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A GENERAL FAMILY OF DUAL TO RATIO-CUM-PRODUCT ESTIMATOR IN SAMPLE SURVEYS

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ABSTRACT

This paper presents a family of dual to ratio-cum-product estimators for the finite population mean. Under simple random sampling without replacement (SRSWOR) scheme, expressions of the bias and mean-squared error (MSE) up to the first order of approximation are derived. We show that the proposed family is more efficient than usual unbiased estimator, ratio estimator, product estimator, Singh estimator (1967), Srivenkataramana (1980) and Bandyopadhyaya estimator (1980) and Singh et al. (2005) estimator. An empirical study is carried out to illustrate the performance of the constructed estimator over others.

Key words: Family of estimators, auxiliary variables, bias, mean-squared error.

1. Introduction

It is common practice to use the auxiliary variable for improving the precision of the estimate of a parameter. Out of many ratio and product methods of estimation are good examples in this context. When the correlation between the study variate and auxiliary variates is positive (high), ratio method of estimation is quite effective. On the other hand, when this correlation is negative (high), product method of estimation can be employed effectively. Let U be a finite population consisting of N units $U_1, U_2, \ldots, U_N$. Let y and (x, z) denote the study variate and auxiliary variates taking the values y_i and (x_i, z_i), respectively, on the unit U_i $(i=1,2\ldots,N)$, where x is positively correlated with y and z is negatively correlated with y. We wish to estimate the population mean $\overline{Y} = \frac{1}{N}\sum_{i=1}^{N} y_i$ of y,

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assuming that the population means $\left(\overline{X}, \overline{Z}\right)$ of (x, z) are known. Assume that a simple random sample of size n is drawn without replacement from U. The classical ratio and product estimators for estimating $\overline{Y}$ are:

$$\overline{y}_R = \frac{\overline{y}}{\overline{x}}\overline{X} \tag{1.1}$$

and

$$\overline{y}_P = \overline{y}\frac{\overline{z}}{\overline{Z}} \tag{1.2}$$

respectively, where $\overline{y} = \frac{1}{n}\sum_{i=1}^{n} y_i$, $\overline{x} = \frac{1}{n}\sum_{i=1}^{n} x_i$ and $\overline{z} = \frac{1}{n}\sum_{i=1}^{n} z_i$ are the sample means of y, x and z respectively.

Using the transformation $x_i^* = \frac{\left(N\overline{X} - nx_i\right)}{\left(N-n\right)}$, and $z_i^* = \frac{(N\overline{Z} - nz_i)}{(N-n)}$, (i=1,2…N) Srivenkataramana (1980) and Bandyopadhyaya (1980) suggested a dual to ratio and product estimator as:

$$\overline{y}_R^* = \overline{y}\frac{\overline{x}^*}{\overline{X}} \tag{1.3}$$

and $$\overline{y}_P^* = \overline{y}\frac{\overline{Z}}{\overline{z}^*} \tag{1.4}$$

where $\overline{x}^* = \frac{N\overline{X} - n\overline{x}}{(N-n)}$ and $\overline{z}^* = \frac{N\overline{Z} - n\overline{z}}{(N-n)}$.

In some survey situations, information on a secondary auxiliary variate z, correlated negatively with the study variate y, is readily available. Let $\overline{Z}$ be the known population mean of z. For estimating $\overline{Y}$, Singh (1967) considered a ratio-cum-product estimator

$$\overline{y}_{RP} = \overline{y}\left(\frac{\overline{X}}{\overline{x}}\right)\left(\frac{\overline{z}}{\overline{Z}}\right) \tag{1.5}$$

Using a simple transformation $x_i^* = \left(1+g\right)\overline{X} - gx_i$ and $z_i^* = (1+g)\overline{Z} - gz_i$, i=1,2…,N, with $g = \frac{n}{\left(N-n\right)}$, Singh et al. (2005) proposed a dual to usual ratio-cum-product estimator

$$\bar{y}^*_{RP} = \bar{y}\left(\frac{\bar{x}^*}{\bar{X}}\right)\left(\frac{\bar{Z}}{\bar{z}^*}\right) \tag{1.6}$$

where $\bar{x}^* = (1+g)\bar{X} - g\bar{x}$ and $\bar{z}^* = (1+g)\bar{Z} - g\bar{z}$.

