

**A GENERAL THEOREM FOR THE CHARACTERIZATION  
OF N PRIME NUMBERS SIMULTANEOUSLY**

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1. **ABSTRACT.** This article presents a necessary and sufficient theorem for  $N$  numbers, coprime two by two, to be prime simultaneously.

It generalizes V. Popa's theorem [3], as well as I. Cucurezeanu's theorem ([1], p. 165), Clement's theorem, S. Patrizio's theorems [2], etc.

Particularly, this General Theorem offers different characterizations for twin primes, for quadruple primes, etc.

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2. **INTRODUCTION.** It is evidently the following:

Lemma 1. Let  $A, B$  be nonzero integers. Then:

$AB \equiv 0 \pmod{pB} \Leftrightarrow A \equiv 0 \pmod{p} \Leftrightarrow A/p$  is an integer.

Lemma 2. Let  $(p,q) \sim 1$ ,  $(a,p) \sim 1$ ,  $(b,q) \sim 1$ .

Then:

$A \equiv 0 \pmod{p}$  and  $B \equiv 0 \pmod{q} \Leftrightarrow aAq + bBp \equiv 0 \pmod{pq} \Leftrightarrow aA + bBp/q \equiv 0 \pmod{p} \Leftrightarrow aA/p + bB/q$  is an integer.

Proof:

The first equivalence:

We have  $A = K_1p$  and  $B = K_2q$ , with  $K_1, K_2 \in \mathbb{Z}$ , hence

$$aAq + bBp = (aK_1 + bK_2)pq.$$

Reciprocal:  $aAq + bBp = Kpq$ , with  $K \in \mathbb{Z}$ , it results that  $aAq \equiv 0 \pmod{p}$  and  $bBp \equiv 0 \pmod{q}$ , but from our assumption we find  $A \equiv 0 \pmod{p}$  and  $B \equiv 0 \pmod{q}$ .

The second and third equivalence result from lemma 1.

By induction we extend this lemma to

LEMMA 3: Let  $p_1, \dots, p_n$  be coprime integers two by two, and let  $a_1, \dots, a_n$  be integer numbers such that  $(a_i, p_i) \sim 1$  for all  $i$ . Then:

$$\begin{aligned} & A_1 \equiv 0 \pmod{p_1}, \dots, A_n \equiv 0 \pmod{p_n} \Leftrightarrow \\ & \Leftrightarrow \sum_{i=1}^n a_i A_i \cdot \prod_{j \neq i} p_j \equiv 0 \pmod{p_1 \dots p_n} \Leftrightarrow \\ & \Leftrightarrow (P/D) \cdot \sum_{i=1}^n (a_i A_i / p_i) \equiv 0 \pmod{P/D}, \end{aligned}$$

where  $P = p_1 \dots p_n$  and  $D$  is a divisor of  $p$ ,  $\Leftrightarrow$

$$\Leftrightarrow \sum_{i=1}^n a_i A_i / p_i \text{ is an integer.}$$

3. From this last lemma we can immediately find a

**GENERAL THEOREM:**

Let  $P_{ij}$ ,  $1 \leq i \leq n$ ,  $1 \leq j \leq m_i$ , be coprime integers two by two, and let  $r_1, \dots, r_n, a_1, \dots, a_n$  be integer numbers such that  $a_i$  be coprime with  $r_i$  for all  $i$ .

The following conditions are considered:

(i)

$p_{i_1}, \dots, p_{i_{n_1}}$ , are simultaneously prime if and only

if  $c_i \equiv 0 \pmod{r_i}$ , for all  $i$ .

Then:

The numbers  $p_{ij}$ ,  $1 \leq i \leq n$ ,  $1 \leq j \leq m_i$ , are simultaneously prime if and only if

$$(*) \quad (R/D) \cdot \sum_{i=1}^n (a_i c_i / r_i) \equiv 0 \pmod{R/D},$$

where  $P = \prod_{i=1}^n r_i$  and  $D$  is a divisor of  $R$ .

Remark.

Often in the conditions (i) the module  $r_i$  is equal to

$\prod_{j=1}^{m_i} p_{ij}$ , or to a divisor of it, and in this case the

relation of the General Theorem becomes:

$$(P/D) \cdot \sum_{i=1}^n (a_i c_i / \prod_{j=1}^{m_i} p_{ij}) \equiv 0 \pmod{P/D},$$

where

$$P = \prod_{i,j=1}^{n, m_i} p_{ij} \text{ and } D \text{ is a divisor of } P.$$

### Corollaries.

We easily obtain that our last relation is equivalent to:

$$\sum_{i=1}^n a_i c_i \cdot (P / \prod_{j=1}^{m_i} p_{ij}) \equiv 0 \pmod{P},$$

and

$$\sum_{i=1}^n (a_i c_i / \prod_{j=1}^{m_i} p_{ij}) \text{ is an integer,}$$

etc.

The imposed restrictions for the numbers  $p_{ij}$  from the General Theorem are very wide, because if there were two non-coprime distinct numbers, then at least one from these would not be prime, hence the  $m_1 + \dots + m_n$  numbers might

not be prime.

