FIVE INTEGER NUMBER ALGORITHMS TO SOLVE LINEAR SYSTEMS

Florentin Smarandache, Ph D
Associate Professor
Chair of Department of Math & Sciences
University of New Mexico
200 College Road
Gallup, NM 87301, USA
E-mail: smarand@unm.edu

This chapter further extends the results obtained in chapters 4 and 5 (from linear equation to linear systems). Each algorithm is thoroughly proved and then an example is given.

Five integer number algorithms to solve linear systems are further given.

Algorithm 1. (Method of substitution)

(Although simple, this algorithm requires complex computations but is, nevertheless, easy to implement into a computer program).

Some integer number equation are initially solved (which is usually simpler) by means of one of the algorithms 4 or 5. (If there is an equation of the system which does not have integer systems, then the integer system does not have integer systems, then Stop.) The general integer solution of the equation will depend on $n-1$ integer number parameters (see [5]):

$$(p_1) \quad x_{i_1} = f_{i_1}^{(1)}\left(k_1^{(1)},\dots,k_{n-1}^{(1)}\right),\ i=\overline{1,n},$$

where all functions $f_{i_1}^{(1)}$ are linear and have integer number coefficients.

This general integer number system (p_1) is introduced into the other $m-1$ equations of the system. We obtain a new system of $m-1$ equations with $n-1$ unknown variables:

$$k_{i_1}^{(1)},\ i_1=\overline{1,n-1},$$

which is also to be solved with integer numbers. The procedure is similar. Solving a new equation, we obtain its general integer solution:

$$(p_2) \quad k_{i_2}^{(1)} = f_{i_2}^{(2)}\left(k_1^{(2)},\dots,k_{n-2}^{(2)}\right),\ i_2=\overline{1,n-1},$$

where all functions $f_{i_2}^{(2)}$ are linear, with integer number coefficients. (If, along this algorithm we come across an equation, which does not have integer solutions, then the initial system does not have integer solution. Stop.)

In the case that all solved equations had integer system at step (j), $1\le j\le r$ (r being of the same rank as the matrix associated to the system) then:

$$(p_j) \quad k_{i_j}^{(j-1)} = f_{i_j}^{(j)}\left(k_1^{(j)},\dots,k_{n-j}^{(j)}\right),\ i_j=\overline{1,n-j+1},$$

$f_{i_j}^{(j)}$ are linear functions and have integer number coefficients.

Finally, after r steps, and if all solved equations had integer solutions, we obtain the integer solution of one equation with $n-r+1$ unknown variables.

The system will have integer solutions if and only if in this last equation has integer solutions.

If it does, let its general integer solution be:

$$(p_r) \qquad k_{i_r}^{(r-1)} = f_{i_r}^{(r)} \; k_1^{(r)},...,k_{n-1}^{(r)} \;\;, \; i_r = \overline{1, n-r+1},$$

where all $f_{i_r}^{(r)}$ are linear functions with integer number coefficients.

We'll present now the reverse procedure as follows.

We introduce the values of $k_{i_r}^{(r-1)}$, $i_r = \overline{1, n-r+1}$, at step p_r in the values of

$$k_{i_{r-1}}^{(r-2)}, \; i_{r-1} = \overline{1, n-r+2}$$

from step (p_{r-1}).

It follows:

$$k_{i_{r-1}}^{(r-2)} = f_{i_{r-1}}^{(r-1)} \;\; f_1^{(r)} \;\; k_1^{(r)},...,k_{n-r}^{(r)} \;\; ,..., \; f_{n-r+1}^{(r)} \;\; k_1^{(r)},...,k_{n-r}^{(r)} \;\;\; = g_{i_{r-1}}^{(r-1)} \;\; k_1^{(r)},...,k_{n-r}^{(r)} \;\; ,$$

$i_{r-1} = \overline{1, n-r-1}$, from which it follows that $g_{i_r}^{(r-1)}$ are linear functions with integer number coefficients.

Then follows those (p_{r-2}) in (p_{r-e}) and so on, until we introduce the values obtained at step (p_2) in those from the step (p_1).

It will follow:

$$x_{i_j} = g_i^1 \left(k_1^{(r)},...,k_{n-r}^{(r)} \right)$$

notation $g_{i_1} \;\; k_1,...,k_{n-r} \;\;, \; i = \overline{1, n}$, with all g_{i_1} most obviously, linear functions with integer number coefficients (the notation was made for simplicity and aesthetical aspects). This is, thus, the general integer solution, of the initial system.

The correctness of algorithm 1.

The algorithm is finite because it has r steps on the forward way and $r-1$ steps on the reverse, $(r < +\infty)$. Obviously, if one equation of one system does not have (integer number) solutions then the system does not have solutions either.

Writing S for the initial system and S_j the system resulted from step (p_j), $1 \le j \le r-2$, it follows that passing from (p_j) to (p_{j+1}) we pass from a system S_j to a system S_{j+1} equivalent from the point of view of the integer number solution, i.e.

$$k_{i_j}^{(j-1)} = t_{i_j}^0, \;\; i_j = \overline{1, n-j+1},$$

which is a particular integer solution of the system S_j if and only if

$$k_{i_{j+1}}^{(j)} = h_{i_{j+1}}^0, \; i_{j+1} = \overline{1, n-j},$$

is a particular integer solution of the system S_{j+1} where

$$k_{i_{j+1}}^0 = f_{i_{j+1}}^{(j+1)} \left(t_1^0,...,t_{n-j+1}^0 \right), \; i_{j+1} = \overline{1, n-j}.$$

Hence, their general integer solutions are also equivalent (considering these substitutions). Such that, in the end, resolving the initial system S is equivalent with solving the equation (of the system consisting of one equation) S_{r-1} with integer number coefficients. It follows that the system S has integer number solution if and only if the systems S_j have integer number solution, $1 \le j \le r-1$.

Example 1. By means of algorithm 1, let us calculate the integer number solution of the following system:

$$(S) \qquad \begin{cases} 5x - 7y - 2z + 6w = 6 \\ -4x + 6y - 3z + 11w = 0 \end{cases}$$

Solution: We solve the first integer number equation. We obtain the general solution (see [4] or [5]):

$$(p_1) \qquad \begin{cases} x = t_1 + 2t_2 \\ y = t_1 \\ z = -t_1 + 5t_2 + 3t_3 - 3 \\ w = t_3 \end{cases}$$

where $t_1,\ t_2,\ t_3 \in \mathbf{Z}$.

Substituting in the second, we obtain the system:

$$(S_1) \qquad 5t_1 - 23t_2 + 2t_3 + 9 = 0.$$

Solving this integer equation we obtain its general integer solution:

$$(p_2) \qquad \begin{cases} t_1 = k_1 \\ t_2 = k_1 + 2k_2 + 1 \\ t_3 = 9k_1 + 23k_2 + 7 \end{cases}$$

where $k_1,\ k_2 \in \mathbf{Z}$.

The reverse way. Substituting (p_2) in (p_1) we obtain:

$$\begin{cases} x = 3k_1 + 4k_2 + 2 \\ y = k_1 \\ z = 31k_1 + 79k_2 + 23 \\ w = 9k_1 + 23k_2 + 7 \end{cases}$$

where $k_1,\ k_2 \in \mathbf{Z}$, which is the general integer solution of the initial system (S). Stop.