To the first degree of approximation

$$V(\bar{y}) = \theta\bar{Y}C_y^2 \tag{1.7}$$

$$MSE(\bar{y}_R) = \theta\bar{Y}^2\left[C_y^2 + C_x^2(1 - 2K_{yx})\right] \tag{1.8}$$

$$MSE(\bar{y}_P) = \theta\bar{Y}^2\left[C_y^2 + C_x^2(1 + 2K_{yz})\right] \tag{1.9}$$

$$MSE(\bar{y}_{RP}) = \theta\bar{Y}^2\left[C_y^2 + C_x^2(1 + 2K_{yz}) + C_x^2(1 - 2K)\right] \tag{1.10}$$

$$MSE(\bar{y}^*_R) = \theta\bar{Y}^2\left[C_y^2 + gC_x^2(g - 2K_{yx})\right] \tag{1.11}$$

$$MSE(\bar{y}^*_P) = \theta\bar{Y}^2\left[C_y^2 + gC_z^2(g + 2K_{yz})\right] \tag{1.12}$$

$$MSE(\bar{y}^*_{RP}) = \theta\bar{Y}^2\left[C_y^2 + gC_z^2(g + 2K_{yz}) + gC_x^2(g - 2gK_{zx} - 2K_{yx})\right] \tag{1.13}$$

where MSE (.) stands for mean square error (MSE) of (.).

$$\theta = \frac{1-f}{n}, f = \frac{n}{N}, C_y = \frac{S_y}{\bar{Y}}, C_x = \frac{S_x}{\bar{X}}, C_z = \frac{S_z}{\bar{Z}}, K_{yx} = \rho_{yx}\frac{C_y}{C_x}, K_{yz} = \rho_{yz}\frac{C_y}{C_z},$$

$$K_{zx} = \rho_{xz}\frac{C_z}{C_x},\ K = K_{yx} + K_{zx},\ \rho_{yx} = \frac{S_{yx}}{S_y S_x},\ \rho_{yz} = \frac{S_{yz}}{S_y S_z},\ \rho_{xz} = \frac{S_{xz}}{S_x S_z},$$

$$S_y^2 = \frac{1}{(N-1)}\sum_{i=1}^{N}(y_i - \bar{Y})^2,\quad S_x^2 = \frac{1}{(N-1)}\sum_{i=1}^{N}(x_i - \bar{X})^2, S_z^2 = \frac{1}{(N-1)}\sum_{i=1}^{N}(z_i - \bar{Z})^2,$$

$$S_{yx} = \frac{1}{(N-1)}\sum_{i=1}^{N}(y_i - \bar{Y})(x_i - \bar{X}),\qquad S_{yz} = \frac{1}{(N-1)}\sum_{i=1}^{N}(y_i - \bar{Y})(z_i - \bar{Z}),$$

and $S_{xz} = \frac{1}{(N-1)}\sum_{i=1}^{N}(x_i - \bar{X})(z_i - \bar{Z})$.

In this paper, under SRSWOR, we present a family of dual to ratio-cum-product estimator for estimating the population mean $\bar{Y}$. We obtain the first order approximation of the bias and the MSE for this family of estimators. Numerical illustrations are given to show the performance of the constructed estimator over other estimators.

2. The suggested family of estimators

We define a family of estimators of $\overline{Y}$ as

$$t = \bar{y}\left(\frac{\bar{x}^*}{\overline{X}}\right)^{\alpha_1}\left(\frac{\overline{Z}}{\bar{z}^*}\right)^{\alpha_2} \tag{2.1}$$

where α_i′s (i=1,2) are unknown constants to be suitably determined.

