An Important Application of the Computation of the Distances between Remarkable Points in the Triangle Geometry

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In this article we'll prove through computation the Feuerbach's theorem relative to the tangent to the nine points circle, the inscribed circle, and the ex-inscribed circles of a given triangle.

Let ABC a given random triangle in which we denote with O the center of the circumscribed circle, with I the center of the inscribed circle, with H the orthocenter, with I_a the center of the A ex-inscribed circle, with O_9 the center of the nine points circle, with $p=\dfrac{a+b+c}{2}$ the semi-perimeter, with R the radius of the circumscribed circle, with r the radius of the inscribed circle, and with r_a the radius of the A ex-inscribed circle.

Proposition

In a triangle ABC are true the following relations:

(i) $OI^2 = R^2 - 2Rr$ Euler's relation

(ii) $OI_a^2 = R^2 + 2Rr_a$ Feuerbach's relation

(iii) $OH^2 = 2r^2 - 2p^2 + 9R^2 + 8Rr$

(iv) $IH^2 = 3r^2 - p^2 + 4R^2 + 4Rr$

(v) $I_aH^2 = r^2 - p^2 + 2r_a^{\,2} + 4R^2 + 4Rr$

Proof

(i) The positional vector of the center I of the inscribed circle of the given triangle ABC is

$$\overrightarrow{PI} = \frac{1}{2p}\left(a\overrightarrow{PA} + b\overrightarrow{PB} + c\overrightarrow{PC}\right)$$

For any point P in the plane of the triangle ABC.
We have

$$\overrightarrow{OI} = \frac{1}{2p}\left(a\overrightarrow{OA} + b\overrightarrow{OB} + c\overrightarrow{OC}\right)$$

We compute $\overrightarrow{OI} \times \overrightarrow{OI}$, and we obtain:

$$OI^2 = \frac{1}{4p^2}\left(a^2OA^2 + b^2OB^2 + c^2OC^2 + 2ab\overrightarrow{OA}\times\overrightarrow{OB} + 2bc\overrightarrow{OB}\times\overrightarrow{OC} + 2ca\overrightarrow{OC}\times\overrightarrow{OA}\right)$$

From the cosin's theorem applied in the triangle OBC we get

$$\overrightarrow{OB}\times\overrightarrow{OC}=R^2-\frac{a^2}{2}$$

and the similar relations, which substituted in the relation for OI^2 we find

$$OI^2=\frac{1}{4p^2}\left(R^2\cdot 4p^2-abc\cdot 2p\right)$$

Because $abc=4Rs$ and $s=pr$ it results (i)

(ii) The position vector of the center I_a of the A ex-inscribed circle is give by:

$$\overrightarrow{PI_a}=\frac{1}{2(p-a)}\left(-a\overrightarrow{PA}+b\overrightarrow{PB}+c\overrightarrow{PC}\right)$$

We have:

$$\overrightarrow{OI_a}=\frac{1}{2(p-a)}\left(-a\overrightarrow{OA}+b\overrightarrow{OB}+c\overrightarrow{OC}\right)$$

Computing $\overrightarrow{OI_a}\cdot\overrightarrow{OI_a}$ we obtain

$$\overrightarrow{OI_a}^2=R^2\cdot\frac{a^2+b^2+c^2}{2(p-a)^2}-\frac{ab}{2(p-a)^2}\overrightarrow{OA}\times\overrightarrow{OB}+\frac{bc}{2(p-a)^2}\overrightarrow{OB}\times\overrightarrow{OC}-\frac{ac}{2(p-a)^2}\overrightarrow{OA}\times\overrightarrow{OC}$$

Because $\overrightarrow{OB}\times\overrightarrow{OC}=R^2-\frac{a^2}{2}$ and $s=r_a(p-a)$, executing a simple computation we obtain the Feuerbach's relation.

(iii) In a triangle it is true the following relation

$$\overrightarrow{OH}=\overrightarrow{OA}+\overrightarrow{OB}+\overrightarrow{OC}$$

This is the Sylvester's relation.

We evaluate $\overrightarrow{OH}\times\overrightarrow{OH}$ and we obtain:

$$OH^2 = 9R^2-(a^2+b^2+c^2).$$

We'll prove that in a triangle we have:

$$ab+bc+ca=p^2+r^2+4Rr$$

and

$$a^2+b^2+c^2=2p^2-2r^2-8Rr$$

We obtain

$$\frac{s^2}{p}=(p-a)(p-b)(p-c)=-p^3+p(ab+bc+ca)-abc$$

Therefore

$$\frac{s^2}{p^2}=-p^2+ab+bc+ca-\frac{4Rs}{p}$$

We find that

$$ab+bc+ca=p^2+r^2+4Rr$$

Because

$$a^2+b^2+c^2=(a+b+c)^2-2(ab+bc+ca)$$

it results that

$$a^2+b^2+c^2=2p^2-2r^2-8Rr$$

which leads to (iii).