Algorithm 2.

Input

A linear system (1) without all $a_{ij} = 0$.

Output

We decide on the possibility of an integer solution of this system. If it is possible, we obtain its general integer solution.

Method

1. $t = 1,\ h = 1,\ p = 1$

2. (A) Divide each equation by the largest co-divisor of the coefficients of the unknown variables. If you do not obtain an integer quotient for at least one equation, then the system does not have integer solutions. Stop.

(B) If there is an inequality in the system, then the system does not have integer solutions. Stop.

(C) If repetition of more equations occurs, keep one and if an equation is an identity, remove it from the system.

3. If there is (i_0, j_0) such that $\left|a_{i_0 j_0}\right| = 1$ then obtain the value of the variable x_{j_0} from equation i_0; statement (T_t).

Substitute this statement (where possible) in the other equations of the system and in the statement (T_{t-1}), (H_h) and (P_p) for all i, h, and p. Consider $t := t+1$, remove equation (i_0) from the system. If there is no such a pair, go to step 5.

4. Does the system (left) have at least one unknown variable? If it does, consider the new data and go on to step 2. If it does not, write the general integer solution of the system substituting $k_1, k_2, ...$ for all variables from the right term of each expression which gives the value of the unknowns of the initial system. Stop.

5. Calculate

$$a = \min_{i, j_1, j_2} \left\{ |r| \,\middle|\, a_{ij_1} \equiv r \mod a_{ij_2} \ , \ 0 < |r| < \left|a_{ij_2}\right| \right\}$$

and determine the indices i, j_1, j_2 as well as the r for which this minimum can be calculated. (If there are more variables, choose one arbitrarily.)

6. Write: $x_{j_2} = t_h - \dfrac{a_{ij_1} - r}{a_{ij_2}} x_{ij_2}$, statement (H_h). Substitute this statement (where possible in all the equations of the system and in the statements (T_t), (H_h) and (P_p) for all t, h, and p.

7. (A) If $a \neq 1$, consider $x_{j_2} := t_h$, $h := h+1$, and go on to step 2.

(B) If $a = 1$, then obtain the value of x_{j_1} from the equation (i); statement (P_p). Substitute this statement (where possible in the other equations of the system and in the relations (T_t), (H_h) and (P_{p-1}) for all t, h, and p.

Remove the equation (i) from the system.

Consider $h := h+1$, $p := p+1$, and go back to step 4.

The correctness of algorithm 2. Let consider system (1).

Lemma 1. We consider the algorithm at step 5. Also, let

$$M = \left\{ |r|, \ a_{ij_1} \equiv r \mod a_{ij_2} \ , \ 0 < |r| < \left|a_{ij_2}\right|, \ i, j_1, j_2 = 1, 2, 3, ... \right\}.$$

Then $M \neq \varnothing$.

Proof:

Obviously, M is finite and $M \subset \mathrm{N}^*$. Then, M has a minimum if and only if $M \neq \varnothing$. We suppose, conversely, that $M = \varnothing$. Then

$$a_{ij_2} \equiv 0 (\mod a_{ij_2}), \ \forall \ i, j_1, j_2.$$

It follows as well that

$$a_{ij_2} \equiv 0 (\mod a_{ij_1}), \ \forall \ i, j_1, j_2.$$

That is

$$|a_{ij_1}| = |a_{ij_2}|,\ \forall\ i, j_1, j_2 .$$

We consider an i_0 arbitrary but fixed. It is clear that

$$(a_{i_0 1},...,a_{i_0 n}) : \ a_{i_0 j} \neq 0,\ \forall j$$

(because the algorithm has passed through the sub-steps 2(B) and 2(C). But, because it has also passed through step 3, it follows that

$$|a_{i_0 j}| \neq 1,\ \forall j ,$$

but as it previously passed through step 2(A), it would result that

$$|a_{i_0 j}| = 1,\ \forall j .$$

This contradiction shows that the assumption is false.

Lemma 2. Let's consider $a_{i_0 j_1} \equiv r (\bmod a_{ij_2})$. Substitute

$$x_{j_2} = t_h - \frac{a_{i_0 j} - r}{a_{i_0 j_2}} x_{j_1}$$

in system (A) obtaining system (B). Then

$$x_j = x_j^0,\ j = \overline{1,n}$$

is the particular integer solution of (A) if and only if

$$x_j = x_j^0,\ j \neq j_2 \ \text{and}\ t_h = x_{j_2}^0 - \frac{a_{i_0 j_1} - r}{a_{i_0 j_2}}$$

is the particular integer solution of (B).

Lemma 3. Let $a_1 \neq$ and a_2 obtained at step 5.

Then $0 < a_2 < a_1$

Proof:

It is sufficient to show that $a_1 < |a_{ij}|,\ \forall\ i, j$ because in order to get a_2, step 6 is obligatory, when the coefficients if the new system are calculated, a_1 being equal to a coefficient form the new system (equality of modules), the coefficient on $(i_0\ j_1)$.

Let $a_{i_0 j_0}$ with the property $|a_{i_0 j_0}| \leq a_1$.

Hence, $a_1 \geq |a_{i_0 j}| = \min\ |a_{i_0 j}|$. Let $a_{i_0 j_s}$ with $|a_{i_0 j_s}| > |a_{ij_m}|$; there is such an element because $|a_{i_0 j_m}|$ is the minimum of the coefficients in the module and not all $|a_{i_0 j}|,\ j = \overline{1,n}$ are equal (conversely, it would result that $(a_{i_0 j},....,a_{i_0 n}) \square\ a_{i_0 j},\ \forall j \in \overline{1,r}$, the algorithm passed through sub-step 2(A) has simplified each equation by the maximal co-divisor of its coefficients; hence, it would follow that $|a_{i_0 j}| = 1,\ \forall j = \overline{1,n}$, which, again, cannot be real because the algorithm also passed through step 3). Out of the coefficients $a_{i_0 j_m}$ we choose one with the property $a_{i_0 j_{s_0}} \neq M a_{i_0 j_m}$ there is such an element (contrary, it would result $(a_{i_0 j},...,a_{i_0 n}) \square\ |a_{i_0 j_m}|$ but the algorithm has also passed through step 2(A) and it would mean that $\left|a_{i_0 j_m}\right| = 1$ which contradicts step 3 through which the algorithm has also passed).

Considering $q_0 = \left[a_{i_0 j_{s_0}} / a_{i_0 j_m} \right] \in \mathbf{Z}$ and $r = a_{i_0 j_{s_0}} - q_0 a_{i_0 j_m} \in \mathbf{Z}$, we have $a_{i_0 j_{s_0}} \equiv r_0 (\bmod a_{i_0 j_m})$ and $0 < |r_0| < \left|a_{i_0 j_m}\right| < \left|a_{i_0 j_0}\right| \leq a_1$. We have, thus, obtained an r_0 with $|r_0| < a_1$, which is in contradiction with the very definition of a_1. Thus $a_1 < \left|a_{ij}\right|,\ \forall i, j$.

Lemma 4. Algorithm 2 is finite.