To obtain the bias and MSE of t, we write

$$\bar{y} = \overline{Y}(1+e_0),\ \bar{x} = \overline{X}(1+e_1),\ \bar{z} = \overline{Z}(1+e_2)$$

such that

$$E\ (e_o) = E\ (e_1) = E\ (e_2) = 0$$

and

$$E(e_0^2) = \theta C_y^2,\ E(e_1^2) = \theta C_x^2,\ E(e_2^2) = \theta C_z^2,$$
$$E(e_0 e_1) = \theta\rho_{xy} C_y C_x,\ E(e_0 e_2) = \theta\rho_{yz} C_y C_z,\ E(e_1 e_2) = \theta\rho_{xz} C_x C_z.$$

Expressing t in terms of e's, we have

$$t = \overline{Y}(1+e_0)(1-ge_1)^{\alpha_1}(1-ge_2)^{-\alpha_2} \tag{2.2}$$

We assume that $|ge_1|<1$, $|ge_2|<1$, so that $(1-ge_1)^{\alpha_1}$ and $(1-ge_2)^{-\alpha_2}$ are expandable. Expanding the right hand side of (2.2) and retaining terms up to second powers of e's (up to the first order of approximation), we have

$$t = \overline{Y}[1 + e_0 - \alpha_1 ge_1 - \alpha_1 ge_0 e_1 + \frac{\alpha_1(\alpha_1-1)}{2} g^2 e_1^2 + \alpha_2 ge_2 + \alpha_2 ge_0 e_2$$
$$- \alpha_1\alpha_2 g^2 e_1 e_2 + \frac{\alpha_2(\alpha_2-1)}{2} g^2 e_2^2] \tag{2.3}$$

Taking expectations of both sides in (2.3) and then subtracting $\overline{Y}$ from both sides, we get the bias of the estimator t, up to the first order of approximation, as

$$B(t) = E(t - \overline{Y})$$

$$= \overline{Y} g\theta[(\alpha_2 K_{yz} C_z^2 - \alpha_1 K_{yx} C_x^2) + \frac{\alpha_1(\alpha_1-1)}{2} gC_x^2 - \alpha_2 gC_z^2\{\alpha_1 K_{xz} - \frac{(\alpha_2-1)}{2}\}] \tag{2.4}$$

where $K_{xz} = \rho_{xz}\dfrac{C_x}{C_z}$.

From (2.3), we have

$$\left(t-\overline{Y}\right)\cong\overline{Y}\left[e_0-g(\alpha_1 e_1-\alpha_2 e_2)\right] \tag{2.5}$$

Squaring both sides of (2.5), we have

$$\left(t-\overline{Y}\right)^2=\overline{Y}^2[e_0^2+g^2\{\alpha_1^2 e_1^2+\alpha_2^2 e_2^2-2\alpha_1\alpha_2 e_1 e_2\}-2g(\alpha_1 e_0 e_1-\alpha_2 e_0 e_2)] \tag{2.6}$$

Taking expectations of both sides of (2.6), we get the MSE of t to the first degree of approximation as

$$MSE(t)=\theta\overline{Y}^2[C_y^2+g\alpha_2 C_z^2(g\alpha_2+2K_{yz})+C_x^2 g(g\alpha_1^2-2\alpha_1\alpha_2 gK_{zx}-2\alpha_1 K_{yx})] \tag{2.7}$$

Minimization of (2.7) with respect to α_1 and α_2 yields their optimum values as

$$\left.\begin{aligned}\alpha_1&=\frac{K_{yx}-K_{zx}K_{yz}}{g\left(1-\rho_{xz}^2\right)}\\ \alpha_2&=\frac{-(K_{yz}-K_{xz}K_{yx})}{g(1-\rho_{xz}^2)}\end{aligned}\right\} \tag{2.8}$$

Substitution of (2.8) in (2.7) yields the minimum value of MSE (t) as

$$\min.MSE(t)=\theta\overline{Y}^2 C_y^2(1-\rho_{y.xz}^2) \tag{2.9}$$

where $\rho_{y.xz}^2=\dfrac{\rho_{yx}^2+\rho_{yz}^2-2\rho_{yx}\rho_{yz}\rho_{xz}}{(1-\rho_{xz}^2)}$ is the multiple correlation coefficient of y on x and z.

***Remark 2.1**:* For (α_1,α_2)=(1,0), the estimator t reduces to the 'dual to ratio' estimator

$$\overline{y}_R^*=\overline{y}\left(\frac{\overline{x}^*}{\overline{X}}\right)$$

While for (α_1,α_2)=(0,1) it reduces to the 'dual to product' estimator

$$\overline{y}_P^*=\overline{y}\left(\frac{\overline{Z}}{\overline{z}^*}\right)$$

It coincides with the estimator in Singh et al. (2005) when (α_1,α_2)=(1,1).

