Further Generalization of *n-D* Distance and *n-D* Dependent Function in Extenics

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Abstract. Prof. Cai Wen [1] defined the *1-D* Distance and *1-D* Dependent Function in 1983. F. Smarandache [6] generalized them to *n-D* Distance and *n-D* Dependent Function respectively in 2012 during his postdoc research at Guangdong University of Technology in Guangzhou. O. I. Şandru [7] extended the last results in 2013. Now [2015], as a further generalization, we unify all these results into a single formula for the *n-D* Distance and respectively for the *n-D* Dependent Function.

Keywords: Extenics, extension distance, dependent function, attraction point, position indicator.

1 Extension Distance in *1-D* Space

Let's use the notation $<a, b>$ for any kind of closed, open, or half-closed interval *{ [a, b], (a, b), (a, b], [a, b) }*. Prof. Cai Wen has defined the extension distance between a point x_0 and a real interval $S = <a, b>$, by

$$\rho(x_0, S) = | x_o - \frac{a+b}{2} | - \frac{b-a}{2} \tag{1}$$

where in general $\rho : (R, R^2) \rightarrow [-(b-a)/2, +\infty)$. (2)

2 Principle of the Extension *1-D* Distance

Geometrically studying this extension distance, we find the following principle that Prof. Cai has used in 1983 defining it:

$\rho(x_0, S)$ = *the geometric distance between the point* x_0 *and the closest extremity point of the interval* $<a, b>$ *to it (going in the direction that connects* x_0 *with the optimal point O), distance taken as negative if* $x_0 \in <a, b>$*, and as positive if* $x_0 \notin <a, b>$.

3 Dependent Function in *1-D* Space

Prof. Cai Wen defined in 1983 in *1D* the Dependent Function *K(y)*.

If one considers two intervals S_1 and S_2, that have no common end points, and $S_1 \subset S_2$, for any y ϵ R one has:

$$K(y) = \frac{\rho(y, S_2)}{\rho(y, S_2) - \rho(y, S_1)}. \qquad (3)$$

4 Definition of Attraction Point Principle in *n-D* Space

F. Smarandache [2012] introduced the *Attraction Point Principle in n-D-Space*, as follows.

Let S be a given n-D set in the universe of discourse *U*, and the optimal point $O \in S$. Then each point $P(x_1, x_2, \ldots, x_n)$ from the universe of discourse tends towards, or is attracted by, the optimal point *O*, because the optimal point *O* is an ideal of each point.

There are spaces where the attraction phenomena undergo linearly or non-linearly, by upon some specific linear or non-linear curves.

That's why one computes the extension *n-D*-distance between the point *P* and the set *S* as $\rho((x_1, x_2, \ldots, x_n), S)$ on the direction determined by the point *P* and the optimal point *O*, or on the curve *PO*. It is a kind of convergence/attraction of each point towards the optimal point.

There are classes of examples where such attraction point principle works.

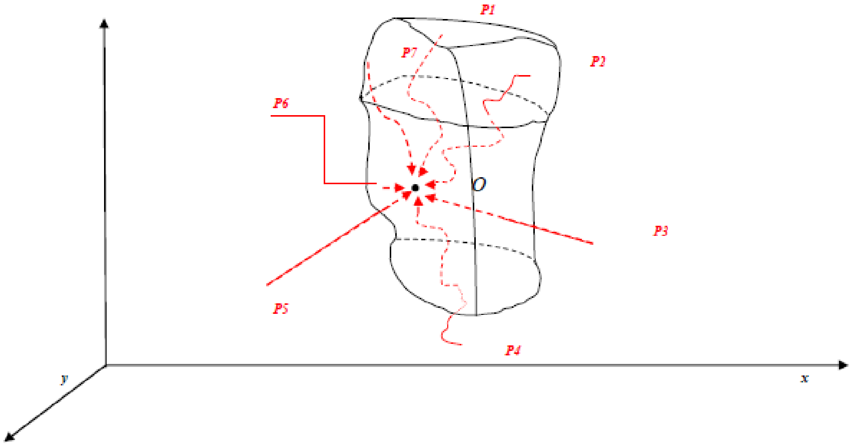

Fig. 1. Linear or Non-Linear Attraction Point Principle for any bounded 3D-body.

