

GENERALIZATION OF THE THEOREM OF MENELAUS USING A SELF-RECURRENT METHOD

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Abstract.

This generalization of the Theorem of Menelaus from a triangle to a polygon with n sides is proven by a self-recurrent method which uses the induction procedure and the Theorem of Menelaus itself.

The **Theorem of Menelaus for a Triangle** is the following:

If a line (d) intersects the triangle $\Delta A_1A_2A_3$ sides A_1A_2 , A_2A_3 , and A_3A_1 respectively in the points M_1 , M_2 , M_3 , then we have the following equality:

$$\frac{M_1A_1}{M_1A_2} \cdot \frac{M_2A_2}{M_2A_3} \cdot \frac{M_3A_3}{M_3A_1} = 1$$

where by M_1A_1 we understand the (positive) length of the segment of line or the distance between M_1 and A_1 ; similarly for all other segments of lines.

Let's generalize the Theorem of Menelaus for any n -gon (a polygon with n sides), where $n \geq 3$, using our Recurrence Method for Generalizations, which consists in doing an induction and in using the Theorem of Menelaus itself.

For $n = 3$ the theorem is true, already proven by Menelaus.

The **Theorem of Menelaus for a Quadrilateral**.

Let's prove it for $n = 4$, which will inspire us to do the proof for any n .

Suppose a line (d) intersects the quadrilateral $A_1A_2A_3A_4$ sides A_1A_2 , A_2A_3 , A_3A_4 , and A_4A_1 respectively in the points M_1 , M_2 , M_3 , and M_4 , while its diagonal A_2A_4 into the point M [see Fig. 1 below].

We split the quadrilateral $A_1A_2A_3A_4$ into two disjoint triangles (3-gons) $\Delta A_1A_2A_4$ and $\Delta A_4A_2A_3$, and we apply the Theorem of Menelaus in each of them, respectively getting the following two equalities:

$$\frac{M_1A_1}{M_1A_2} \cdot \frac{MA_2}{MA_4} \cdot \frac{M_4A_4}{M_4A_1} = 1$$

and

$$\frac{MA_4}{MA_2} \cdot \frac{M_2A_2}{M_2A_3} \cdot \frac{M_3A_3}{M_3A_4} = 1.$$

Now, we multiply these last two relationships and we obtain the Theorem of Menelaus for $n = 4$ (a quadrilateral):

$$\frac{M_1A_1}{M_1A_2} \cdot \frac{M_2A_2}{M_2A_3} \cdot \frac{M_3A_3}{M_3A_4} \cdot \frac{M_4A_4}{M_4A_1} = 1.$$

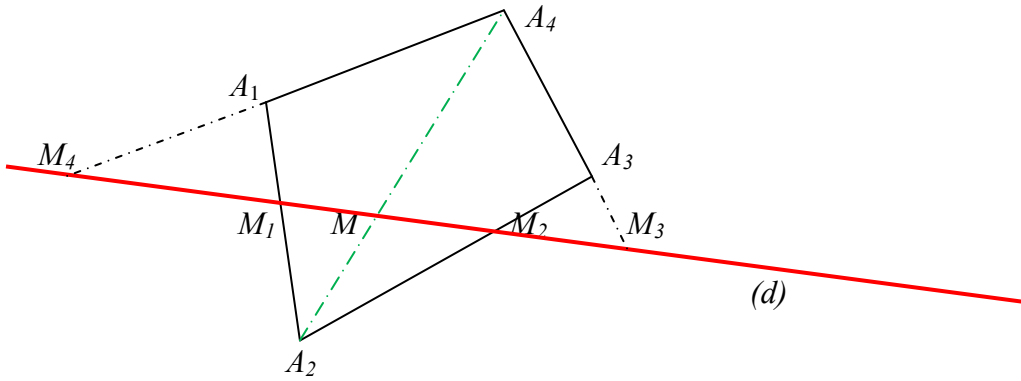


Fig. 1

Let's suppose by induction upon $k \geq 3$ that the Theorem of Menelaus is true for any k -gon with $3 \leq k \leq n-1$, and we need to prove it is also true for $k = n$.

Suppose a line (d) intersects the n -gon $A_1A_2 \dots A_n$ sides A_iA_{i+1} in the points M_i , while its diagonal A_2A_n into the point M {of course by A_nA_{n+1} one understands A_nA_1 } – see Fig. 2.

We consider the n -gon $A_1A_2 \dots A_{n-1}A_n$ and we split it similarly as in the case of quadrilaterals in a 3-gon $\Delta A_1A_2A_n$ and an $(n-1)$ -gon $A_nA_2A_3 \dots A_{n-1}$ and we can respectively apply the Theorem of Menelaus according to our previously hypothesis of induction in each of them, and we respectively get:

$$\frac{M_1A_1}{M_1A_2} \cdot \frac{MA_2}{MA_n} \cdot \frac{MnAn}{MnA_1} = 1$$

and

$$\frac{MA_n}{MA_2} \cdot \frac{M_2A_2}{M_2A_3} \cdot \dots \cdot \frac{M_{n-2}A_{n-2}}{M_{n-2}A_{n-1}} \cdot \frac{M_{n-1}A_{n-1}}{M_{n-1}A_n} = 1$$

whence, by multiplying the last two equalities, we get

the **Theorem of Menelaus for any n -gon**:

$$\prod_{i=1}^n \frac{M_iA_i}{M_iA_{i+1}} = 1.$$

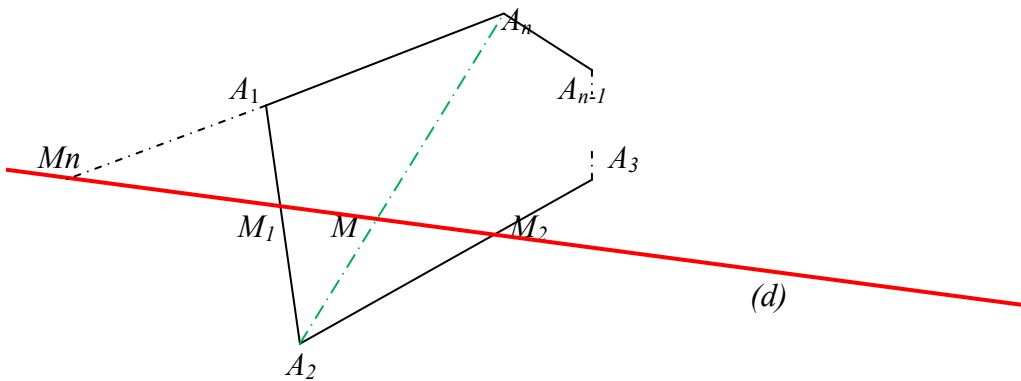


Fig. 2

Conclusion.

We hope the reader will find useful this self-recurrence method in order to generalize known scientific results by means of themselves!

{Translated from French by the Author.}

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