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Introduction to the Symbolic Plithogenic Algebraic Structures (revisited)

Florentin Smarandache[1]*

[1] University of New Mexico, Mathematics Department, Gallup, NM 87301, USA
* Correspondence: smarand@unm.edu

Abstract: In this paper, we recall and study the new type of algebraic structures called Symbolic Plithogenic Algebraic Structures. Their operations are given under the Absorbance Law and the Prevalence Order.

Keywords: Absorbance Law; Prevalence Order; Neutrosophic Quadruple Numbers; Plithogenic Set; Type-k Neutrosophic Set; Type-k Plithogenic Set; Hybridization of Classical, Fuzzy and Fuzzy Extension Sets; Symbolic Plithogenic Components; Symbolic Plithogenic Operations; Plithogenic Numbers; Symbolic Plithogenic Algebraic Structures; Symbolic Plithogenic Group; Symbolic Plithogenic Ring.

1. Introduction

The plithogeny, plithogenic set, plithogenic logic, plithogenic probability, plithogenic statistics, and the symbolic plithogenic algebraic structures were introduced in 2018-2019 by Smarandache [1, 2, 3, 4, 5].

Plithogeny is the genesis or origination, creation, formation, development, and evolution of new entities from dynamics and organic fusions of contradictory and/or neutrals and/or non-contradictory multiple old entities.

And **plithogenic** means what is pertaining to plithogeny.

Plithogeny is an extension of neutrosophy, which is an extension of paradoxism.

While **paradoxism** [6] is based on using opposite ideas, contradictions, paradoxes in arts, letters, and science creations,

neutrosophy is based on the dynamics of a pair of opposites (<A> ,<antiA>) and their neutral (indeterminacy) <neutA>,

but **plithogeny** on the dynamics of many pairs of opposites (<A_1>, <antiA_1>), …, (<A_k>, <antiA_k>) and their neutralities <neutA_1>, …, <neutA_k>, for $k \geq 2$ ["plitho" means "many" in Greek language].

Plithogenic Set was extended to Type-n Plithogenic Set, for integer $n \geq 1$.

Symbolic Plithogenic Numbers are generalizations of Neutrosophic Quadruple Numbers, Refined Neutrosophic Quadruple Numbers, and Symbolic Turiyam Numbers.

Consequently, the Symbolic Plithogenic Algebraic Structures (semigroup, group, ring, etc.) are generalization of the corresponding algebraic structures built on these particular cases described above.

2. Informal Definition of Plithogenic Set

A plithogenic set (PS) is a set whose elements are characterized, as in our real world, by many attributes (parameters): P_1, P_2, …, P_n.

$PS = \{x(P_1, P_2, ..., P_n), x \in U\}$, where U is a universe of discourse.

A generic element x belongs to the plithogenic set PS in a certain degree $d(P_i)$ with respect to each attribute (parameter) P_i. The degree of appurtenance of an element to the plithogenic set may be: classical, fuzzy, intuitionistic fuzzy, neutrosophic, refined neutrosophic, and any other type of extended fuzzy.

In a better descriptive way, emphasizing the degrees, we may re-write it as:

$$PS = \{x(d(P_1), d(P_2), ..., d(P_n)), x \in U\}$$

The attributes (parameters) P_1, P_2, …, P_n may be independent, dependent, or partially independent and partially dependent of each other - according to each application.

This is also called Type-1 Plithogenic Set.

3. Type-k Plithogenic Set

The Type-k Neutrosophic Set [13] has been extended to Type-k Plithogenic Set.

If the parameters $P_i, 1 \le i \le n,$ depend on sub-parameters $P_{i1}, P_{i2}, ..., P_{im_i}$ for $m_i \ge 1$ then one gets a Type-2 Plithogenic Set.

Afterward, if the sub-parameters $P_{ij}, 1 \le i \le n, 1 \le j \le m_i$ are formed by sub-sub-parameters $P_{ij1}, P_{ij2}, ..., P_{ijm_j}$ for $m_j \ge 1$ then one gets a Type-3 Plithogenic Set.

And so on, up to Type-k Plithogenic Set.

Passing to degrees, one may write:

$$PS_1 = \{x(d_1(P_1), d_1(P_2), ..., d_1(P_n)), x \in U\}$$

Type-2 Plithogenic Set

$$PS_2 = \{x(d_2(d_1(P_1)), d_2(d_1(P_2)), ..., d_2(d_1(P_n))), x \in U\}$$

In general, Type-n Plithogenic Set

$$PS_k = \{x(d_k(...d_2(d_1(P_1))...), d_k(...d_2(d_1(P_2))...), ..., d_k(...d_2(d_1(P_n))...), x \in U\}.$$

4. Hybridization of Classical, Fuzzy, and Fuzzy Extension Sets

The real applications require many times to deal with multiple types of classical, fuzzy, and fuzzy extension sets.