5 Extension Linear or Non-Linear *n-D* Distance

F. Smarandache [2012] defined the Extension Linear or Non-Linear n-D-Distance between point *P* and set *S* as follows:

$$\rho_c(P,S)=\begin{cases}-\inf\limits_{P'\in Fr(S)}\{d_c(P,P'),\} & P\neq O, P\in c(OP');\\ \inf\limits_{P'\in Fr(S)}\{d_c(P,P')\}, & P\neq O, P'\in c(OP);\\ -\sup\limits_{Q\in Fr(S), Q\in c(O)}\{d_c(P,Q)\}; & P=O;\end{cases} \tag{4}$$

where:

- *c* means a family of given curves;
- $\rho_c(P,S)$ means the n-D distance as measured along the family of curves *c*;
- the lines are considered as particular cases of curves;
- *O* is the optimal point (or curvedly attraction point);
- the points are attracting by the optimal point on trajectories described by the family of curves *c*;
- *c(O)* means the family of curves passing through point *O*;
- $d_c(P,P')$ means the curvedly *n-D*-distance between two points *P* and *P’,* or the arc length of the curve *c* between the points *P* and *P’*;
- inf{$d_c(P,P')$} means the infimum arc length between the points P and P’, i.e. among the curves passing between the points P and P’ one takes that curve which has the infimum arc length;
- *Fr(S)* means the frontier of set *S*;
- *P’* lays on the frontier of *S*;
- $P\in c(OP')$ means that point P lays on the curve c that passes through the points O and P’, and P is in between O and P’;
- similarly $P'\in c(OP)$ means that point P’ lays on the curve c that passes through the points O and P, and P’ is in between O and P;
- and c(*OP’)* means the curve arc length between the points *O* and *P’* (the extremity points *O* and *P’* included), therefore $P\in c(OP')$ means that *P* lies on the curve *c* in between the points *O* and *P’*; similarly *c(OP);*
- for *P* coinciding with *O*, one defined the n-D distance between the optimal point *O* and the set *S* as the negatively supremum curvilinear arc length (to be in concordance with the *1-D* definition).

6 Extension Linear or Non-Linear Dependent *n-D* Function

F. Smarandache [2012] defined the extension linear or non-linear dependent n-D function as follows.

In general, in a universe of discourse *U*, let's have a nest of two *n-D*-sets, $S_1 \subset S_2$, with no common end points, S_1 and S_2 included in *U*, and a point $P(x_1, x_2, \ldots, x_n)$ in the universe of discourse *U*.

Then the Extension Linear or Non-Linear Dependent n-D-Function K_{nD} referring to point $P(x_1, x_2, \ldots, x_n)$ along the family of curves *c* is:

$$K_{nD}(P)_c = \frac{\rho_c(P, S_2)}{\rho_c(P, S_2) - \rho_c(P, S_1)} \quad (5)$$

where $\rho_c(P, S_1)$ and $\rho_c(P, S_2)$ are the previous extension linear or non-linear n-D-distances between the point *P* and the *n-D*-set S_1 , respectively between the point P and the n-D-set S_2, along the family of given curves *c*.

7 Point-Set Position Indicator

O. I. Şandru [2013] defined the indicator [called extension distance by Cai Wen] between a point P(x) and a set *S*, without referring to an optimal or attraction point *O*, as follows.

First, the classical mathematical distance between a point *x* and a set *A* is:

$$\delta(x, A) = \inf\{d(x, a), a \in A\}, \quad (6)$$

for any point $x \in R^n$ and any set $S \subset R^n$, where *d* is the Euclidean distance on R^n.

Whence,

$$\rho(x, S) = \begin{Bmatrix} \delta(x, S), x \in CS \\ -\delta(x, CS), x \in S \end{Bmatrix}, \quad (7)$$

where CA represents the absolute complement of A, i.e. $CA = R^n \setminus A$.

8 Point-Two Sets Position Indicator

Then, O. I. Şandru [2013] defined the Point-Two Sets Position Indicator, for any two nested sets without common ending points from R^n, such that $S_1 \subset S_2$, where $\rho(x, S_1)$ and $\rho(x, S_2)$ are the previous point-set position indicators, as follows:

$$K(x, S_1, S_2) = \frac{\rho(x, S_2)}{\rho(x, S_2) - \rho(x, S_1)}. \quad (8)$$

9 Further Generalization of the *n-D* Distance (or Unification of *n-D* Distances)

We further generalize the defined Extension Linear or Non-Linear *n-D* Distance and Point-Set Position Indicator between point *P* and set *S* as follows.