(iv) In the triangle ABC we have

$$\overrightarrow{IH}=\overrightarrow{OH}-\overrightarrow{OI}$$

We compute IH^2, and we obtain:

$$IH^2=OH^2+OI^2-2\overrightarrow{OH}\cdot\overrightarrow{OI}$$

$$\overrightarrow{OH}\times\overrightarrow{OI}=\left(\overrightarrow{OA}+\overrightarrow{OB}+\overrightarrow{OC}\right)\cdot\frac{1}{2p}\left(a\overrightarrow{OA}+b\overrightarrow{OB}+c\overrightarrow{OC}\right)$$

$$\overrightarrow{OH}\times\overrightarrow{OI}=\frac{1}{2p}\left[R^2(a+b+c)+(a+b)\times\overrightarrow{OA}\times\overrightarrow{OB}+(b+c)\times\overrightarrow{OB}\times\overrightarrow{OC}+(c+a)\times\overrightarrow{OC}\times\overrightarrow{OA}\right]=$$

$$=3R^2-\frac{a^3+b^3+c^3}{2(a+b+c)}-\frac{a^2+b^2+c^2}{2}.$$

$$IH^2=4R^2-2Rr-\frac{a^3+b^3+c^3}{a+b+c}$$

To express $a^3+b^3+c^3$ in function of p,r,R we'll use the identity:

$$a^3+b^3+c^3-3abc=(a+b+c)\left(a^2+b^2+c^2-ab-bc-ca\right).$$

and we obtain

$$a^3+b^3+c^3=2p\left(p^2-3r^2-6Rr\right)$$

Substituting in the expression of IH^2, we'll obtain the relation (iv)

(v) We have

$$\overrightarrow{HI_a}=\frac{1}{2(p-a)}\left(-a\overrightarrow{HA}+b\overrightarrow{HB}+c\overrightarrow{HC}\right)$$

We'll compute $\vec{HI}_a\times\vec{HI}_a$

$$HI_a^{\,2}=\frac{1}{4(p-a)^2}\left(a^2HA^2+b^2HB^2+c^2HC^2-2ab\overrightarrow{HA}\times\overrightarrow{HB}-2ac\overrightarrow{HA}\times\overrightarrow{HC}+2bc\overrightarrow{HB}\times\overrightarrow{HC}\right)$$

If A_1 is the middle point of (BC) it is known that $\overrightarrow{AH}=2\overrightarrow{OA_1}$, therefore

$$AH^2=4R^2-a^2$$

also,

$$\overrightarrow{HA}\times\overrightarrow{HB}=\left(\overrightarrow{OB}+\overrightarrow{OC}\right)\left(\overrightarrow{OC}+\overrightarrow{OA}\right)$$

We obtain:

$$\overrightarrow{HA}\times\overrightarrow{HB}=4R^2-\frac{1}{2}\left(a^2+b^2+c^2\right)$$

Therefore

$$a^2+b^2+c^2=2\left(p^2-r^2-4Rr\right)$$

It results

$$\overrightarrow{HA}\times\overrightarrow{HB}=r^2-p^2+4R^2+4Rr$$

Similarly,

$$\overrightarrow{HB}\times\overrightarrow{HC}=\overrightarrow{HC}\times\overrightarrow{HA}=r^2-p^2+4R^2+4Rr$$

$$HI_a^{\ 2}=\frac{1}{4\left(p-a\right)^2}\left[4R^2\left(a^2+b^2+c^2\right)-\left(a^4+b^4+c^4\right)+\left(r^2-p^2+4R^2+4Rr\right)\left(2bc-2ab-2ac\right)\right]$$

Because $b+c-a=2\left(p-a\right)$, it results

$$2bc-2ab-2ac=4\left(p-a\right)^2-\left(a^2+b^2+c^2\right)$$

$$HI_a^{\ 2}=\frac{1}{4\left(p-a\right)^2}\left[\left(a^2+b^2+c^2\right)\left(p^2-r^2-4Rr\right)+4\left(p-a\right)^2\left(r^2-p^2+4R^2+4Rr\right)-\left(a^4+b^4+c^4\right)\right]$$

It is known that

$$16s^2=2a^2b^2+2b^2c^2+2c^2a^2-a^4-b^4-c^4$$

From which we find

$$a^2b^2+b^2c^2+c^2a^2=\left(ab+bc+ca\right)^2-2abc\left(a+b+c\right)=\left(r^2+p^2+4Rr\right)^2-4pabc$$

Substituting, and after several computations we obtain (v).

Theorem (K. Feuerbach)

In a given triangle the circle of the nine points is tangent to the inscribed circle and to the ex-inscribed circles of the triangle.

Proof

We apply the median's theorem in the triangle OIH and we obtain

$$4IO_9^2=2\left(OI^2+IH^2\right)-OH^2$$

We substitute OI^2,IH^2,OH^2 with the obtained formulae in function of r,R,p and after several simple computations we'll obtain

$$IO_9=\frac{R}{2}-r$$

This relation shows that the circle of the nine points (which has the radius $\frac{R}{2}$) is tangent to inscribed circle.

We apply the median's theorem for the triangle OI_aH, and we obtain

$$4I_aO_9^2=2\left(OI_a^{\ 2}+I_aH^2\right)-OH^2$$

We substitute OI_a,I_aH,OH and we'll obtain

$$I_aO_9 = \frac{R}{2} + r_a$$

This relation shows that the circle of the nine points and the A- ex-inscribed circle are tangent in exterior.

Note

In an article published in the Gazeta Matematică, no. 4, from 1982, the late Romanian Professor Laurențiu Panaitopol asked for the finding of the strongest inequality of the type $kR^2 + hr^2 \geq a^2 + b^2 + c^2$ and proves that this inequality is

$$8R^2 + 4r^2 \geq a^2 + b^2 + c^2.$$

Taking into consideration that

$$IH^2 = 4R^2 + 2r^2 - \frac{a^2 + b^2 + c^2}{2}$$

and that $IH^2 \geq 0$ we re-find this inequality and its geometrical interpretation.

References

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