $\alpha(G_1 \cup G_2) = \alpha(G_1) + \alpha(G_2)$. □

Theorem 4.7. *Let G_1 and G_2 be two bipolar neutrosophic graphs then $\alpha(G_1 + G_2) = \alpha(G_1) \vee \alpha(G_2)$.*

Proof. Let N_1 and N_2 be maximal bipolar neutrosophic independent sets. Since every vertex of G_1 dominates every vertex of G_2 in $G_1 + G_1$ Hence, maximal bipolar neutrosophic independent set of $G_1 + G_2$ is either N_1 or N_2. Thus, $\alpha(G_1 + G_2) = \alpha(G_1) \vee \alpha(G_2)$. □

Theorem 4.8. *Let N_1 and N_2 be the maximal bipolar neutrosophic independent sets of G_1 and G_2, respectively and $X_1 \cap X_2 = \emptyset$. Then $\alpha(G_1 \Box G_2) = |N_1 \times N_2| + |N|$ where, $N = \{(x_i, y_i) : x_i \in X_1 \setminus N_1, y_i \in X_2 \setminus N_2, x_ix_{i+1} \in E_1, y_iy_{i+1} \in E_2, \ \ i = 1, 2, 3, \cdots\}$.*

Proof. N_1 and N_2 are maximal independent sets of G_1 and G_2, respectively. Clearly, $N_1 \times N_2$ is an independent set of $G_1 \Box G_2$ since no vertex of $N_1 \times N_2$ dominates any other vertex of $N_1 \times N_2$.
Consider the set of vertices $N = \{(x_i, y_i) : x_i \in X_1 \setminus N_1, y_i \in X_2 \setminus N_2, x_i x_{i+1} \in E_1, y_i y_{i+1} \in E_2\}$. It can be seen that no vertex $(x_i, y_i) \in N$ for each $i = 1, 2, 3, \cdots$ dominates $(x_{i+1}, y_{i+1}) \in N$ for each $i = 1, 2, 3, \cdots$. Hence $N' = (N_1 \times N_2) \cup N$ is an independent set of $G_1 \Box G_2$.
Assume that $S = N' \cup \{(x_i, y_j)\}$, for some $i \neq j$, $x_i \in X_1 \setminus N_1$ and $y_j \in X_2 \setminus N_2$, is a maximal independent set. Without loss of generality, assume that j=i+1 then (x_i, y_j) is dominated by (x_i, y_i). A contradiction, hence N' is a maximal independent set and $\alpha(G_1 \Box G_2) = |N'| = |N_1 \times N_2| + |N|$ □

Theorem 4.9. *Let D_1 and D_2 be two minimal dominating sets of G_1 and G_2 , respectively. Then $X_1 \times X_2 \setminus D_1 \times D_2$ is a maximal independent set of $G_1 \times G_2$ and $\alpha(G_1 \times G_2) = n_1 n_2 - \lambda(G_1 \times G_2)$ where, n_1 and n_2 are the number of vertices in G_1 and G_2.*

The proof is obvious.

Theorem 4.10. *A bipolar neutrosophic independent set of a bipolar neutrosophic graph $G = (C, D)$ is maximal if and only if it is independent and dominating.*

Proof. Let N be a maximal independent set of G, then for every $x \in X \setminus N$, $N \cup \{x\}$ is not an independent set. For every vertex $x \in X \setminus N$, there exists some $y \in N$ such that

$$
\begin{aligned}
t_D^p(xy) &= t_C^p(x) \wedge t_C^p(y), \quad & I_D^p(xy) &= I_C^p(x) \vee I_C^p(y), \quad & f_D^p(xy) &= f_C^p(x) \vee f_C^p(y),\\
t_D^n(xy) &= t_C^n(x) \vee t_C^n(y), \quad & I_D^n(xy) &= I_C^n(x) \wedge I_C^n(y), \quad & f_D^n(xy) &= f_C^n(x) \wedge f_C^n(y).
\end{aligned}
$$

Thus y dominates x and hence N is both independent and dominating set.
Conversely, assume that D is both independent and dominating set but not maximal independent set, so there exists a vertex $x \in X \setminus N$ such that $N \cup \{x\}$ is an independent set i.e., no vertex in N dominates x, a contradiction to the fact that N is a dominating set. Hence N is maximal. □

Theorem 4.11. *Every maximal independent set in a bipolar neutrosophic set is a minimal dominating set.*

Proof. Let N be a maximal independent set in a bipolar neutrosophic graph then by theorem 4.10, N is a dominating set. Suppose N is not a minimal dominating set, there exists at least one $y \in N$ for which $N \setminus \{y\}$ is a dominating set. But if $N \setminus \{y\}$ dominates $X \setminus \{N \setminus \{y\}\}$, then at least one vertex in $N \setminus \{y\}$ dominates y. A contradiction to the fact that N is a bipolar neutrosophic independent set of G. Hence N is a minimal dominating set. □

5 Applications

In this section, we present a method for the identification of risk in decision support systems. The method is explained by an example for prevention of accidental hazards in chemical industry. The application of domination in bipolar neutrosophic graphs is given for the construction of transmission stations.