Assume that, starting from a neutrosopohic element of the form x(T, I, F), with $0 \le T + I + F \le 3$, one has be combined it with a Picture_Fuzzy form (T, N, F), with $0 \le T + N + F \le 1$, then one gets: the neutrosophic-picture_fuzzy hybrid form: $((TT, TN, TF), (IT, IN, IF), (FT, FN, FF))$,

with $0 \le TT + TN + TF \le 1, 0 \le IT + IN + IF \le 1, 0 \le FT + FN + FF \le 1,$

where T was split into TT, TN, TF representing the confidence in T, neutral-confidence in T, and nonconfidence in T respectively; similarly for I and F.

Further on, let's combine the result with the Spherical_Fuzzy Set, where the sum of squares of components is between 0 and 1, then one obtains a neutrosophic-picture_fuzzy-spherical_fuzzy hybrid form: $((TT, TN, TF), (IT, IN, IF), (FT, FN, FF))$, with

$$0 \le TT^2 + TN^2 + TF^2 \le 1, 0 \le IT^2 + IN^2 + IF^2 \le 1, 0 \le FT^2 + FN^2 + FF^2 \le 1.$$

The hybridization chain may be as long as needed, and may deal with various types of classical, fuzzy, and fuzzy extension sets – including repeated types.

5. Definitions of Symbolic Plithogenic Set & Symbolic Plithogenic Algebraic Structures

Let SPS be a non-empty set, included in a universe of discourse U, defined as follows:

$SPS = \{x \mid x = a_0 + a_1P_1 + a_2P_2 + ... + a_nP_n, n \ge 1, a_i \in R$ or $a_i \in C$ or a_i belong to some given algebraic structure, for $0 \le i \le n$, where *R* = the set of real numbers, *C* = the set of complex numbers, and all P_i are letters (or variables) and are called *Symbolic (Literal) Plithogenic Components (Variables)*}, where *1*, $P_1, P_2, \ldots, P_n$ act like a base for the elements of the above set *SPS*.

$a_0, a_1, a_2, \ldots, a_n$ are called coefficients.

SPS is called a Symbolic Plithogenic Set. And the algebraic structures defined on this set are called *Symbolic Plithogenic Algebraic Structures.*

In general, Symbolic (or Literal) Plithogenic Theory is referring to the use of abstract symbols {i.e. the letters/parameters) $P_1, P_2, \ldots, P_n$, representing the plithogenic components (variables) as above} in some theory.

6. Definition of Plithogenic Numbers *(PN)*

The numbers of the form $PN = a_0 + a_1P_1 + a_2P_2 + ... + a_nP_n$ defined as above are called Plithogenic Numbers, where a_nP_n is called the *leading (strongest) term*.

7. Prevalence Order *(PO)*

The experts establish a *prevalence order* [1], or total order, according to the importance of each attribute/parameter (P_i) into the application. To obtain a total order among the symbolic plithogenic components $\{P_1, P_2, ..., P_n\}$, one defines some relationships (laws) between them.

The most used one is the absorbance law.

8. Absorbance Law

We recall and use now our 2015 Absorbance Law [1], simply defined as:

the greater absorbs the smaller [the bigger fish eats the smaller fish].

9. Multiplication and Power of Symbolic Plithogenic Components under the Absorbance Law

We assume that in the above definition of the plithogenic numbers, the symbolic plithogenic components are ranked increasingly, or

$$P_1 < P_2 < ... < P_n \quad (\textit{prevalence order})$$

where "<" may signify: smaller, less important, under, inferior, etc.

Whence, the multiplication and power of symbolic plithogenic components are:

$P_i \cdot P_j = P_{\max\{i,j\}}$, whence $(P_i)^2 = P_i$.

In general, $P_{i_1} \cdot P_{i_2} \cdot ... \cdot P_{i_k} = P_{\max\{i_1, i_2, ..., i_k\}}$ and $(P_i)^m = P_i$ for integer $m \geq 1$.

Negative powers of Symbolic Plithogenic Components do not exist, $\left(P_i\right)^{-m} = \dfrac{1}{\left(P_i\right)^m}$ does not exist. For example, $\left(P_i\right)^{-1} = \dfrac{1}{P_i}$ does not exist.

And P_i to the power zero is equal to 1 by definition: $\left(P_i\right)^0 \triangleq 1$.

10. m-th Root of the Symbolic Plithogenic Components

$\sqrt[m]{P_i} = P_i, 1 \leq i \leq n$, for integer $m \geq 2$, because $\left(\sqrt[m]{P_i}\right)^m = (P_i)^m$, or $P_i = P_i$.

$\left(\sqrt[m]{P_i}\right)^m$ cannot be equal to P_{i-1} or lower, nor P_{i+1} or upper, because the last two raised to the power *m* would not give P_i.

Examples: $\sqrt{P_1} = P_1$, $\sqrt[3]{P_7} = P_7$, $\sqrt[4]{16P_9} = \sqrt[4]{16} \cdot \sqrt[4]{P_9} = 2P_9$.

