

Introduction to biquaternion number, Schrödinger equation, and fractal graph

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1. Introduction

It is known that quaternion number has wide application in theoretical physics and engineering fields alike, in particular to describe Maxwell electrodynamics. In the meantime, recently this quaternion number has also been used to draw fractal graph. The present note is intended as an introduction to this very interesting study, i.e. to find linkage between quaternion/biquaternion number, quantum mechanical equation (Schrödinger equation) and fractal graph. Hopefully this note will be found useful for subsequent study.

2. An alternative derivation of Schrödinger-type equation

In this section we will make an attempt to re-derive a Schrödinger-type equation, but with a new definition of total energy.

In this regard, it seems worth noting here that it is more proper to use Noether's expression of total energy in lieu of standard derivation of Schrödinger's equation ($E = \bar{p}^2 / 2m$). According to Noether's theorem [4], the total energy of the system corresponding to the time translation invariance is given by:

$$E = mc^2 + (cw/2) \cdot \int_0^\infty (\gamma^2 \cdot 4\pi r^2 \cdot dr) = k\mu c^2 \quad (1)$$

where k is *dimensionless* function. It could be shown, that for low-energy state the total energy could be far less than $E = mc^2$. In this regard, interestingly Bakhoun [5] has also argued in favor of using $E = mv^2$ for expression of total energy, which expression could be traced back to Leibniz. Therefore it seems possible to argue that expression $E = mv^2$ is more generalized than the standard expression of special relativity, in particular because the total energy now depends on actual velocity [4].

We start with Bakhoun's assertion $E = mv^2$, instead of more convenient form $E = mc_s^2$. This notion would imply [5]:

$$H^2 = p^2 \cdot c^2 - m_o^2 \cdot c^2 \cdot v^2. \quad (2)$$

Therefore, for photon speed in the limit $p \rightarrow 0$, we write [6]:

$$E(p) \equiv c_s \cdot p. \quad (3)$$

In the first approximation, we could derive Klein-Gordon-type relativistic equation from equation (2), as follows. By introducing a new parameter:

$$\zeta = i(v/c), \quad (4)$$

then we can rewrite equation (2) in the known procedure of Klein-Gordon equation:

$$E^2 = p^2 \cdot c^2 + \zeta^2 m_o^2 \cdot c^4, \quad (5)$$

where $E = mv^2$. [5] By using known substitution:

$$E = i\hbar \cdot \partial / \partial t, \quad p = \hbar \nabla / i, \quad (6)$$

and dividing by $(\hbar c)^2$, we get Klein-Gordon-type relativistic equation:

$$-c^{-2} \cdot \partial \Psi / \partial t + \nabla^2 \Psi = k_o'^2 \cdot \Psi, \quad (7)$$

where

$$k'_o = \zeta m_o c / \hbar . \quad (8)$$

One could derive Dirac-type equation using similar method. Nonetheless, the use of new parameter (4) seems to be indirect solution, albeit it simplifies the solution, because here we can use the same solution from Klein-Gordon equation.

Alternatively, one could derive a new quantum relativistic equation, by noting that expression of total energy $E = mv^2$ is already relativistic equation. We will derive here an alternative approach using Ulrych's [7] method to get relativistic wave function from this expression of total energy [4].

$$E = mv^2 = p.v \quad (9)$$

Taking square of this expression, we get:

$$E^2 = p^2 .v^2 \quad (10)$$

or

$$p^2 = E^2 / v^2 \quad (11)$$

Now we use Ulrych's substitution [7]:

$$\left[(P - qA)_\mu (\bar{P} - qA)^\mu \right] = p^2 , \quad (12)$$

and introducing standard substitution in Quantum Mechanics (6), one gets:

$$\left[(P - qA)_\mu (\bar{P} - qA)^\mu \right] \Psi = v^{-2} . (i\hbar . \partial / \partial t)^2 \Psi , \quad (13)$$

or

$$\left[(-i\hbar \nabla_\mu - qA_\mu) (-i\hbar \nabla^\mu - qA^\mu) - (i\hbar / v . \partial / \partial t)^2 \right] \Psi = 0 . \quad (14)$$

This equation is comparable to Schrödinger equation for a charged particle interacting with an external electromagnetic field [8]:

$$\left[(-i\hbar \nabla_\mu - qA_\mu) (-i\hbar \nabla^\mu - qA^\mu) \right] \Psi = [-i2m . \partial / \partial t + 2mU(x)] \Psi . \quad (15)$$

In other words, we could re-derive Schrödinger-type equation for a charged particle from Ulrych's approach [7].