Proof:

The functioning of the algorithm is meant to transform a linear system of m equations and n unknowns into one of $m_1 \times n_1$ with $m_1 < m$, $n_1 < n$, thus, successively into a final linear equation with $n - r + 1$ unknowns (where r is the rank of the associated matrix). This equation is solved by means of the same algorithm (which works as [5]). The general integer solution of the system will depend on the $n-1$ integer number independent parameters (see [6] – similar properties can be established also the general integer solution of the linear system). The reduction of equations occurs at steps 2, 3 and sub-step 7(B). Step 2 and 3 are obvious and, hence, trivial; they can reduce the equation of the system (or even put an end to it) but only under particular conditions. The most important case finds its solution at step 7(B), which always reduces one equation of the system. As the number of equations is finite we come to solve a single integer number equation. We also have to show that the transfer from one system $m_i \times n_i$ to another $m_{i+1} \times n_{i+1}$ is made in a finite interval of time: by steps 5 and 6 permanent substitution of variables are made until we to $a = 1$ (we to $a = 1$ because, according to lemma 3, all $a - s$ are positive integer numbers and form a strictly decreasing row).

Theorem of correctness.

Algorithm 2 correctly calculates the general integer solution of the linear system.

Proof:

Algorithm 2 is finite according to lemma 4. Steps 2 and 3 are obvious (see also [4], [5]). Their part is to simplify the calculations as much as possible. Step 4 tests the finality of the algorithm; the substitution with the parameters $k_1, k_2, \ldots$ has systematization and aesthetic reasons. The variables t, h, p are counter variables (started at step 1) and they are meant to count the statement of the type T, H, P (numbering required by the substitutions at steps 3, 6 and sub-step 7(B); h also counts the new (auxiliary) variables introduced in the moment of decomposition of the system. The substitution from step 6 does not affect the general integer solution of the system (it follows from lemma 2). Lemma 1 shows that at step 5 there is always a, because $\emptyset \neq M \subset \mathrm{N}^*$.

The algorithm performs the transformation of a system $m_i \times n_i$ into another $m_{i+1} \times n_{i+1}$, equivalent to it, preserving the general solution (taking into account, however, the substitutions) (see also lemma 2).

Example 2. Calculate the integer solution of:

$$\begin{cases} 12x - 7y + 9z = 12 \\ -5y + 8z + 10w = 0 \\ 0z + 0w = 0 \\ 15x + 21z + 69w = 3 \end{cases}$$

Solution:

We apply algorithm 2 (we purposely selected an example to be passed through all the steps of this algorithm):

1. $t = 1,\ h = 1,\ p = 1$
2. (A) The fourth equation becomes $5x + 7z + 23w = 1$
   (B) –
   (C) Equation 3 is removed.
3. No; go on to step5.
5. $a = 2$ and $i = 1, j_1 = 2, j_2 = 3$, and $r = 2$.
6. $z = t_1 + y$, the statement (H_1). Substituting it in the

$$\begin{cases} 12x - 2y + 9t_1 = 12 \\ 3y + 9t_1 + 10w = 0 \\ 5x + 7y + 7t_1 + 23w = 1 \end{cases}$$

7. $a \neq 1$ consider $z = t_1, h := 2$, and go back to step 2.
2. –
3. No. Step 5.
5. $a = 1$ and $i = 2, j_1 = 4, j_2 = 2$, and $r = 1$.
6. $y = t_2 - 3w$, the statement (H_2). Substituting in the system:

$$\begin{cases} -12x + 2t_2 + 9t_1 - 6w = 12 \\ 3t_2 + 8t_1 + w = 0 \\ 5x + 7t_2 + 7t_1 + 2w = 1 \end{cases}$$

Substituting it in statement (H_1), we obtain:

$z = t_1 + t_2 - 3w$, statement $(H_1)'$.

7. $w = -3t_2 - 8t_1$ statement (P_1).

Substituting it in the system, we obtain:

$$\begin{cases} -12x - 20t_2 + 57t_1 = 12 \\ 5x + t_2 - 9t_1 = 1 \end{cases}$$

Substituting it in the other statements, we obtain:

$z = 10t_2 + 25t_1,\ (H_1)''$

$y = 10t_2 + 24t_1,\ (H_2)''$

$h := 3,\ p := 2$, and go back to step 4.

4. Yes.
2. –
3. $t_2 = 1 - 5x + 9t_1$, statement (T_1).

Substituting it (where possible) we obtain:

$$\{-112x + 237t_1 = -8 \text{ (the new system);}$$
$$z = 10 - 50x + 115t_1, \quad (H_1)'''$$
$$y = 10 - 50x + 114t_1, \quad (H_2)''$$
$$w = -3 + 15x + 35t_1, \quad (P_1)'$$

Consider $t := 2$ go on to step 4.

4. Yes. Go back to step 2. (From now on algorithm 2 works similarly with that from [5], with the only difference that the substitution must also be made in the statements obtained up to this point).

2. –

3. No. Go on to step 5.

5. $a = 13$ (one three) and $i = 1, j_1 = 2, j_2 = 1$, and $r = 13$.

6. $x = t_3 + 2t_1$, statement (H_3).

After substituting we obtain:

$$\{-112t_3 + 13t_1 = -8 \text{ (the system)}$$
$$z = 10 - 50t_3 + 15t_1, \quad (H_1)^{IV};$$
$$y = 10 - 50t_3 + 14t_1, \quad (H_2)''';$$
$$w = -3 + 15t_3 - 5t_1, \quad (P_1)'';$$
$$t_2 = 1 - 5t_3 - t_1, \quad (T_1)';$$

7. $x := t_3,\ h := 4$ and go on to step 2.

2. –

3. No. go on to step 5.

5. $a = 5$ and $i = 1, j_1 = 1, j_2 = 2$ and $r = 5$

6. $t_1 = t_4 + 9t_3$, statement (H_4).

Substituting it, we obtain :

$$5t_3 + 13t_4 = -8 \text{ (the system).}$$
$$z = 10 + 85t_3 + 15t_4, \quad (H_1)^V;$$
$$y = 10 + 76t_3 + 14t_4, \quad (H_2)^{IV};$$
$$x = 19t_3 + 2t_4, \quad (H_3)';$$
$$w = -3 - 30t_3 - 5t_4, \quad (P_1)''';$$
$$t_2 = 1 - 14t_3 - t_4, \quad (T_1)'';$$

7. $t_1 := t_4; h := 5$ and go back to step 2.

2. –

3. No. Step 5.

5. $a = 2$ and $i = 1, j_1 = 2, j_2 = 1$ and $r = -2$.