(1) An outranking approach for safety analysis using bipolar neutrosophic sets

The proposed methodology can be implemented in various fields in different ways e.g., multi-criteria decision making problems with bipolar neutrosophic information. However, our main focus is the identification of risk assessments in industry which is described in the following steps.
The bipolar neutrosophic information consists of a group of risks\alternatives $R = \{r_1, r_2, \cdots, r_n\}$ evaluated on the basis of criteria $C = \{c_1, c_2, \cdots, c_m\}$. Here $r_i,\ i = 1, 2, \cdots, n$ is the possibility for the criteria $c_k,\ k = 1, 2, \cdots, m$ and r_{ik} are in the form of bipolar neutrosophic values. This method is suitable if we have a small set of data and experts are able to evaluate the data in the form of bipolar neutrosophic information. Take the values of r_{ik} as $r_{ik} = (t^p_{ik}, I^p_{ik}, f^p_{ik}, t^n_{ik}, I^n_{ik}, f^n_{ik})$.
Step 1. Construct the table of the given data.
Step 2. Determine the average values using the following bipolar neutrosophic average operator,

$$A_i = \frac{1}{n}(\sum_{j=1}^{m} t^p_{ij} - \prod_{j=1}^{m} t^p_{ij},\ \prod_{j=1}^{m} I^p_{ij},\ \prod_{j=1}^{m} f^p_{ij},\ \prod_{j=1}^{m} t^n_{ij},\ \sum_{j=1}^{m} I^n_{ij} - \prod_{j=1}^{m} I^n_{ij},\ \sum_{j=1}^{m} f^n_{ij} - \prod_{j=1}^{m} f^n_{ij}), \tag{5.1}$$

for each $i = 1, 2, \cdots, n$.
Step 3. Construct the weighted average matrix.
Choose the weight vector $\mathbf{w} = (w_1, w_2, \cdots, w_n)$. According to the weights for each alternative, the weighted average table can be calculated by multiplying each average value with the corresponding weight as:

$$\beta_i = A_i w_i, \qquad i = 1, 2, \cdots, n.$$

Step 4. Calculate the normalized value for each alternative\risk β_i using the formula,

$$\alpha_i = \sqrt{(t^p_i)^2 + (I^p_i)^2 + (f^p_i)^2 + (1 - t^n_i)^2 + (-1 + I^n_i)^2 + (-1 + f^n_i)^2}, \tag{5.2}$$

for each $i = 1, 2, \cdots, n$. The resulting table indicate the preference ordering of the alternatives\risks. The alternative\risk with maximum α_i value is most dangerous or more preferable.

Example 5.1. Chemical industry is a very important part of human society. These industries contain large amount of organic and inorganic chemicals and materials. Many chemical products have a high risk of fire due to flammable materials, large explosions and oxygen deficiency etc. These accidents can cause the death of employs, damages to building, destruction of machines and transports, economical losses etc. Therefore, it is very important to prevent these accidental losses by identifying the major risks of fire, explosions and oxygen deficiency.
A manager of a chemical industry Y wants to prevent such types of accidents that caused the major loss to company in the past. He collected data from witness reports, investigation teams and near by chemical industries and found that the major causes could be the chemical reactions, oxidizing materials, formation of toxic substances, electric hazards, oil spill, hydrocarbon gas leakage and energy systems. The witness reports, investigation teams and industries have different opinions. There is a bipolarity in people's thinking and judgement. The data can be considered as bipolar neutrosophic information. The bipolar neutrosophic information about company Y old accidents is given in Table 1:

Table 1: Bipolar neutrosophic Data

	Fire	Oxygen Deficiency	Large Explosion
Chemical Exposures	(0.5,0.7,0.2,-0.6,-0.3,-0.7)	(0.1,0.5,0.7,-0.5,-0.2,-0.8)	(0.6,0.2,0.3,-0.4,0.0,-0.1)
Oxidizing materials	(0.9,0.7,0.2,-0.8,-0.6,-0.1)	(0.3,0.5,0.2,-0.5,-0.5,-0.2)	(0.9,0.5,0.5,-0.6,-0.5,-0.2)
Toxic vapour cloud	(0.7,0.3,0.1,-0.4,-0.1,-0.3)	(0.6,0.3,0.2,-0.5,-0.3,-0.3)	(0.5,0.1,0.2,-0.6,-0.2,-0.2)
Electric Hazard	(0.3,0.4,0.2,-0.6,-0.3,-0.7)	(0.9,0.4,0.6,-0.1,-0.7,-0.5)	(0.7,0.6,0.8,-0.7,-0.5,-0.1)
Oil Spill	(0.7,0.5,0.3,-0.4,-0.2,-0.2)	(0.2,0.2,0.2,-0.7,-0.4,-0.4)	(0.9,0.2,0.7,-0.1,-0.6,-0.8)
Hydrocarbon gas leakage	(0.5,0.3,0.2,-0.5,-0.2,-0.2)	(0.3,0.2,0.3,-0.7,-0.4,-0.3)	(0.8,0.2,0.1,-0.1,-0.9,-0.2)
Ammonium Nitrate	(0.3,0.2,0.3,-0.5,-0.6,-0.5)	(0.9,0.2,0.1,0.0,-0.6,-0.5)	(0.6,0.2,0.1,-0.2,-0.3,-0.5)

By applying the bipolar neutrosophic average operator 5.1 on Table 1, the average values are given in Table.2.

Table 2: Bipolar neutrosophic average normalized table

	Average Value
Chemical Exposures	(0.39,0.023,0.014,-0.04,-0.167,-0.515)
Oxidizing materials	(0.619,0.032,0.001,-0.08,-0.483,-0.165)
Toxic vapour cloud	(0.53,0.003,0.001,-0.04,-0.198,-0.261)
Electric Hazard	(0.570,0.032,0.032,-0.014,-0.465,-0.422)
Oil Spill	(0.558,0.007,0.014,-0.009,-0.384,-0.445)
Hydrocarbon gas leakage	(0.493,0.004,0.002,-0.011,-0.543,-0.229)
Ammonium Nitrate	(0.546,0.003,0.001,0.0,-0.464,-0.417)

With regard to the weight vector $(0.35, 0.80, 0.30, 0.275, 0.65, 0.75, 0.50)$ associated to each cause of accident, the weighted average values are obtained by multiplying each average value with corresponding weight and are given in Table 3.

Table 3: Bipolar neutrosophic weighted average table

	Average Value
Chemical Exposures	(0.1365,0.0081,0.0049,-0.0140,-0.0585,-0.1803)
Oxidizing materials	(0.4952,0.0256,0.0008,-0.0640,-0.3864,-0.1320)
Toxic vapour cloud	(0.1590,0.0009,0.0003,-0.012,-0.0594,-0.0783)
Electric Hazard	(0.2850,0.0160,0.0160,-0.0070,-0.2325,-0.2110)
Oil Spill	(0.1535,0.0019,0.0039,-0.0025,-0.1056,-0.1224)
Hydrocarbon gas leakage	(0.3205,0.0026,0.0013,-0.0072,-0.3530,-0.1489)
Ammonium Nitrate	(0.4095,0.0023,0.0008,0.0,-0.3480,-0.2110)

Using formula 5.2, the resulting normalized values are shown in Table 4.

Table 4: Normalized values

	Normalized value
Chemical Exposures	1.5966
Oxidizing materials	1.5006
Toxic vapour cloud	1.6540
Electric Hazard	1.6090
Oil Spill	1.4938
Hydrocarbon gas leakage	1.6036
Ammonium Nitrate	1.5089

The accident possibilities can be placed in the following order: Toxic vapour cloud $\succ$ Electric Hazard $\succ$ Hydrocarbon gas leakage $\succ$ Chemical Exposures $\succ$ Ammonium Nitrate $\succ$ Oxidizing materials $\succ$ Oil Spill where, the symbol $\succ$ represents partial ordering of objects. It can be easily seen that the formation of toxic vapour clouds, electrical and energy systems and hydrocarbon gas leakage are the major dangers to the chemical industry. There is a very little danger due to oil spill. Chemical Exposures, oxidizing materials and ammonium nitrate has an average accidental danger. Therefore, industry needs special precautions to prevent the major hazards that could happen due the formation of toxic vapour clouds.