Alternatively, one can use similar assertion as Schrödinger described in his original equation:

$$E = mv^2 = p^2 / m \quad (16)$$

Using the same method (equation 12), we get:

$$\left[(-i\hbar \nabla_\mu - qA_\mu) (-i\hbar \nabla^\mu - qA^\mu) - m . (i\hbar . \partial / \partial t) \right] \Psi = 0 . \quad (17)$$

For $m \rightarrow 1$, one recovers standard Schrödinger equation [8].

3. Introduction to Quaternion number

Let us begin with a few definitions of numbers. It is known that complex numbers are an extension to the real numbers. They can be seen as two dimensional vectors where also multiplication is defined. We define it as $z = a + bi$, where a real part, b imaginary part, and $i^2 = -1$.

Complex: $z = t + xi$
 Quaternion: $z = t + xi + yj + zk$
 Octonion: $z = t + xi + yj + zk + aE + bI + cJ + wK$

Where: $z = (t, V)$, $t = \text{scalar}$, $V = \text{vector}$.

Furthermore, we can define conjugate complex of z :

$$\bar{z} = a - bi$$

which has properties

$$\bar{z}z = a^2 + b^2 = |z|^2$$

For application of these numbers in quantum physics, see [7][9][10].

4. Introduction to Biquaternion number

Biquaternion numbers are an extension of quaternion to four dimensions [11]. They can be seen as four dimensional vectors (with one scalar and a vector in three space). In physics they are also used in relativity; it is also very useful to describe Maxwell electrodynamics in its original form [7][12].

We could define:

$$z = a + bi + cj + dk \quad \text{where } i^2=j^2=-1 \text{ and } k^2=jk=1$$

For those not familiar with the matrices of Biquaternion and quaternion algebra, here are the tables:

Biquaternion math table

	i	j	k
i	-1	k	-j
j	k	-1	-i
k	-j	-i	1

Quaternion math table

	i	j	k
i	-1	k	-j
j	-k	-1	i
k	j	-i	-1

In both quaternion and Biquaternion math $i^2 = -1$. The Biquaternion rules provide for one real variable, two complex variables (i and j) and one variable which Charles Muses refers to as countercomplex (k). In quaternion math there is one real variable and three complex variables. In Biquaternion math, unlike quaternion math, the commutative law holds, that is reversing the order of multiplication doesn't change the product.

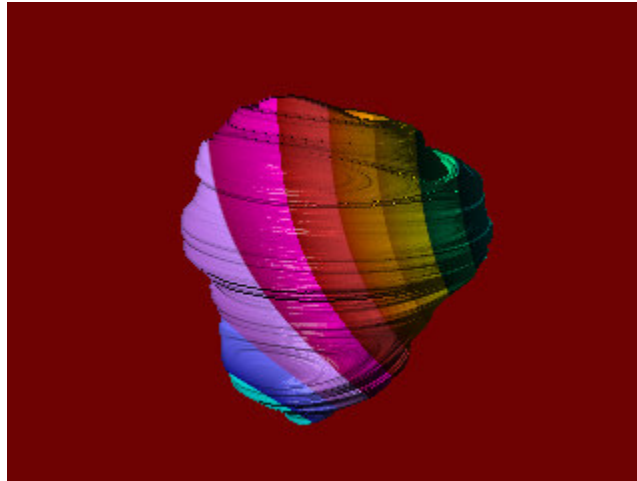
One other concept that mathematicians like to dwell on is the idea of a "ring". There is one ring in quaternion and Biquaternion math, "ijk". If you start anywhere in this ring and proceed to multiply three variables in a loop, backwards or forwards, you get the same number, 1 for Biquaternion, -1 for quaternion [13].