a) When the optimal or attraction point does exist, one has:

$$\rho_M(P,S)=\begin{cases}-\inf\limits_{P'\in Fr(S)\cap M}\{d_M(P,P')\}, & P\neq O, P\in c(OP')\subseteq M;\\ \inf\limits_{P'\in Fr(S)\cap M}\{d_M(P,P')\}, & P\neq O, P'\in c(OP)\subseteq M;\\ -\sup\limits_{Q\in Fr(S)\cap M, Q\in c(O)\subseteq M}\{d_M(P,Q)\}; & P=O;\end{cases} \tag{9}$$

where:

- *M* is a subspace of the universe of discourse *U*, i.e. $M\subseteq U$;
- $d_M(P, P')$ means the distance between points *P* and *P'* defined on given curves included in subspace *M*;
- the other notations are identical to those in section 6.

b) When the optimal or attraction point does not exist, one simply has:

$$\rho_M(P,S)=\left\{\begin{array}{ll}-\inf\limits_{P'\in Fr(S)\cap M}\{d_M(P,P')\}, & P\in c(OP')\subseteq M;\\ \inf\limits_{P'\in Fr(S)\cap M}\{d_M(P,P')\}, & P'\in c(OP)\subseteq M;\end{array}\right\} \tag{10}$$

By language abuse, we may consider that formula (9) works also for the case when there is no optimal or attraction point *O*, since of course in the formula of $\rho_M(P,S)$ an existing point *P* is different from a non-existing point *O* for the first and second pieces where one has $P\neq O$ and respectively $P'\neq O$, and an existing point *P* cannot be equal to a non-existing point *O* in the third piece {therefore the third piece of formula (9) is discarded}. Therefore, formula (9) is a generalization of formula (10).

Now, if in formulas (9) and (10) one replaces *M* by a given family of curves passing through an attracting *O*, one obtains the *n-D* distance formula (defined in 2012).

But, if in formula (9) and (10) one replaces *M* by *U* (the universe of discourse), and one considers all possible curves in *U*, one obtains the point-set position indicator (defined in 2013).

c) Similarly, formula *(7)* becomes:

$$\rho_M(x,S)=\left\{\begin{array}{l}\delta(x,S), x\in CS\cap M\\ -\delta(x,CS), x\in S\cap M\end{array}\right\} \tag{11}$$

where the Euclidean distances $\delta(x,S)$ and $\delta(x,CS)$ are computed with respect to the points *x* that belong to *M* too.

If in formula (11) one replace *M* by a given family of curves passing through an attracting point *O*, one obtains the *n-D* distance formula (defined in 2012).

But, if in formula (11) one replaces *M* by *U*, one obtains and one considers all possible curves in *U*, one obtains the point-set position indicator (defined in 2013).

10 Further Generalization of *n-D* Function (or Unification of *n-D* Dependent Functions)

In general, in an universe of discourse *U*, let's have a nest of two *n-D* sets, $S_1 \subset S_2$, with no common end points, S_1 and S_2 included in *U*, and a point $P(x_1, x_2, \ldots, x_n)$ in the universe of discourse *U*.

Then the Generalization of *n-D* Function $K_{nD}(P)_M$ and Point-Two Sets Indicator referring to point $P(x_1, x_2, \ldots, x_n)$ along the family of curves included in *M* is:

$$K_{nD}(P)_M = \frac{\rho_M(P,S_2)}{\rho_M(P,S_2)-\rho_M(P,S_1)} \tag{12}$$

where $\rho_M(P,S_1)$ and $\rho_M(P,S_2)$ are the previous generalization of *n-D*-distance {formula (9)} and point-set indicator {formula (10)} between the point *P* and the *n-D* set S_1 , respectively between the point *P* and the *n-D* set S_2, along the family of given curves from *M*.

And formula (8) is generalized to:

$$K_M(x,S_1,S_2) = \frac{\rho_M(x,S_2)}{\rho_M(x,S_2)-\rho_M(x,S_1)} \tag{13}$$

The properties of newly generalized *n-D* distance and *n-D* dependent function remain the same as in *1D* case.

11 Conclusion

In this paper we have listed the previously defined in the scientific literature formulas for the *1-D* and *n-D* distance and dependent function, respectively the point-set position indicator and point-two-set position indicator.

We took into consideration problems with optimal or attraction point, and problems without such optimal or attraction point.

