A Generalized Numeration Base

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Abstract. Á Generalized Numeration Base is defined in this paper, and then particular cases are presented, such as Prime Base, Square Base, m-Power Base, Factorial Base, and operations in these bases.

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Introduction.

The following bases are important for partitions of integers into primes, squares, cubes, generally into m-powers, also into factorials, and into any strictly increasing sequence.

1) The Prime Base:

0,1,10,100,101,1000,1001,10000,10001,10010,10100,100000,100001,1000000,
1000001,1000010,1000100,10000000,10000001,100000000,100000001,100000010,
100000100,1000000000,1000000001,1000000010,1000000100,1000000101,... .
(Each number n written in the Prime Base.)

(We define over the set of natural numbers the following infinite base: $p_0 = 1$, and for $k >= 1$ p_k is the k-th prime number.)

He proved that every positive integer A may be uniquely written in the Prime Base as:

$$A = \overline{(a_n \ldots a_1 a_0)}_{(SP)} \overset{def}{===} \sum_{i=0}^{n} a_i p_i, \text{ with all } a_i = 0 \text{ or } 1, \text{ (of course } a_n = 1),$$

in the following way:
- if $p_n <= A < p_{n+1}$ then $A = p_n + r_1$;
- if $p_m <= r_1 < p_{m+1}$ then $r_1 = p_m + r_2$, m < n;
and so on until one obtains a rest $r_j = 0$.

Therefore, any number may be written as a sum of prime numbers + e, where e = 0 or 1.

If we note by p(A) the superior prime part of A (i.e. the largest
prime less than or equal to A), then
A is written in the Prime Base as:

    A = p(A) + p(A-p(A)) + p(A-p(A)-p(A-p(A))) + ...

This base is important for partitions with primes.

2)The Square Base:
    0,1,2,3,10,11,12,13,20,100,101,102,103,110,111,112,1000,1001,1002,1003,
    1010,1011,1012,1013,1020,10000,10001,10002,10003,10010,10011,10012,10013,
    10020,10100,10101,100000,100001,100002,100003,100010,100011,100012,100013,
    100020,100100,100101,100102,100103,100110,100111,100112,101000,101001,
    101002,101003,101010,101011,101012,101013,101020,101100,101101,101102,
    1000000,... .
(Each number n written in the Square Base.)

(We define over the set of natural numbers the following infinite
 base: for k >= 0 s_k = k^2.)

We prove that every positive integer A may be uniquely written in
the Square Base as:

$$A = (\overline{a_n \ ... \ a_1 a_0})_{(S2)} \overset{def}{===} \sum_{i=0}^{n} a_i s_i, \text{ with } a_i = 0 \text{ or } 1 \text{ for } i >= 2,$$

 0 <= a_0 <= 3, 0 <= a_1 <= 2, and of course a_n = 1,
in the following way:
  - if s_n <= A < s_{n+1} then A = s_n + r_1 ;
  - if s_m <= r_1 < p_{m+1} then r_1 = s_m + r_2 , m < n;
   and so on until one obtains a rest r_j = 0.

Therefore, any number may be written as a sum of squares (1 not counted
as a square -- being obvious) + e, where e = 0, 1, or 3.

If we note by s(A) the superior square part of A (i.e. the
largest square less than or equal to A), then A is written in the
Square Base as:

    A = s(A) + s(A-s(A)) + s(A-s(A)-s(A-s(A))) + ...

This base is important for partitions with squares.

3)The m-Power Base (generalization):

(Each number n written in the m-Power Base,
 where m is an integer >= 2.)

(We define over the set of natural numbers the following infinite
 m-Power Base: for k >= 0 t_k = k^m.)

He proved that every positive integer A may be uniquely written in
the m-Power Base as:

$$A = (\overline{a_n \ldots a_1 a_0})_{(SM)} \overset{def}{===} \sum_{i=0}^{n} a_i t_i \text{, with } a_i = 0 \text{ or } 1 \text{ for } i >= m,$$

$$0 <= a_i <= \lfloor ((i+2)^m - 1) / (i+1)^m \rfloor \text{ (integer part)}$$

for i = 0, 1, ..., m-1, a_i = 0 or 1 for i >= m, and of course a_n = 1,

in the following way:
   - if t_n <= A < t_{n+1} then A = t_n + r_1 ;
   - if t_m <= r_1 < t_{m+1} then r_1 = t_1 + r_2 , m < n;
   and so on until one obtains a rest r_j = 0.

Therefore, any number may be written as a sum of m-powers (1 not counted
as an m-power -- being obvious) + e, where e = 0, 1, 2, ..., or 2^m-1.