From this viewpoint, we can find further extension of Schrödinger type equation described above to biquaternion form [7][10].

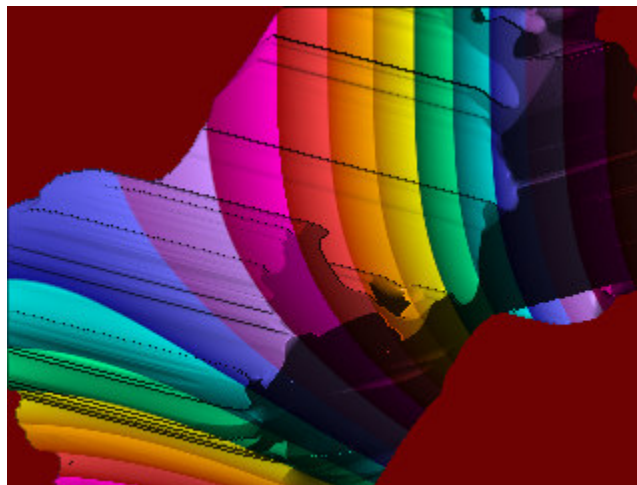
Now we're ready to find simulation of this number via fractal graph. [11]

5. Fractal graph (examples)

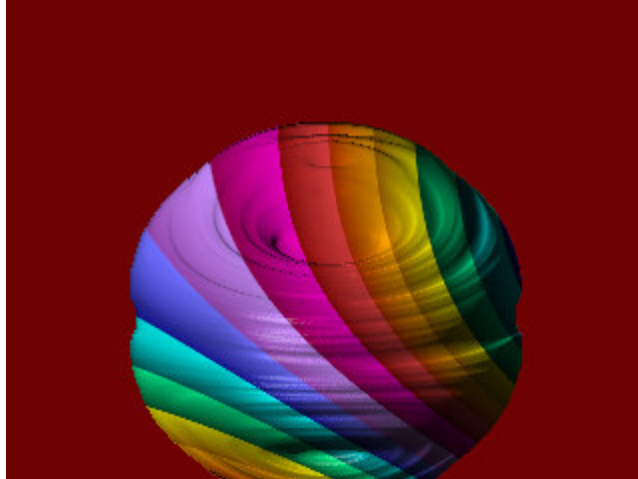
A few examples of fractal graph from quaternion number can be found at www.fraktalstudio.de, and www.bugman123.com. The following graphs were drawn with Dofo-Zon (www.mysticfractal.com), and FractalExplorer ().



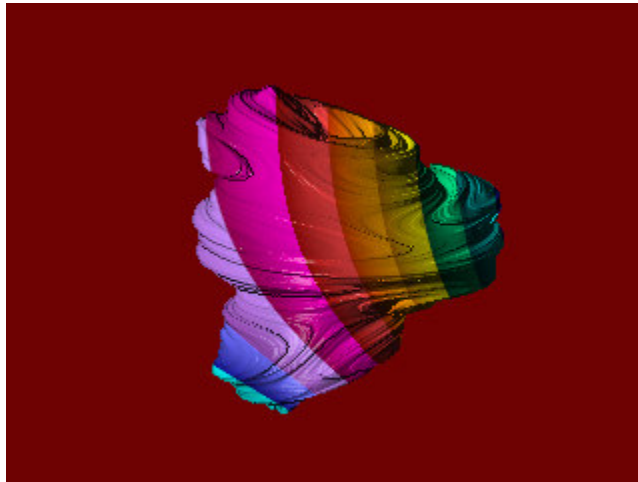
Picture 1. Random quaternion (Dofo-Zon)