6. $t_3 = t_5 - 3t_4$ statement (H_5). After substituting, we obtain:

$$5t_5 - 2t_4 = -8 \text{ (the system).}$$

$$z = 10 + 85t_5 - 240t_4, \quad (H_1)^{VI};$$
$$y = 10 + 76t_5 - 214t_4, \quad (H_2)^{V};$$
$$x = \quad 19t_5 - 55t_4, \quad (H_3)^{IV};$$
$$w = -3 - 30t_5 + 85t_4, \quad (P_1)^{IV};$$
$$t_2 = -1 - 14t_5 + 41t_4, \quad (T_1)''';$$
$$t_1 = \quad 9t_5 + 26t_4, \quad (H_4)';$$

7. $t_3 := t_6, h := 6$ and go back to step 2.
2. –
3. No. Step 5.
5. $a = 1$ and $i = 1, j_1 = 2, j_2, r = 1$.
6. $t_4 = t_6 + 2t_5$ statement (H_6). After substituting, we obtain:

$$t_5 - 2t_6 = -8 \text{ (the system)}$$
$$z = 10 - 395t_5 - 240t_6, \quad (H_1)^{VII};$$
$$y = 10 - 392t_5 - 214t_6, \quad (H_2)^{IV};$$
$$x = \quad -91t_5 - 55t_6, \quad (H_3)''';$$
$$w = -3 + 140t_5 + 85t_6, \quad (P_1)^{V};$$
$$t_2 = \quad 1 + 68t_5 + 41t_6, \quad (T_1)^{IV};$$
$$t_1 = \quad -43t_5 - 26t_6, \quad (H_4)'';$$
$$t_3 = \quad -5t_5 - 3t_6, \quad (H_5);$$

7. $t_5 = 2t_6 - 8$ statement (P_2). Substituting it in the system we obtain: 0=0. Substituting it in the other statements, it follows:

$$z = -1030t_6 + 3170$$
$$y = -918t_6 + 2826$$
$$x = -237t_6 + 728$$
$$w = \ 365t_6 - 1123$$
$$\left.\begin{aligned} t_2 &= 177t_6 - 543 \\ t_1 &= 112t_6 + 344 \\ t_3 &= \ 13t_6 + \ 40 \\ t_4 &= \ 5t_6 - \ 16 \end{aligned}\right\} \text{ statements of no importance.}$$

Consider $h := 7, p := 3$, and go back to step 4. $t_6 \in \mathbf{Z}$

4. No. The general integer solution of the system is:

$$\begin{cases} x = -237k_1 + 728 \\ y = -918k_1 + 2826 \\ z = 1030k_1 + 3170 \\ w = \ 365k_1 - 1123 \end{cases}$$

where k_1 is an integer number parameter.
Stop.

Algorithm 3.
Input
A linear system (1)
Output
We decide on the possibility of an integer solution of this system. If it is possible, we obtain its general integer solution.

Method

1. Solve the system in $\mathbb{Q}^n$. If it does not have solutions in $\mathbb{Q}^n$, it does not have solutions in $\mathbf{Z}^n$ either. Stop.
2. $f=1,\ t=1,\ h=1,\ g=1$
3. Write the value of each main variable x_i under the form:

$$\left(E_{f,\,i}\right)_i : x_i = \sum_j q_{ij}x_j^{'} + q_i + \left(\sum_j r_{ij}x_j^{'} + r_i\right) / \Delta_i$$

with all $q_{ij},\ q_i,\ r_{ij},\ r_i,\ \Delta_i$ in $\mathbf{Z}$ such that all $\left|r_{ij}\right| < \left|\Delta_i\right|,\ \Delta_i \neq 0,\ \left|r_i\right| < \left|\Delta_i\right|$ (where all $x_j^{'}$ of the right term are integer number variables: either of the secondary variables of the system or other new variables introduced with the algorithm). For all i, we write

$$r_{ij_f} \equiv \Delta_i .$$

4. $\left(E_{f,\,i}\right)_i : \sum_j r_{ij}x_j^{'} - r_{ij_f}Y_{f,\,i} + r_i = 0$ where $\left(Y_{f,\,i}\right)_i$ are auxiliary integer number variables. We remove all the equations $\left(F_{f,\,i}\right)$ which are identities.
5. Does at least one equation $\left(F_{f,\,i}\right)$ exists? If it does not, write the general integer solution of the system substituting $k_1, k_2, \ldots$ for all the variables from the right term of each expression representing the value of the initial unknowns of the system. Stop.
6. (A) Divide each equation $\left(F_{f,\,i}\right)$ by the maximal co-divisor of the coefficients of their unknowns. If the quotient is not an integer number for at least one i_0 the system does not have integer solutions. Stop.

   (B) Simplify –as in m - all the fractions from the statements $\left(E_{f,\,i}\right)_i$.
7. Does $r_{i_0 j_0}$ exist having the absolute value 1? If it does not, go on to step 8. If it does, find the value of $x_j^{'}$ from the equation $\left(F_{f,\,i_0}\right)$; write $\left(T_t\right)$ for this statement, and substitute it (where it is possible) in the statements $\left(E_{f,\,i}\right)_i$, $\left(T^{t-1}\right)$, $\left(H_h\right)$, $\left(G_g\right)$ for all $i,\ t,\ h$ and g. Remove the equation $\left(F_{f,\,i_0}\right)$. Consider $f := f+1,\ t := t+1$, and go back to step3.
8. Calculate

$$a = \min_{i, j_1, j_2} |r|,\ r_{ij_1} \equiv r(\mathrm{mod}\, r_{ij_2}),\ 0 < |r| < |r_{ij_2}|$$

and determine the indices i_m, j_1, j_2 as well as the r for which this minimum can be obtained. (When there are more variables, choose only one).

9. (A) Write $x'_{j_2} = z_h - \frac{a_{i_m j_1} - r}{a_{j_m j_2}} x'_{j_1}$, where z_h is a new integer variable; statement (H_h).

   (B) Substitute the letter (where possible) in the statements $(E_{f,i})$, $(F_{f,i})$, (T_t), (H_{h-1}), (G_g) for all i, t, h and g.

   (C) Consider $h := h + 1$.

10. (A) If $a \neq 1$ go back to step 4.

    (B) If $a = 1$ calculate the value of the variable x'_j from the equation $(F_{f,i})$; relation (G^1_g). Substitute it (where possible) in the statements $(E_{f,i})$, (T_t), (H_h), (G_{g-1}) for all i, t, h, and g. Remove the equation $(F_{f,i})$. Consider $g := g + 1, f := f + 1$ and go back to step 3.

The correctness of algorithm 3

Lemma 5. Let i be fixed. Then $\left(\sum_{j=n_1}^{n_2} r_{ij} x'_j + r \right) | \Delta_i$ (with all $r_{ij}, r_i, \Delta_i, n_1, n_2$ being integers, $n_1 \leq n_2$, $\Delta_i \neq 0$ and all x'_j being integer variables) can have integer values if and only if $r_{in_1}, ..., r_{in_2}, \Delta_i \ | r_i$.

Proof:

The fraction from the lemma can have integer values if and only if there is a $z \in \mathbf{Z}$ such that

$$\left(\sum_{j=n_1}^{n_2} r_{ij} x'_j + r_i \right) | \Delta_i = z \Leftrightarrow \sum_{j=n_1}^{n_2} r_{ij} x'_j - \Delta_i z + r_i = 0,$$

which is a linear equation. This equation has integer solution $\Leftrightarrow$ $r_{in_1}, ..., r_{in_2}, \Delta_i \ | r_i$.

Lemma 6. The algorithm is finite. It is true. The algorithm can stop at steps 1,5 or sub-steps 6(A). (It rarely stops at step 1).