(2) Domination in bipolar neutrosophic graphs

Domination has a wide variety of applications in communication networks, coding theory, fixing surveillance cameras, detecting biological proteins and social networks etc. Consider the example of a TV channel that wants to set up transmission stations in a number of cities such that every city in the country get access to the channel signals from at least one of the stations. To reduce the cost for building large stations it is required to set up minimum number of stations. This problem can be represented by a neutrosophic graph in which vertices represent the cities and there is an edge between two cities if they can communicate directly with each other. Consider the network of ten cities $\{C_1, C_2, \cdots, C_{10}\}$. In the bipolar neutrosophic graph, the degree of each vertex represents the level of signals it can transmit to other cities and the bipolar neutrosophic value of each edge represents the degree of communication between the cities. The graph is shown in Figure.9. $D = \{C_8, C_{10}\}$ is the minimum dominating set. It is concluded that building only two large transmitting stations in C_8 and C_{10}, a high economical benefit can be achieved.

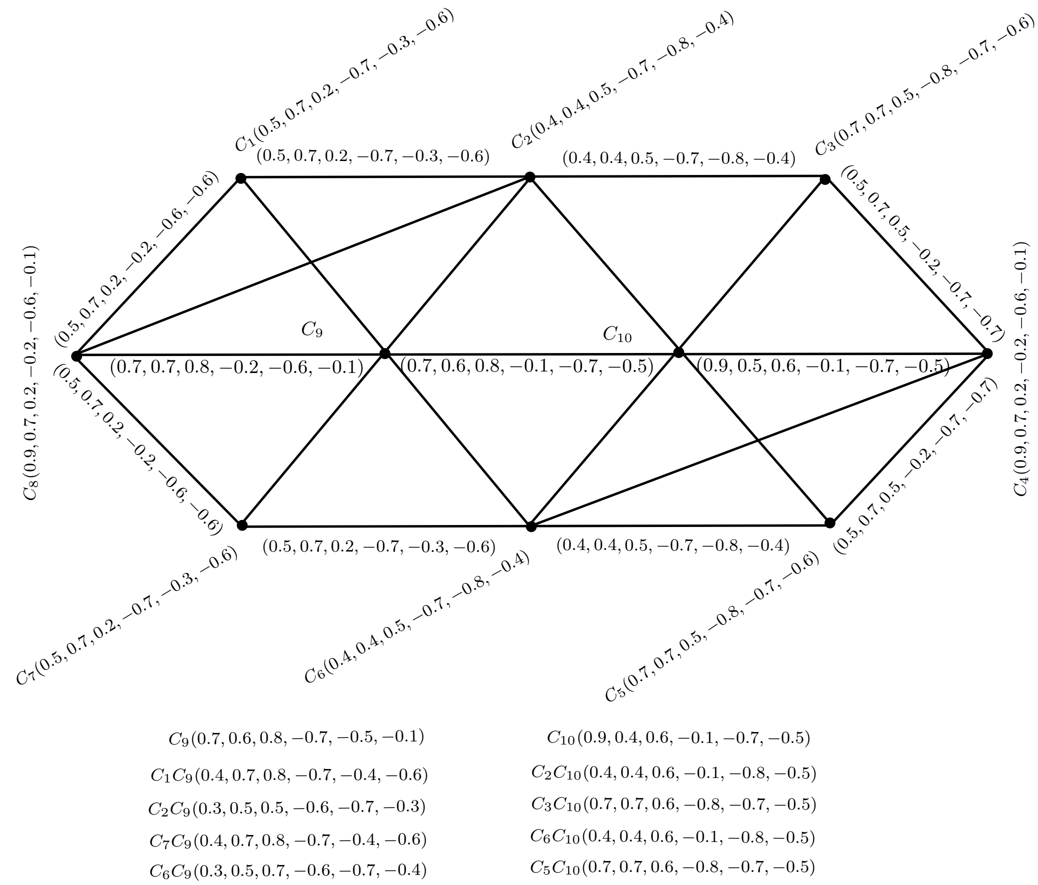


Figure 9: Domination in bipolar neutrosophic graph

6 Conclusion

Bipolar fuzzy graph theory has many applications in science and technology, especially in the fields of neural networks, operations research, artificial intelligence and decision making. A bipolar neutrosophic graph is a generalization of the notion bipolar fuzzy graph. We have introduced the idea of bipolar neutrosophic graph and operations on bipolar neutrosophic graphs. Some properties of regular, totally regular, irregular and totally irregular bipolar neutrosophic graphs are discussed in detail. We have investigated the dominating and independent sets of certain graph products. Two applications of bipolar neutrosophic sets and bipolar neutrosophic graphs are studied in chemical industry and construction of radio channels. We are planing to extend our research work to (1) $m-$polar fuzzy neutrosophic graphs, (2) Roughness in neutrosophic graphs, (3) $m-$polar fuzzy soft neutrosophic graphs.

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