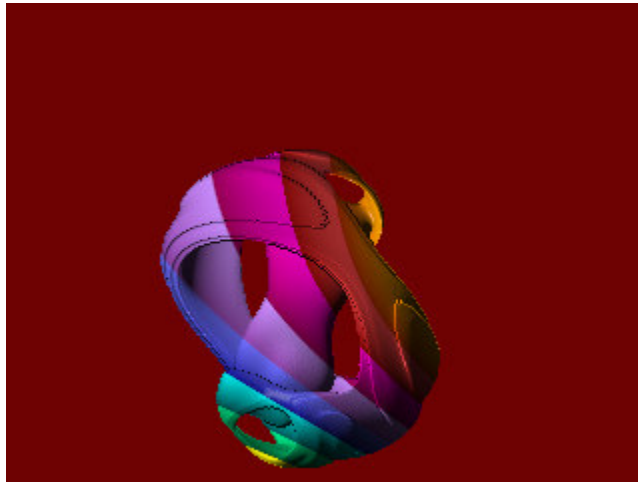
Picture 2. Random quaternion (Dofo-Zon)



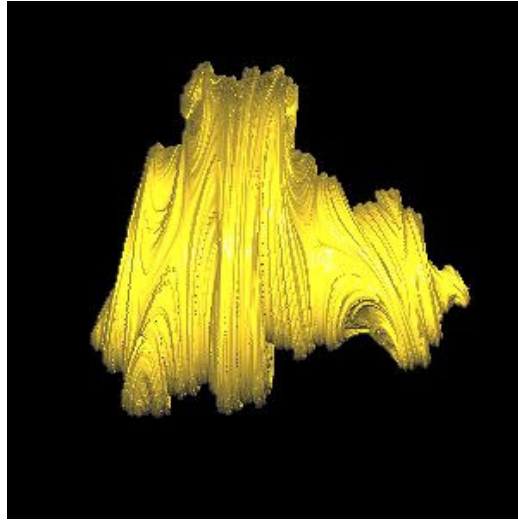
Picture 3. Random quaternion (Dof-Zon)



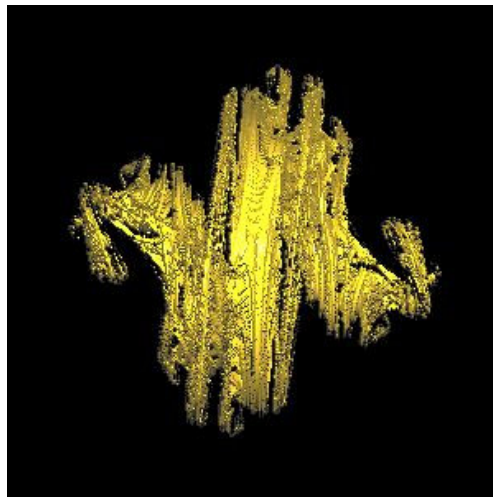
Picture 4. Random quaternion (Dof-Zon)



Picture 5. Random quaternion (Dof-Zon)



Picture 6. Quaternion, $Q*Q*Q+Q'$ (Fractal Explorer)



Picture 6. Quaternion, $Q*Q*Q*Q+Q'$ (Fractal Explorer)

Concluding remarks

We have explored some of those stunning images created using the notion of quaternion numbers to draw fractal graphs. This application of quaternion numbers in physics are known, therefore it could be expected that such quaternion/biquaternion fractal graphs will also be found useful in theoretical physics alike. This will be the subject of further exploration.

Dec. 14th, 2005

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- [13] www.mysticfractal.com; [13a] www.ultrafractal.com
- [14] <http://www.student.math.uwaterloo.ca/~pmat370/FractalRefs.html>

Fractal links

Some useful links for drawing fractal from quaternion numbers:

Name	Description	URL link
QuaSZ QuaSZ Mac	Primarily for exploring quaternions, hypernions, octonions, cubics, and complexified quats.	www.mysticfractal.com
Hydra	This program graphs 3-D slices of formulas based on 4-D complex number planes, currently supporting quaternion, hypernion, and user-customized quad types of the Mandelbrot set and Julia sets.	www.mysticfractal.com
Fractal Agent Fractal Commander	Freeware programs originally written to draw escape-type fractals using every conceivable complex math function. And now convergent and orbit-trap types, and extended basic complex math to hypercomplex and quaternion math	http://www.geocities.com/SoHo/Lofts/5601
Quaternion Julia Set VRML Server	A CGI engine used to generate VRML quaternion Julia sets.	http://www.ecs.wsu.edu/~hart

Source: http://home.att.net/~Paul.N.Lee/Fractal_Software.html