A GENERALIZATION OF A THEOREM OF CARNOT

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Theorem of Carnot: Let M be a point on the diagonal AC of an arbitrary quadrilateral $ABCD$. Through M one draws a line which intersects AB in α and BC in β. Let us draw another line, which intersects CD in γ and AD in δ. Then one has:

$$\frac{A\alpha}{B\alpha} \cdot \frac{B\beta}{C\beta} \cdot \frac{C\gamma}{D\gamma} \cdot \frac{D\delta}{A\delta} = 1 .$$

Generalization: Let $A_1 ... A_n$ be a polygon. On a diagonal $A_1 A_k$ of this polygon one takes a point M through which one draws a line d_1 which intersects the lines $A_1A_2, A_2A_3, ..., A_{k-1}A_k$ respectively in the points $P_1, P_2, ..., P_{k-1}$ and another line d_2 intersects the other lines $A_kA_{k+1}, , ..., A_{n-1}A_n, A_nA_1$ respectively in the points $P_k, ..., P_{n-1}, P_n$. Then one has:

$$\prod_{i=1}^{n} \frac{A_i P_i}{A_{\varphi(i)} P_i} = 1 ,$$

where φ is the circular permutation

$$\begin{pmatrix} 1 & 2 & ... & n-1 & n \\ 2 & 3 & ... & n & 1 \end{pmatrix} .$$

Proof:

Let us have $1 \le j \le k-1$. One easily shows that:

$$\frac{A_j P_j}{A_{j+1} P_j} = \frac{D(A_j, d_1)}{D(A_{j+1}, d_1)}$$

where $D(A, d)$ represents the distance from the point A to the line d, since the triangles $P_j A_j A_j^{'}$ and $P_j A_{j+1} A_{j+1}^{'}$ are similar. (One notes with $A_j^{'}$ and $A_{j+1}^{'}$ the projections of the points A_j and A_{j+1} on the line d_1).

It results from it that:

$$\frac{A_1P_1}{A_2P_1} \cdot \frac{A_2P_2}{A_3P_2} \cdots \frac{A_{k-1}P_{k-1}}{A_kP_{k-1}} = \frac{D(A_1, d_1)}{D(A_2, d_1)} \cdot \frac{D(A_2, d_1)}{D(A_3, d_1)} \cdots \frac{D(A_{k-1}, d_1)}{D(A_k, d_1)} = \frac{D(A_1, d_1)}{D(A_k, d_1)}$$

In a similar way, for $k \le h \le n$ one has:

$$\frac{A_h P_h}{A_{\varphi(h)} P_h} = \frac{D(A_h, d_2)}{D(A_{\varphi(h)}, d_2)}$$

and

$$\prod_{h=k}^{n} \frac{A_h P_h}{A_{\varphi(h)} P_h} = \frac{D(A_k, d_2)}{D(A_1, d_2)}$$

The product of the theorem is equal to:

$$\frac{D(A_1, d_1)}{D(A_k, d_1)} \cdot \frac{D(A_k, d_2)}{D(A_1, d_2)},$$

but

$$\frac{D(A_1, d_1)}{D(A_k, d_1)} = \frac{A_1 M}{A_k M}$$

since the triangles $MA_1A_1^{'}$ and $MA_kA_k^{'}$ are similar. In the same way, because the triangles $MA_1A_1^{''}$ and $MA_kA_k^{''}$ are similar (one notes with $A_1^{''}$ and $A_k^{''}$ the respective projections of A_1 and A_k on the line d_2), one has:

$$\frac{D(A_k, d_2)}{D(A_1, d_2)} = \frac{A_k M}{A_1 M}.$$

The product from the statement is therefore equal to 1.

Remark: If one replaces n by 4 in this theorem, one finds the theorem of Carnot.