The General Theorem has many variants in accordance with the assigned values for the parameters  $a_1, \dots, a_n$ , and  $r_1, \dots, r_m$ , the parameter  $D$ , as well as in accordance with the congruences  $c_1, \dots, c_n$  which characterize either a prime number or many other prime numbers simultaneously.

We can start from the theorems (conditions  $c_i$ ) which characterize a single prime number [see Wilson, Leibniz, Smarandache [4], or Simionov ( $p$  is prime if and only if  $(p-k)!(k-1)! - (-1)^k \equiv 0 \pmod{p}$ ), when  $p \geq k \geq 1$ ; here, it is preferable to take  $k = \lfloor (p+1)/2m \rfloor$ , where  $\lfloor x \rfloor$  represents the greatest integer number  $\leq x$ , in order that the number  $(p-k)!(k-1)!$  be the smallest possible] for obtaining, by means of the General Theorem, conditions  $c_j'$ , which characterize many prime numbers simultaneously. Afterwards, from the conditions  $c_i, c_j'$ , using the General Theorem again, we find new conditions  $c_n''$  which characterize prime numbers simultaneously. And this method can be continued analogically.

#### Remarks.

Let  $m_i = 1$  and  $c_i$  represent the Simionov's theorem for all  $i$ .

(a) If  $D = 1$  it results in V. Popa's theorem, which generalizes in its turn Cucurezeanu's theorem and the last one generalizes in its turn Clement's theorem!

(b) If  $D = P/p_2$  and choosing conveniently the parameters  $a_i, k_i$  for  $i = 1, 2, 3$ , it results in S. Patrizio's theorem.

**Several EXAMPLES:**

1. Let  $p_1, p_2, \dots, p_n$  be positive integers  $> 1$ , coprime integers two by two, and  $1 \leq k_i \leq p_i$  for all  $i$ .

Then:

$p_1, p_2, \dots, p_n$  are simultaneously prime if and only if:

$$(T) \quad \sum_{i=1}^n [(p_i - k_i)! (k_i - 1)! - (-1)^{k_i}] \cdot \prod_{j \neq i} p_j \equiv 0 \pmod{p_1 p_2 \dots p_n}$$

or

$$(U) \quad \left( \sum_{i=1}^n [(p_i - k_i)! (k_i - 1)! - (-1)^{k_i}] \cdot \prod_{j \neq i} p_j \right) / (p_{s+1} \dots p_n) \equiv 0 \pmod{p_1 \dots p_s}$$

or

$$(V) \quad \sum_{i=1}^n [(p_i - k_i)! (k_i - 1)! - (-1)^{k_i}] p_j / p_i \equiv 0 \pmod{p_j}$$

or

$$(W) \quad \sum_{i=1}^n [(p_i - k_i)! (k_i - 1)! - (-1)^{k_i}] / p_i \text{ is an integer.}$$

2. Another relation example (using the first theorem from [4]):  $p$  is a prime positive integer if and only if  $(p-3)! - (p-1)/2 \equiv 0 \pmod{p}$ .

$$\sum_{i=1}^n [(p_i - 3)! - (p_i - 1)/2] p_i / p_i \equiv 0 \pmod{p_i}.$$

3. The odd numbers  $p$  and  $p + 2$  are twin prime if and only if:

$$(p-1)! (3p+2) + 2p + 2 \equiv 0 \pmod{p(p+2)}$$

or

$$(p-1)! (p-2) - 2 \equiv 0 \pmod{p(p+2)}$$

or

$[(p-1)! + 1] / p + [(p-1)! - 2 + 1] / (p+2)$  is an integer.

These twin prime characterizations differ from Clement's theorem  $((p-1)! - 4 + p + 4 \equiv 0 \pmod{p(p+2)})$ .

4. Let  $(p, p+k) \sim 1$ , then:  $p$  and  $p + k$  are prime simultaneously if and only if  $(p-1)! (p+k) + (p+k-1)! p +$

$2p + k \equiv 0 \pmod{p(p+k)}$ , which differs from I.

Cucurezeanu's theorem ([1], p. 165):  $k \cdot k! [(p-1)!+1] + [k! - (-1)^k] p \equiv 0 \pmod{p(p+k)}$ .

5. Look at a characterization of a quadruple of primes for  $p, p + 2, p + 6, p + 8$ :  $[(p-1)!+1]/p + [(p-1)!2!+1]/(p+2) + [(p-1)!6!+1]/(p+6) + [(p-1)!8!+1]/(p+8)$  be an integer.

6. For  $p - 2, p, p + 4$  coprime integers two by two, we find the relation:  $(p-1)!+p[(p-3)!+1]/(p-2)+p[(p+3)!+1]/(p+4) \equiv -1 \pmod{p}$ , which differ from S. Patrizio's theorem ( $8[(p+3)!/(p+4)] + 4[(p-3)!/(p-2)] \equiv -11 \pmod{p}$ ).

#### References:

- [1] Cucurezeanu, I., "Probleme de aritmetica si teoria numerelor", Ed. Tehnica, Bucuresti, 1966.
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[4] Smarandache, Florentin, "Criterii ca un numar natural sa fie prim", *Gazeta Matematica*, Nr. 2, pp. 49-52; 1981; see *Mathematical Reviews (USA)*: 83a: 10007.

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