

A GENERALIZATION OF THE INEQUALITY OF HÖLDER

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One generalizes the inequality of Hölder thanks to a reasoning by recurrence. As particular cases, one obtains a generalization of the inequality of Cauchy-Buniakovski-Schwartz, and some interesting applications.

Theorem: If $a_i^{(k)} \in \mathbb{R}_+$ and $p_k \in]1, +\infty[$, $i \in \{1, 2, \dots, n\}$, $k \in \{1, 2, \dots, m\}$, such that, $\frac{1}{p_1} + \frac{1}{p_2} + \dots + \frac{1}{p_m} = 1$, then:

$$\sum_{i=1}^n \prod_{k=1}^m a_i^{(k)} \leq \prod_{k=1}^m \left(\sum_{i=1}^n (a_i^{(k)})^{p_k} \right)^{\frac{1}{p_k}} \text{ with } m \geq 2.$$

Proof:

For $m = 2$ one obtains exactly the inequality of Hölder, which is true. One supposes that the inequality is true for the values which are strictly smaller than a certain m .

Then,:

$$\sum_{i=1}^n \prod_{k=1}^m a_i^{(k)} = \sum_{i=1}^n \left(\left(\prod_{k=1}^{m-2} a_i^{(k)} \right) \cdot (a_i^{(m-1)} \cdot a_i^{(m)}) \right) \leq \left(\prod_{k=1}^{m-2} \left(\sum_{i=1}^n (a_i^{(k)})^{p_k} \right)^{\frac{1}{p_k}} \right) \cdot \left(\sum_{i=1}^n (a_i^{(m-1)} \cdot a_i^{(m)})^p \right)^{\frac{1}{p}}$$

where $\frac{1}{p_1} + \frac{1}{p_2} + \dots + \frac{1}{p_{m-2}} + \frac{1}{p} = 1$ and $p_h > 1$, $1 \leq h \leq m-2$, $p > 1$;

but

$$\sum_{i=1}^n (a_i^{(m-1)})^p \cdot (a_i^{(m)})^p \leq \left(\sum_{i=1}^n ((a_i^{(m-1)})^p)^{t_1} \right)^{\frac{1}{t_1}} \cdot \left(\sum_{i=1}^n ((a_i^{(m)})^p)^{t_2} \right)^{\frac{1}{t_2}}$$

where $\frac{1}{t_1} + \frac{1}{t_2} = 1$ and $t_1 > 1$, $t_2 > 2$.

It results from it:

$$\sum_{i=1}^n (a_i^{(m-1)})^p \cdot (a_i^{(m)})^p \leq \left(\sum_{i=1}^n (a_i^{(m-1)})^{pt_1} \right)^{\frac{1}{p_1}} \cdot \left(\sum_{i=1}^n (a_i^{(m)})^{pt_2} \right)^{\frac{1}{p_2}}$$

with $\frac{1}{pt_1} + \frac{1}{pt_2} = \frac{1}{p}$.

Let us note $pt_1 = p_{m-1}$ and $pt_2 = p_m$. Then $\frac{1}{p_1} + \frac{1}{p_2} + \dots + \frac{1}{p_m} = 1$ is true and one has $p_j > 1$ for $1 \leq j \leq m$ and it results the inequality from the theorem.

Note: If one poses $p_j = m$ for $1 \leq j \leq m$ and if one raises to the power m this inequality, one obtains a generalization of the inequality of Cauchy-Buniakovski-Schwartz:

$$\left(\sum_{i=1}^n \prod_{k=1}^m a_i^{(k)} \right)^m \leq \prod_{k=1}^m \sum_{i=1}^n (a_i^{(k)})^m.$$

Application:

Let $a_1, a_2, b_1, b_2, c_1, c_2$ be positive real numbers.

Show that:

$$(a_1 b_1 c_1 + a_2 b_2 c_2)^6 \leq 8(a_1^6 + a_2^6)(b_1^6 + b_2^6)(c_1^6 + c_2^6)$$

Solution:

We will use the previous theorem. Let us choose $p_1 = 2$, $p_2 = 3$, $p_3 = 6$; we will obtain the following:

$$a_1 b_1 c_1 + a_2 b_2 c_2 \leq (a_1^2 + a_2^2)^{\frac{1}{2}} (b_1^3 + b_2^3)^{\frac{1}{3}} (c_1^6 + c_2^6)^{\frac{1}{6}},$$

or more:

$$(a_1 b_1 c_1 + a_2 b_2 c_2)^6 \leq (a_1^2 + a_2^2)^3 (b_1^3 + b_2^3)^2 (c_1^6 + c_2^6),$$

and knowing that

$$(b_1^3 + b_2^3)^2 \leq 2(b_1^6 + b_2^6)$$

and that

$$(a_1^2 + a_2^2)^3 = a_1^6 + a_2^6 + 3(a_1^4 a_2^2 + a_1^2 a_2^4) \leq 4(a_1^6 + a_2^6)$$

since

$$a_1^4 a_2^2 + a_1^2 a_2^4 \leq a_1^6 + a_2^6 \text{ (because: } -(a_2^2 - a_1^2)^2 (a_1^2 + a_2^2) \leq 0 \text{)}$$

it results the exercise which was proposed.