We have then generalized (united) them into the formulas (9) and (11) for the *n-D* distance between a point and a set, and into the formulas (12) and (13) for the *n-D* dependent function.

Let S be an arbitrary *n-D* set in the universe of discourse U of any dimension, and the optimal or attraction point $O \in S$. Then each point $P(x_1, x_2, \ldots, x_n)$, $n \geq 1$, from the universe of discourse (linearly or non-linearly) tends towards, or is attracted by, the optimal point O, because the optimal point O is an ideal of each point.

The following **properties occur for the generalized (united) definition of n-D-distance** [formulas (9), (10) and (11)]:

a) It is obvious from the generalized (united) definition of the extension *n-D*-distance between a point P in the universe of discourse and the extension *n-D*-set S that:
   - i) Point $P(x_1, x_2, \ldots, x_n) \in \mathrm{Int}(S)$ iff $\rho_M((x_1, x_2, \ldots, x_n), S) < 0$;
   - ii) Point $P(x_1, x_2, \ldots, x_n) \in \mathrm{Fr}(S)$ iff $\rho_M((x_1, x_2, \ldots, x_n), S) = 0$;
   - iii) Point $P(x_1, x_2, \ldots, x_n) \notin S$ iff $\rho_M((x_1, x_2, \ldots, x_n), S) > 0$.

b) Let S_1 and S_2 be two extension sets, in the universe of discourse U, such that they have no common end points, and $S_1 \subset S_2$. We assume they have the same optimal points $O_1 \equiv O_2 \equiv O$ located in their center of symmetry. Then for any point $P(x_1, x_2, \ldots, x_n) \in U$ one has:

$$\rho_M((x_1, x_2, \ldots, x_n), S_1) \geq \rho_M((x_1, x_2, \ldots, x_n), S_2). \tag{14}$$

The following **properties occur for the generalized (united) definition of n-D-dependent function** {formulas (12) and (13)}:

a) If point $P(x_1, x_2, \ldots, x_n) \in \mathrm{Int}(S_1)$, then $K_{nD}(x_1, x_2, \ldots, x_n)_M > 1$;

b) If point $P(x_1, x_2, \ldots, x_n) \in \mathrm{Fr}(S_1)$, then $K_{nD}(x_1, x_2, \ldots, x_n)_M = 1$;

c) If point $P(x_1, x_2, \ldots, x_n) \in \mathrm{Int}(S_2\text{-}S_1)$, then $K_{nD}(x_1, x_2, \ldots, x_n)_M \in (0, 1)$;

d) If point $P(x_1, x_2, \ldots, x_n) \in \mathrm{Int}(S_2)$, then $K_{nD}(x_1, x_2, \ldots, x_n)_M = 0$;

e) If point $P(x_1, x_2, \ldots, x_n) \notin \mathrm{Int}(S_2)$, then $K_{nD}(x_1, x_2, \ldots, x_n)_M < 0$.

References

1. Cai Wen: Extension Set and Non-Compatible Problems [J]. Journal of Scientific Exploration, (1), pp. 83-97 (1983)
2. Cai Wen: Extension Set and Non-Compatible Problems [A]. Advances in Applied Mathematics and Mechanics in China [C]. Peking: International Academic Publishers, pp. 1-21 (1990)
3. Cai Wen: Extension theory and its application, [J]. Chinese Science Bulletin, 44 (7), pp. 673-682 (1999)
4. Yang Chunyan, Cai Wen: Extension Engineering [M]. Beijing: Public Library of Science (2007)

5. Wu Wenjun et al.: Research on Extension theory and its application. Expert Opinion, 2; http://web.gdut.edu.cn/~extenics/jianding.htm (2004)
6. Xiangshan Science Conferences Office. Scientific Significance and Future Development of Extenics – No. 271 Academic Discussion of Xiangshan Science Conferences, Brief Report of Xiangshan Science Conferences, Period 260, 1 (2006)
7. Florentin Smarandache: Generalizations of the Distance and Dependent Function in Extenics to 2D, 3D, and n-D, in his book Extenics in Higher Dimensions, Guangdong University of Technology, Education Publ. Columbus, Ohio, pp. 22-38 (2012)
8. Ovidiu Ilie Șandru, Florentin Smarandache, Alexandra Șandru: A Position Indicator with Applications in the Field of Designing Forms with Artificial Intelligence, Sci. Bull., Series A, Vol. 75, Iss. 2, pp. 133-138 (2013)