One equation after another are gradually eliminated at step 4 and especially 7 and 10(B) $(F_{f,i})$ - the number of equation is finite.

If at steps 4 and 7 the elimination of equations may occur only in special cases elimination of equations at sub step 10 (B) is always true because, through steps 8 and 9 we get to $a = 1$ (see [5]) or even lemma 4 (from the correctness of algorithm 2). Hence, the algorithm is finite.

Theorem of Correctness.

The algorithm 3 correctly calculates the general integer solution of the system (1).

Proof:

The algorithm if finite according to lemma 6. It is obvious that the system does not have solution in $\mathbb{R}^n$ it does not have in $\mathbf{Z}^n$ either, because $\mathbf{Z}^n \subset \mathbb{R}^n$ (step 1).

The variables f, t, h, g are counter variables and are meant to number the statements of the type E, F, T, H and G, respectively. They are used to distinguish between the statements and make the necessary substitutions (step 2).

Step 3 is obvious. All coefficients of the unknowns being integers, each main variable x_i will be written:

$$x_i = \left(\sum_j c_{ij} x_j^{'} + c_i \right) | \Delta_i$$

which can assume the form and conditions required in this step.

Step 4 is obtained from 3 by writing each fraction equal to an integer variable $Y_{f,\, i}$ (this being $x_i \in \mathbf{Z}$).

Step 5 is very close to the end. If there is no fraction among all $\left(E_{f,\, i}\right)$ it means that all main variables x_i already have values in $\mathbf{Z}$, while the secondary variables of the system can be arbitrary in $\mathbf{Z}$, or can be obtained from the statements T, H or G (but these have only integer expressions because of their definition and because only integer substitutions are made). The second assertion of this step is meant to systematize the parameters and renumber; it could be left out but aesthetic reasons dictate its presence. According to lemma 5 the step 6(A) is correct. (If a fraction depending on certain parameters (integer variables) cannot have values in $\mathbf{Z}$, then the main variable which has in the value of its expression such a fraction cannot have values in $\mathbf{Z}$ either; hence, the system does not have integer system). This 6(A) also has a simplifying role. The correctness of step 7, trivial as it is, also results from [4] and the steps 8-10 from [5] or even from algorithm 2 (lemma 4).

Ther initial system is equivalent to the "system" from step 3 (in fact, $\left(E_{f,\, i}\right)$ as well, can be considered a system) Therefore, the general integer solution is preserved (the changes of variables do not prejudice it (see [4], [5], and also lemma 2 from the correctness of algorithm 2)). From a system $m_i \times n_i$ we form an equivalent system $m_{i+1} \times n_{i+1}$ with $m_{i+1} < m_i$ and $n_{i+1} < n_i$. This algorithm works similarly to algorithm 2.

Example 3. Employing algorithm 3, find an integer solution of the following system:

$$\begin{cases} 3x_1 + 4x_2 \quad + 22x_4 - 8x_5 = 25 \\ 6x_1 \qquad\qquad + 46x_4 - 12x_5 = 2 \\ \qquad 4x_2 + 3x_3 - x_4 + 9x_5 = 26 \end{cases}$$

Solution

1. Common resolving in $\mathbb{R}^3$ it follows:

$$\begin{cases} x_1 = \dfrac{23x_4 - 6x_5 - 1}{-3} \\ x_2 = \dfrac{x_4 + 2x_5 + 24}{4} \\ x_3 = \dfrac{11x_5 + 2}{3} \end{cases}$$

2. $f = 1,\ t = 1,\ h = 1,\ g = 1$

3.
$$\begin{cases} x_1 = -7x_4 + 2x_5 + \dfrac{2x_4 - 1}{-3} & (E_{1,\,1}) \\ x_2 = \qquad\quad 6 + \dfrac{x_4 + 3x_5}{4} & (E_{1,\,2}) \\ x_3 = \qquad -4x_5 + \dfrac{x_5 + 2}{3} & (E_{1,\,3}) \end{cases}$$

4.
$$\begin{cases} 2x_4 \qquad + 3y_{11} - 1 = 0 & F_{1,\,1} \\ x_4 + 2x_5 - 4y_{12} \quad = 0 & F_{1,\,2} \\ \qquad x_5 - 3y_{13} + 2 = 0 & F_{1,\,3} \end{cases}$$

5. Yes.
6. –
7. Yes: $|r_{35}| = 1$. Then $x_5 = 3y_{13} - 2$ the statement (T_1). Substituting it in the others, we obtain:

$$\begin{cases} x_1 = -7x_4 + 6y_{13} - 4 + \dfrac{2x_4 - 1}{-3} & (E_{1,\,1}) \\ x_2 = \qquad\qquad 6 + \dfrac{x_4 + 6y_{13} - 4}{4} & (E_{1,\,2}) \\ x_3 = \qquad -12y_{13} + 8 + \dfrac{3y_{13} - 2 + 2}{3} & (E_{1,\,3}) \end{cases}$$

Remove the equation $(F_{1,\,3})$.

Consider $f := 2,\ t := 2$; go back to step 3.

3
$$\begin{cases} x_1 = -7x_4 + 6y_{13} - 4 + \dfrac{2x_4 - 1}{-3} & (E_{2,\,1}) \\ x_2 = \qquad\quad y_{13} + 5 + \dfrac{x_4 + 2y_{13}}{4} & (E_{2,\,2}) \\ x_3 = \qquad -11y_{13} + 8 & (E_{2,\,3}) \end{cases}$$

4. $$\begin{cases} 2x_4 \qquad + 3y_{21} - 1 = 0 & F_{2,\,1} \\ x_4 + 2y_{13} - 4y_{22} \quad = 0 & F_{2,\,2} \end{cases}$$

5. Yes.

6. –

7. Yes $|r_{24}| = 1$. We obtain $x_4 = -2y_{13} + 4y_{22}$ statement (T_2). Substituting it in the others we obtain:

$$\begin{cases} x_1 = -28y_{22} + 20y_{13} + \dfrac{-4y_{13} + 8y_{22} - 1}{-3} & E_{2,\,1}{}' \\ x_2 = \quad y_{22} + y_{13} + 5 & E_{2,\,2}{}' \\ x_3 = \quad -11y_{13} + 8 & E_{2,\,3}{}' \end{cases}$$

Remove the equation $(F_{2,\,2})$

Consider $f := 3,\ t := 3$ and go back to step 3.

3.

$$\begin{cases} x_1 = -22y_{13} + 30y_{22} + \dfrac{2y_{13} + 2y_{22} - 1}{-3} & E_{3,\,1} \\ x_2 = \quad y_{13} + y_{22} + 5 & E_{2,\,2} \\ x_3 = \quad -11y_{13} + 8 & E_{3,\,3} \end{cases}$$

4. $2y_{13} + 2y_{22} + 3y_{31} - 1 = 0 \qquad F_{3,\,1}$

5. Yes.

6. –

7. No.

8. $a = 1$ and $i_m = 1, j_1 = 31, j_2 = 22$, and $r = 1$.

9. (A) $y_{22} = z_1 - y_{31}$ (statement (H_1)).