If we note by t(A) the superior m-power part of A (i.e. the
largest m-power less than or equal to A), then A is written in the
m-Power Base as:

   A = t(A) + t(A-t(A)) + t(A-t(A)-t(A-t(A))) + ...

This base is important for partitions with m-powers.

4)The Factorial Base:
0,1,10,11,20,21,100,101,110,111,120,121,200,201,210,211,220,221,300,301,310,
311,320,321,1000,1001,1010,1011,1020,1021,1100,1101,1110,1111,1120,1121,
1200,...
(Each number n written in the Factorial Base.)

(We define over the set of natural numbers the following infinite
 base: for k >= 1 f_k = k!)

He proved that every positive integer A may be uniquely written in
the Factorial Base as:

$$\overline{\qquad\qquad} \overset{def}{} \sum^{n}$$

```
   A = (a  ... a a )      ===  \  a f  , with all a = 0, 1, ..., i for i >= 1.
        n      2 1 (F)        /   i i              i
                             ---
                             i=1
```

in the following way:

```
  - if  f   <= A  < f      then A  = f  + r ;
        n          n+1           n    1
  - if  f   <= r  < f      then r  = f  + r , m < n;
        m    1    m+1           1    m    2
  and so on until one obtains a rest r  = 0.
                                      j
```

What's very interesting: a_1 = 0 or 1; a_2 = 0, 1, or 2; a_3 = 0, 1, 2, or 3,
and so on...

If we note by f(A) the superior factorial part of A (i.e. the
largest factorial less than or equal to A), then A is written in the
Factorial Base as:

    A = f(A) + f(A-f(A)) + f(A-f(A)-f(A-f(A))) + ...

Rules of addition and subtraction in Factorial Base:
For each digit a_I we add and subtract in base i+1, for i >= 1.

For example, an addition:
```
                         base  5 4 3 2
                         ---------------
                               2 1 0 +
                               2 2 1
                         -----------
                               1 1 0 1
```
because: 0+1= 1   (in base 2);
         1+2=10   (in base 3), therefore we write 0 and keep 1;
         2+2+1=11 (in base 4).

Now a subtraction:
```
                         base  5 4 3 2
                         ---------------
                               1 0 0 1 -
                                 3 2 0
                         ---------
                               = = 1 1
```
because: 1-0=1 (in base 2);
         0-2=? it's not possible (in base 3),
                go to the next left unit, which is 0 again (in base 4),
                go again to the next left unit, which is 1 (in base 5),
                therefore 1001 --> 0401 --> 0331
                and then 0331-320=11.

Find some rules for multiplication and division.

In a general case:
  if we want to design a base such that any number

```
                            n
        _____     def ---
  A = (a  ... a a )     === \  a b  , with all a = 0, 1, ..., t  for
       n       2 1 (B)      /  i i            i               i
                           ---
                           i=1
  i >= 1, where all t >= 1, then:
                    i
  this base should be
  b = 1,  b    = (t +1) * b  for i >= 1.
   1       i+1    i       i
```

5)The Generalized Numeration Base:

(Each number n written in the Generalized Numeration Base.)

(We define over the set of natural numbers the following infinite
Generalized Numeration Base: $1 = g_0 < g_1 < ... < g_k < ... .$)

He proved that every positive integer A may be uniquely written in
the Generalized Numeration Base as:

```
                            n
       _____    def ---                      --              --
  A = (a  ... a a )    === \  a g  , with 0 <= a  <= | (g    - 1) / g  |
       n      1 0 (SG)     /  i i            i    -- i+1         i --
                          ---
                          i=0
```

(integer part) for i = 0, 1, ..., n, and of course a_n >= 1,

in the following way:
```
  - if  g  <= A < g     then A  = g  + r ;
         n          n+1             n    1
  - if  g  <= r  < g     then r  = g  + r ,  m < n;
         m    1    m+1          1    m    2
  and so on until one obtains a rest r  = 0.
                                      j
```
If we note by g(A) the superior generalized part of A (i.e. the
largest g_i less than or equal to A), then A is written in the
Generalized Numeration Base as:

    A = g(A) + g(A-g(A)) + g(A-g(A)-g(A-g(A))) + ...

This base is important for partitions: the generalized base may be any
infinite integer set (primes, squares, cubes, any m-powers, Fibonacci/Lucas
   numbers, Bernoully numbers, Smarandache sequences, etc.) those partitions
   are studied.

A particular case is when the base verifies: $2g_i >= g_{i+1}$ for any i,

and g_0 = 1, because all coefficients of a written number in this base

will be 0 or 1.

   Remark: another particular case: if one takes $g_i = p^{i-1}$, i = 1, 2, 3,

..., p an integer >= 2, one gets the representation of a number in the numerical base p {p may be 10 (decimal), 2 (binary), 16 (hexadecimal), etc.}.

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