***Remark 2.2**:* The optimum values of α_{io}'s (i=1,2) are functions of unknown population parameters such as K_{yx}, K_{yz}, K_{yx}. The values of these unknown population parameters can be guessed quite accurately from the past data or experiences gathered in due course of time, for instance, see Srivastava (1967), Reddy (1973), Prasad (1989,p.391), Murthy (1967,pp96-99), Singh et. al. (2007) and Singh et. al. (2009). Also the prior values of K_{yx}, K_{yz}, K_{zx} may be either obtained on the basis of the information from the most recent survey or by conducting a pilot survey, see, Lui (1990,p.3805).

From (1.7) to (1.13) and (2.9) it can be shown that the proposed estimator t at (2.1) is more efficient than usual unbiased estimator $\bar{y}$, usual ratio estimator $\bar{y}_R$, product estimator $\bar{y}_P$, Singh (1967) ratio-cum product estimator $\bar{y}_{RP}$, Srivenkataramana (1980) and Bandyopadhyaya (1980) dual to ratio estimator $\bar{y}_R^*$, dual to product estimator $\bar{y}_P^*$ and Singh et al. (2005) dual to ratio-cum-product estimator $\bar{y}_{RP}^*$ at its optimum conditions.

3. Empirical study

In this section we illustrate the performance of the constructed estimator t over various other estimators $\bar{y}, \bar{y}_R, \bar{y}_P, \bar{y}_{RP}, \bar{y}_R^*, \bar{y}_P^*, \bar{y}_{RP}^*$ through natural data earlier used by Singh (1969, p.377).

y: number of females employed
x: number of females in service
z: number of educated females

$\bar{Y} = 7.46$, $\bar{X} = 5.31$, $\bar{Z} = 179.00$, $C_y^2 = 0.5046$, $C_x^2 = 0.5737$, $C_z^2 = 0.0633$,

$\rho_{yx} = 0.7737$ $\rho_{yz} = -0.2070$, $\rho_{xz} = -0.0033$, N=61 and n=20.

Table 3.1: Range of α_1 and α_2 for which t is better than $\bar{y}_{RP}^*$

α_2	α_1						
	0.94	1	1.25	1.39($\alpha_{1(opt)}$)	1.5	1.7	1.85
2.7	100.81	104.67	116.46	118.52	117.4	109.7	100.6
3.1	101.01	105.02	116.8	118.9	117.8	110.13	100.9
3.34($\alpha_{2(opt)}$)	101.14	105.08	116.96	119.03	117.8	110.19	101.0
3.4	101.2	105.08	116.95	119.02	117.8	110.19	101.04
3.8	101.02	104.8	116.71	118.76	117.6	109.9	100.8
4.5	100.04	103.8	115.38	117.38	116.2	108.7	99.8
5.0	98.83	102.5	113.77	115.71	114.6	107.3	98.6

The percent relative efficiencies (PRE's) of the different estimators with respect to the proposed estimator t are computed by the formula

$$\mathrm{PRE}\,(t,.) = \frac{\mathrm{MSE}(.)}{\mathrm{MSE}(t)} * 100$$

and presented in table 3.2.

Table 3.2: PRE's of various estimators of $\bar{Y}$ with respect to $\bar{y}$

Estimator	**PRE**
$\bar{y}$	100
$\bar{y}_R$	203.43
$\bar{y}_P$	123.78
$\bar{y}_{RP}$	213.64
$\bar{y}_R^*$	214.78
$\bar{y}_P^*$	104.35
$\bar{y}_{RP}^*$	235.68
t_{opt}^*	278.21

4. Conclusion

(i) Table 3.1 shows that there is a wide scope of choosing α_1 and α_2 for which our proposed estimator 't' performs better than $\bar{y}_{RP}^*$.

(ii) Table 3.2 shows clearly that the proposed estimator 't' is more efficient than all other estimators $\bar{y}, \bar{y}_R, \bar{y}_P, \bar{y}_{RP}, \bar{y}_R^*, \bar{y}_P^*$ and $\bar{y}_{RP}^*$ with considerable gain in efficiency.

I this paper we have presented a family of dual to ratio-cum-product estimators for the finite population mean. For future research the family suggested here can be adapted to double sampling scheme using Kumar and Bahl (2006) estimator.

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