(B) Substituting it in the others we obtain:

$$\begin{cases} x_1 = -22y_{13} - 30z_1 + 30y_{31} - 4 + \dfrac{2y_{13} + 2z_1 - 2y_{31} - 1}{-3} & E_{3,\,1}{}' \\ x_2 = \quad y_{13} + \ z_1 \quad - y_{31} \ + 5 & E_{3,\,2}{}' \\ x_3 = -11y_{13} \qquad + 8 & (E_{3,\,3}{}') \end{cases}$$

$2y_{13} + 2z_1 + y_{31} - 1 = 0 \qquad F_{3,\,1}{}'$

$x_4 = -2y_{13} + 4z_1 - 4y_{13} \qquad T_2{}'$

(C) Consider $h := 2$

10. (B) $y_{13} = 1 - 2y_{13} - 2z_1$, statement (G_1).

Substituting it in the others we obtain:

$$
\begin{array}{ll}
x_1 = -40y_{13} - 92z_1 + 27 & E_{3,1} \; '' \\
x_2 = \;\; 3y_{13} + \;\; 3z_1 + 4 & E_{3,2} \; '' \\
x_3 = -11y_{13} \qquad\quad + 8 & E_{3,3} \; '' \\
x_4 = 6y_{13} + 12z_1 - 4 & T_2 \; '' \\
y_{22} = 2y_{13} + 3z_1 \;\; - 1 & H_1 \; '
\end{array}
$$

Remove equation $\left(F_{3,\,1}\right)$.

Consider $g := 2, f := 4$ and go back to step 3.

3.

$$
\begin{cases}
x_1 = -40y_{13} - 92z_1 + 27 & E_{4,1} \\
x_2 = \;\;\; 3y_{13} + \;\; 3z_1 + 4 & E_{4,2} \\
x_3 = -11y_{13} \qquad\quad + 8 & E_{4,3}
\end{cases}
$$

4. -
5. No. The general solution of the initial system is:

$$
\begin{cases}
x_1 = -40k_1 - 92k_2 + 27, & \text{from} \quad E_{4,1} \\
x_2 = \;\;\; 3k_1 + \;\; 3k_2 \;\; + 4, & \text{from} \quad E_{4,2} \\
x_3 = -11k_1 \qquad\quad + 8, & \text{from} \quad E_{4,3} \\
x_4 = \;\; 6k_1 + 12k_2 \;\; - 4, & \text{from} \quad T_2 \; '' \\
x_5 = \;\; 3k_1 \qquad\quad - 2, & \text{from} \quad (T_1)
\end{cases}
$$

where $k_1, k_2 \in \mathbf{Z}$.

Algorithm 4

Input

A linear system (1) with not all $a_{ij} = 0$.

Output

We decide on the possibility of integer solution of this system. If it is possible, we obtain its general integer solution.

Method

1. $h = 1,\ v = 1$.
2. (A) Divide every equation i by the largest co-divisor of the coefficients of the unknowns. If the quotient is not an integer for at least one i_0 then the system does not have integer solutions. Stop.

   (B) If there is an inequality in the system, then it does not have integer solutions

   (C) In case of repetition, retain only one equation of that kind.

(D) Remove all the equations which are identities.

3. Calculate $a = \min_{i,j} |a_{ij}|,\ a_{ij} \neq 0$ and determine the indices i_0, j_0 for which this minimum can be obtained. (If there are more variables, choose one, at random.)
4. If $a \neq 1$ go on to step 6.
   If $a = 1$, then:
   (A) Calculate the value of the variable x_{j_0} from the equation i_0 note this statement (V_v).

   (B) Substitute this statement (where possible) in all the equations of the system as well as in the statements (V_{v-1}), (H_h), for all v and h.

   (C) Remove the equation i_0 from the system.

   (D) Consider $v := v + 1$.
5. Does at least one equation exist in the system?
   (A) If it does not, write the general integer solution of the system substituting $k_1, k_2, ...$ for all the variables from the right term of each expression representing the value of the initial unknowns of the system.
   (B) If it does, considering the new data, go back to step 2.
6. Write all $a_{i_0 j}, j \neq j_0$ and b_{i_0} under the form :

$$a_{i_0 j} = a_{i_0 j_0} q_{i_0 j} + r_{i_0 j}, \text{ with } |r_{i_0 j}| < |a_{i_0 j}|.$$

$$b_{i_0 j} = a_{i_0 j_0} q_{i_0} + r_{i_0}, \text{ with } |r_{i_0}| < |a_{i_0 j_0}|.$$

7. Write $x_{j_0} = -\sum_{j \neq j_0} q_{i_0 j} x_j + q_{i_0} + t_h$, statement (H_h).
   Substitute (where possible) this statement in all the equations of the system as well as in the statement (V_v), (H_h), for all v and h.
8. Consider

$$x_{j_0} := t_h,\ h := h + 1,$$

$$a_{i_0 j} := r_{i_0 j},\ j \neq j_0,$$

$$a_{i_0 j_0} := \pm a_{i_0 j_0},\ b_{i_0} := +r_{i_0},$$

   and go back to step 2

The correctness of Algorithm 4

This algorithm extends the algorithm from [4] (integer solutions of equations to integer solutions of linear systems). The algorithm was thoroughly proved in our previous article; the present one introduces a new cycle – having as cycling variable the number of equations of system – the rest remaining unchanged, hence, the correctness of algorithm 4 is obvious.

Discussion

1. The counter variables h and v count the statements H and V, respectively, differentiating them (to enable the substitutions);
2. Step 2 ((A)+(B) + (C)) is trivial and is meant to simplify the calculations (as algorithm 2);
3. Sub-step 5 (A) has aesthetic function (as all the algorithms described). Everything else has been proved in the previous chapters (see [4], [5], and algorithm 2).

Example 4. Let us use algorithm 4 to calculate the integer solution of the following linear system:

$$\begin{cases} 3x_1 \quad -7x_3+6x_4 \quad =-2 \\ 4x_1+3\,x_2 \quad +6x_4-5x_5 = 19 \end{cases}$$

Solution

1. $h=1,\ v=1$
2. –
3. $a=3$ and $i=1, j=1$
4. $3\neq 1$. Go on to step 6.
6. Then,

$$-7=3\cdot(-3)+2$$
$$6=3\cdot 2+0$$
$$-2=3\cdot 0-2$$

7. $x_1 = 3x_3-2x_4+t_1$ statement (H_1). Substituting it in the second equation we obtain:

$$4t_1+3x_2+12x_3-x_4-5x_5=19$$

8. $x_1 \coloneqq t_1,\ h\coloneqq 2,\ a_{12}\coloneqq 0,\ a_{13}\coloneqq +2,\ a_{14}\coloneqq 0,\ a_{11}\coloneqq +3,\ b\coloneqq -2$.

Go back to step 2.

2. The equivalent system was written:

$$\begin{cases} 3t_1 \quad +3x_3 \quad =-2 \\ 4t_1+3x_2+12x_3-x_4-5x_5=19 \end{cases}$$

3. $a=1,\ i=2,\ j=4$
4. 1=1
   (A) Then: $x_4=4t_1+3x_2+12x_3-5x_5-19$ statement (V_1).
   (B) Substituting it in (H_1), we obtain:
   $x_1=-7t_1-6\,x_2-21x_3+10x_5+38, \quad (H_1)$
   (C) Remove the second equation of the system.
   (D) Consider: $v\coloneqq 2$.
5. Yes. Go back to step 2.

