GENERALIZATION OF THE THEOREM OF MENELAUS USING A SELF-RECURRENT METHOD

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Abstract.
This generalization of the Theorem of Menelaus from a triangle to a polygon with *n* sides is proven by a self-recurrent method which uses the induction procedure and the Theorem of Menelaus itself.

The **Theorem of Menelaus for a Triangle** is the following:

If a line *(d)* intersects the triangle $\Delta A_1A_2A_3$ sides A_1A_2, A_2A_3, and A_3A_1 respectively in the points M_1, M_2, M_3, then we have the following equality:

$$\frac{M_1A_1}{M_1A_2} \cdot \frac{M_2A_2}{M_2A_3} \cdot \frac{M_3A_3}{M_3A_1} = 1$$

where by M_1A_1 we understand the (positive) length of the segment of line or the distance between M_1 and A_1; similarly for all other segments of lines.

Let's generalize the Theorem of Menelaus for any *n-gon* (a polygon with *n* sides), where $n \geq 3$, using our Recurrence Method for Generalizations, which consists in doing an induction and in using the Theorem of Menelaus itself.

For $n = 3$ the theorem is true, already proven by Menelaus.

The **Theorem of Menelaus for a Quadrilateral**.

Let's prove it for $n = 4$, which will inspire us to do the proof for any *n*.

Suppose a line *(d)* intersects the quadrilateral $A_1A_2A_3A_4$ sides A_1A_2, A_2A_3, A_3A_4, and A_4A_1 respectively in the points M_1, M_2, M_3, *and* M_4, while its diagonal A_2A_4 into the point M [see *Fig. 1* below].

We split the quadrilateral $A_1A_2A_3A_4$ into two disjoint triangles (*3-gons*) $\Delta A_1A_2A_4$ and $\Delta A_4A_2A_3$, and we apply the Theorem of Menelaus in each of them, respectively getting the following two equalities:

$$\frac{M_1A_1}{M_1A_2}\cdot\frac{MA_2}{MA_4}\cdot\frac{M_4A_4}{M_4A_1}=1$$

and

$$\frac{MA_4}{MA_2}\cdot\frac{M_2A_2}{M_2A_3}\cdot\frac{M_3A_3}{M_3A_4}=1.$$

Now, we multiply these last two relationships and we obtain the Theorem of Menelaus for $n = 4$ (a quadrilateral):

$$\frac{M_1A_1}{M_1A_2}\cdot\frac{M_2A_2}{M_2A3}\cdot\frac{M_3A_3}{M_3A_4}\cdot\frac{M_4A_4}{M_4A_1}=1.$$

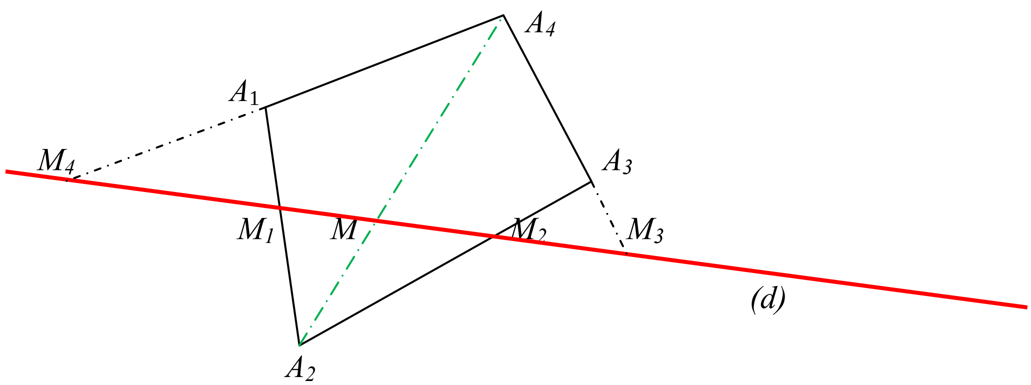

Fig. 1

Let's suppose by induction upon $k \geq 3$ that the Theorem of Menelaus is true for any *k-gon* with $3 \leq k \leq n - 1$, and we need to prove it is also true for $k = n$.

Suppose a line *(d)* intersects the n-gon $A_1A_2...A_n$ sides A_iA_{i+1} in the points M_i, while its diagonal A_2A_n into the point M {of course by A_nA_{n+1} one understands A_nA_1} – see *Fig. 2*.

We consider the *n-gon* $A_1A_2...A_{n-1}A_n$ and we split it similarly as in the case of quadrilaterals in a *3-gon* $\Delta A_1A_2A_n$ and an *(n-1)-gon* $A_nA_2A_3...A_{n-1}$ and we can respectively apply the Theorem of Menelaus according to our previously hypothesis of induction in each of them, and we respectively get:

$$\frac{M_1A_1}{M_1A_2}\cdot\frac{MA_2}{MAn}\cdot\frac{MnAn}{MnA_1}=1$$

and

$$\frac{MA_n}{MA_2} \cdot \frac{M_2A_2}{M_2A_3} \cdot \ldots \cdot \frac{M_{n-2}A_{n-2}}{M_{n-2}A_{n-1}} \cdot \frac{M_{n-1}A_{n-1}}{M_{n-1}A_n} = 1$$

whence, by multiplying the last two equalities, we get

the **Theorem of Menelaus for any *n-gon***:

$$\prod_{i=1}^{n} \frac{M_iA_i}{M_iA_{i+1}} = 1.$$

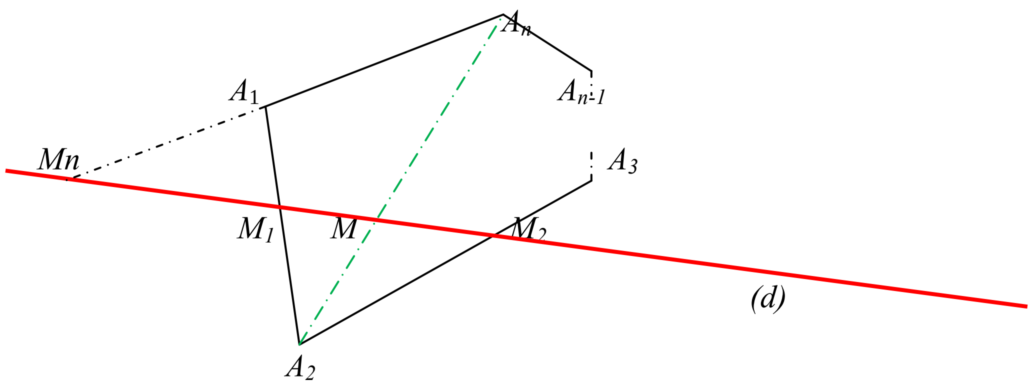

Fig. 2

Conclusion.

We hope the reader will find useful this self-recurrence method in order to generalize known scientific results by means of themselves!

{Translated from French by the Author.}

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