2. The equation $+3t_1 \ +2x_3=-2$ is left.
3. $a=2$ and $i=1, j=3$

4. $2 \neq 2$, go to step 6.
6.

$$+3 = +2 \cdot 2 - 1$$
$$-2 = +2(-1) + 0$$

7. $x_3 = -2t_1 + t_2 - 1$ statement (H_2).
   Substituting it in $(H_1)'$, (V_1), we obtain:

$$x_1 = 35t_1 - 6\,x_2 - 21t_2 + 10x_5 + 59 \qquad H_1\;''$$
$$x_4 = -20t_1 + 3x_2 + 12t_2 - 5x_5 - 31 \qquad V_1\;'$$

8. $x_3 := t_2,\ h := 3,\ a_{11} := -1,\ a_{13} := +2,\ b_1 := 0$, (the others being all = 0). Go back to step 2.
2. The equation $-5t_1 + 2t_2 = 0$ was obtained.
3. $a = 1$, and $i = 1,\ j = 1$
4. 1=1
   (A) Then $t_1 = 2t_2$ statement (V_2).
   (B) After substitution, we obtain:

$$x_1 = 49t_2 - 6x_2 + 10x_5 + 59 \qquad H_1\;''';$$
$$x_4 = -28t_2 + 3x_2 - 5x_5 - 31 \qquad V_1\;'';$$
$$x_3 = -3t_2 \qquad H_2\;';$$

   (C) Remove the first equation from the system.
   (D) $v := 3$
5. No. The general integer solution of the initial system is:

$$\begin{cases} x_1 = 49k_1 - 6k_2 + 10k_3 + 59 \\ x_2 = k_2 \\ x_3 = -3k_1 - 1 \\ x_4 = -28k_1 + 3k_2 - 5k_3 - 31 \\ x_5 = k_3 \end{cases}$$

   where $(k_1, k_2, k_3) \in \mathbf{Z}^3$.
   Stop.

Algorithm 5
Input
A linear system (1)
Output
We decide on the possibility of an integer solution of this system. If it is possible, we obtain its general integer solution.
Method
1. We solve the common system in □^{n}, then it does not have solutions in □^{n}, then it does not have solutions in $\mathbf{Z}^n$ either. Stop.
2. $f = 1,\ v = 1,\ h = 1$
3. Write the value of each main variable x_i under the form:

$$E_{f,i}\ _i : x_i = \sum_j q_{ij} x_j^{'} - q_i + \left(\sum_j r_{ij} x_j^{'} + r_i \right) / \Delta_i ,$$

with all $q_{ij}, q_i, r_{ij}, r_i, \Delta_i$ from $\mathbf{Z}$ such that all $|r_{ij}| < |\Delta_i|$, $|r_i| < |\Delta_i|$, $\Delta_i \neq$ (where all $x_j^{'} - S$ of the right term are integer variables: either from the secondary variables of the system or the new variables introduced with the algorithm). For all i, we write $r_{ij_f} \equiv \Delta_i$

4. $(E_{f,i})_i : \sum_j r_{ij} x_j - r_{i,j_f} y_{f,i} + r_i = 0$ where $(y_{f,i})$ are auxiliary integer variables. Remove all the equations $(F_{f,i})$ which are identities.
5. Does at least one equation $(F_{f,i})$ exist? If it does not, write the general integer solution of the system substituting $k_1, k_2, ...$ for all the variables of the right number of each expression representing the value of the initial unknowns of the system. Stop.
6. (A) Divide each equation $(F_{f,i})$ by the largest co-divisor of the coefficients of their unknowns. If the quotient is an integer for at least one i_0 then the system does not have integer solutions. Stop.
   (B) Simplify – as previously ((A)) all the functions in the relations $(E_{f,i})_i$.
7. Calculate $a = \min_{i,j} |r_{ij}|,\ r_{ij} \neq 0$, and determine the indices i_0, j_0 for which this minimum is obtained.
8. If $a \neq 1$, go on to step 9.
   If $a = 1$, then:
   (A) Calculate the value of the variable $x_{j_0}^{'}$ from the equation $(F_{f,i})$ write (V_v) for this statement.
   (B) Substitute this statement (where possible) in the statement $(E_{f,i})$, (V_{v+1}), (H_h), for all i, v, and h.
   (C) Remove the equation $(E_{f,i})$.
   (D) Consider $v := v+1, f := f+1$ and go back to step 3.
9. Write all $r_{i_0 j}$, $j \neq j_0$ and r_{i_0} under the form:

$$r_{i_0 j} = \Delta_{i_0} \cdot q_{i_0 j} + r_{i_0 j}^{'} \text{, with } |r_{i_0 j}^{'}| < |\Delta_i| ;$$

$$r_{i_0 j} = \Delta_{i_0} \cdot q_{i_0} + r_{i_0}^{'} \text{, with } |r_{i_0}^{'}| < |\Delta_i| .$$

10. (A) Write $x_{j_0}^{'} = -\sum_{j \neq j_0} q_{i_0 j} x_j^{'} + q_{i_0} + t_h$ statement H_h .
    (B) Substitute this statement (where possible) in all the statements $(E_{f,i})$, $(F_{f,i})$, (V_v), (H_{h-1}).

(C) Consider $h := h+1$ and go back to step 4.

The correctness of the algorithm is obvious. It consists of the first part of algorithm 3 and the end part of algorithm 4. Then, steps 1-6 and their correctness were discussed in the case of algorithm 3. The situation is similar with steps 7-10. (After calculating the real solution in order to calculate the integer solution, we resorted to the procedure from 5 and algorithm 5 was obtained). This means that all these insertions were proven previously.

Example 5

Using algorithm 5, let us obtain the general integer solution of the system:

$$\begin{cases} 3x_1 \quad + 6x_3 + 2x_4 \quad = 0 \\ \quad 4x_2 - 2x_3 \quad - 7x_5 = -1 \end{cases}$$

Solution

1. Solving in $\mathbb{R}^5$ we obtain:

$$\begin{cases} x_1 = \dfrac{-6x_3 - 2x_4}{3} \\ x_2 = \dfrac{-2x_3 + 7x_5 - 1}{4} \end{cases}$$

2. $f = 1,\ v = 1,\ h = 1$
3. $(E_{1,1}): x_1 = 2x_3 + \dfrac{-2x_4}{3}$

   $(E_{1,2}): x_2 = \quad x_5 + \dfrac{2x_3 + 3x_5 - 1}{4}$
4. $F_{1,1} \ : -2x_4 - 3y_{11} = 0$

   $(F_{1,2}): 2x_3 + 3x_5 - 4y_{12} - 1 = 0$
5. Yes
6. –
7. $i = 2$ and $i_0 = 2, j_0 = 3$
8. $2 \neq 1$
9. $3 = 2 \cdot 1 + 1$

   $-4 = 2 \cdot (-2)$

   $-1 = 2 \cdot 0 - 1$
10. $x_3 = -x_5 + 2y_{12} + t_1$ statement (H_1). After substitution:

$$E_{1,1}\ ': x_1 = 2x_5 - 4y_{12} - 2t_1 + \frac{-2x_4}{3}$$

$$E_{1,2}\ ': x_2 = x_5 \qquad + \frac{x_5 + 4y_{12} + 2t_1 - 1}{4}$$

$$F_{1,2}\ ': x_5 + 2t_1 - 1 = 0$$

Consider $h := 2$ and go back to step 4.

4. $F_{1,1}$ $': -2x_4 - 3y_{11} = 0$

$F_{1,2}$ $': 2t_1 + x_5 - 1 = 0$

5. Yes.
6. –
7. $a = 1$ and $i_0 = 2$, $j_0 = 5$

(A) $x_5 = -2t_1 + 1$ statement (V_1)

(B) Substituting it, we obtain:

$$E_{1,1} \; '': \; x_1 = -6t_1 + 2 - 4y_{12} + \frac{-2x_4}{3}$$

$$E_{1,2} \; '': \; x_2 = -2t_1 + 1 + y_{12}$$

$$H_1 \; ': \; x_3 = 3t_1 + 1 - 1 + 2y_{12}$$

(C) Remove the equation $(F_{1,2})$.

(D) Consider $v = 2$, $f = 2$ and go back to step 3.

3. $E_{2,1}$: $x_1 = -6t_1 - 4y_{12} + 2 + \frac{-2x_4}{3}$

$(E_{2,2})$: $x_2 = -2t_1 + y_{12} + 1$

4. $F_{2,1}$: $-2x_4 - 3y_{12} = 0$
5. Yes.
6. –
7. $a = 2$ and $i_0 = 1, j_0 = 4$
8. $2 \neq 1$
9. $-3 = -2 \cdot (1) - 1$
10. (A) $x_4 = -y_{21} + t_2$ statement (H_2)

(B) After substitution, we obtain:

$$E_{2,1} \; ': \; x_1 = -6t_1 - 4y_{12} + 2 + \frac{-2y_{21} - 2t_2}{3}$$

$$F_{2,1} \; ': \; -y_{21} - 2t_2 = 0$$

Consider $h := 3$, and go back to step 4.

4. $F_{2,1}$ $': \; -y_{21} - 2t_2 = 0$
5. Yes
6. –
7. $a = 1$ and $i_0 = 1, j_0 = 21$ (two, one).

(A) $y_{21} = -2t_2$ statement (V_2).

(B) After substitution, we obtain:

(C) Remove the equation $F_{2,1}$.

(D) Consider $v = 3, f = 3$ and go back to step 3.

3. $(E_{3,1})$: $x_1 = -6t_1 - 4y_{12} - 2t_2 + 2$

$(E_{3,2})$: $x_2 = -2t_1 + y_{12} \qquad +1$

4. –
5. No. The general integer solution of the system is:

$$\begin{cases} x_1 = -6k_1 - 4k_2 - 2k_3 + 2, & \text{from } E_{3,1}\ ; \\ x_2 = -2k_1 + k_2 \qquad\qquad +1, & \text{from } E_{3,2}\ ; \\ x_3 = 3k_1 + 2k_2 \qquad\qquad -1, & \text{from } H_1\ '; \\ x_4 = \qquad\qquad 3k_3 \qquad , & \text{from } H_2\ '; \\ x_5 = -2k_1 \qquad\qquad\qquad +1, & \text{from } (V_1\ ;) \end{cases}$$

where $(k_1, k_2, k_3) \in \mathbf{Z}$.

Stop.

Note 1. Algorithm 3, 4, and 5 can be applied in the calculation of the integer solution of a linear equation.

Note 2. The algorithms, because of their form, are easily adapted to a computer program.

Note 3. It is up to the reader to decide on which algorithm to use. Good luck!

REFERENCES

[1] Smarandache, Florentin – Rezolvarea ecuațiilor şi a sistemelor de ecuații liniare în numere întregi - diploma paper, University of Craiova, 1979.

[2] Smarandache, Florentin – Généralisations et généralités - Edition Nouvelle, Fes (Maroc), 1984.

[3] Smarandache, Florentin – Problems avec et sans .. problems! Somipress, Fes (Maroc), 1983.

[4] Smarandache, Florentin – General solution proprieties in whole numbers for linear equations – Bul. Univ. Braşov, series C, mathematics, vol. XXIV, pp. 37-39, 1982.

[5] Smarandache, Florentin – Baze de soluții pentru congruențe lineare – Bul. Univ. Braşov, series C, mathematics, vol. XXII, pp. 25-31, 1980, re-published in Buletinul Ştiințific şi Tehnic al Institutului Politehnic "Traian Vuia", Timişoara, series mathematics-physics, tome 26 (40) fascicle 2, pp. 13-16, 1981, reviewed in Mathematical Rev. (USA): 83e:10006.

[6] Smarandache, Florentin – O generalizare a teoremei lui Euler referitoare la congruențe – Bul. Univ. Braşov, series C, mathematics, vol. XXII, pp. 07-12, reviewed in Mathematical Reviews (USA):84j:10006.

[7] Creangă, I., Cazacu, C., Mihuţ, P., Opaiţ, Gh., Corina Reischer – Introcucere în teoria numerelor - Editura Didactică şi Pedagogică, Bucharest, 1965.
[8] Cucurezeanu, Ion – Probleme de aritmetică şi teoria numerelor, Editura Tehnică, Buharest, 1976.
[9] Ghelfond, A. O. – Rezolvarea ecuaţiilor în numere întregi - translation from Russian, Editura Tehnică, Bucharest, 1954.
[10] Golstein, E., Youndin, D. – Problemes particuliers de la programmation lineaire - Edition Mir, Moscou, Traduit de russe, 1973.
[11] Ion, D. Ion, Niţă, C. – Elemente de aritmetică cu aplicaţii în tehnici de calcul, Editura Tehnică, Bucharest, 1978.
[12] Ion, D. Ion, Radu, K. – Algebra - Editura Didactică şi Pedagogică, Bucharest 1970.
[13] Mordell, L. – Two papers on number theory - Veb Deutscher Verlag der Wissenschafen, Berlin, 1972.
[14] Popovici, C. P. – Aritmetica şi teoria numerelor - Editura Didactică şi Pedagogică, Bucharest, 1963.
[15] Popovici, C. P. – Logica şi teoria numerelor - Editura Didactică şi Pedagogică, Bucharest, 1970.
[16] Popovici, C. P. – Teoria numerelor – lecture course, Editura Didactică şi Pedagogică, Bucharest, 1973.
[17] Rusu, E – Aritmetica si teoria numerelor - Editura Didactică şi Pedagogică, Bucharest, 1963.
[18] Rusu, E. – Bazele teoriei numerelor - Editura Tehincă, Bucharest, 1953.
[19] Sierpinski, W. – Ce ştim şi ce nu ştim despre numerele prime – Editura. Ştiinţifică, Bucharest, 1966.
[20] Sierpinski, W. – 250 problemes de theorie elementaires des nombres - Classiques Hachette, Paris, 1972.

[Partly published in “Bulet. Univ. Braşov”, series C, Vol. XXIV, pp. 37-9, 1982, under the title: “General integer solution properties for linear equations”.]