11. Example of Plithogenic Set

Let's have a classical set

S = {John, George, Mary},

and each element is characterized with respect to the attributes: *Weight (W), Tallness (T), Oldness (()), Beauty (B), Health (H).*

Each person/element has some (classical, fuzzy, or any fuzzy extension) degree (*d*) with respect to each attribute (parameter): d(Weight), d(Tallness), d(Oldness), d(Beauty), d(Health).

And thus one transforms the classical set into a plithogenic set:

PS = {John[d(Weight), d(Tallness), d(Oldness), d(Beauty), d(Health)],
George[d(Weight), d(Tallness), d(Oldness), d(Beauty), d(Health)],
Mary[d(Weight), d(Tallness), d(Oldness), d(Beauty), d(Health)]}.

As a numerical example, see below, as evaluated by Expert 1:

PS_1 = *{John(0.5, 0.6, 0.3, 0.1, 0.7), George(0.1, 0.8, 0.3, 0.1, 0.4), Mary(0.9, 0.4, 0.6, 0.1, 0.2)}.*

PS_1 is a *Type-1 Plithogenic Set.*

Which means that on some corresponding scales, John's fuzzy degree of Weight is 0.5, fuzzy degree of Tallness 0.6, fuzzy degree of Oldness 0.3, fuzzy degree of Beauty 0.1, and fuzzy degree of Health 0.7. {Of course, one may consider all kind of degrees: not only fuzzy, but also: classical, intuitionistic fuzzy, neutrosophic, refined neutrosophic, and other fuzzy extensions.}

Similarly for George's and Mary's degrees.

12. Example of Type-2 Plithogenic Set

Assume that Expert 2 is not totally confident on the evaluation of the Expert 1 in the above example. Thus, he decides to evaluate the first evaluation. Expert 2 may, as well, use any types of degrees – according to the expert desire and tools, not necessarily the same as in the previous evaluation.

For the sake of simplicity, let's consider that Expert 2 also uses fuzzy degrees. Now one gets a Type-2 Plithogenic Set:

$$PS_2 = \{John[0.5(0.9), 0.6(0.4), 0.3(1.0), 0.1(0.0), 0.7(0.8)],$$
$$George[0.1(0.3),\ 0.8(0.4),\ 0.3(0.5),\ 0.1(0.7),\ 0.4(0.9)],$$
$$Mary[0.9(0.1),\ 0.4(0.5),\ 0.6(0.6),\ 0.1(0.8),\ 0.2(0.9)]\}$$

Which are interpreted as follows:

0.5(0.9) means that Expert 2 is 0.9 (90%) confident in John's fuzzy degree of Weight of 0.5 assigned by Expert 1;

0.6(0.4) means that Expert 2 is 0.4 (40%) confident in John's fuzzy degree of Tallness of 0.6 assigned by Expert 1;

0.3(1.0) means that Expert 2 is 1.0 (100%) confident in John's fuzzy degree of Oldness of 0.3 assigned by Expert 1;

0.1(0.0) means that Expert 2 is 0.0 (0%) confident in John's fuzzy degree of Beauty of 0.1 assigned by Expert 1;

0.7(0.8) means that Expert 2 is 0.8 (80%) confident in John's fuzzy degree of Health of 0.7 assigned by Expert 1.

And similarly for George's and Mary's second round of degrees.

13. Example of Type-3 Plithogenic Set

The process may go on and have an Expert 3 evaluate the Expert 2. Assume Expert 3 uses neutrosophic degrees.

$$PS_3 = \{John\{0.5[0.9(0.6, 0.7, 0.3)], 0.6[0.4(0.6, 0.7, 0.3)],$$
$$0.3[1.0(0.6, 0.7, 0.3)], 0.1[0.0(0.6, 0.7, 0.3)], 0.7[0.8(0.6, 0.7, 0.3)]\},$$
$$George[0.1[0.3(0.4,0.4,0.06)],\ 0.8[0.4(0.9,0.1,0.03)],\ 0.3[0.5(0.9,1.0,0.2)],$$
$$0.1[0.7(0.7,0.3,0.6)],\ 0.4[0.9(0.1,0.0,0.4)],$$
$$Mary\{0.9[0.1(0.2,0.3,0.4)],\ 0.4[0.5(0.7,0.8,0.7)],$$
$$0.6[0.6(1.0,0.0,0.0)],\ 0.1[0.8(0.1,0.4,0.6)],\ 0.2[0.9(0.0,0.0,0.0)]\}$$

Therefore,
$0.5[0.9(0.6, 0.7, 0.3)]$
means that Expert 3 assigns the neutrosophic degrees of truth = 0.6, indeterminacy = 0.7, and falsehood = 0.3, to the Expert 2's evaluation of 0.9 (90%) confidence on Expert 1's evaluation of 0.5 degree of John's Weight.
And so on for all others.

One may generalize to **Type-k Plithogenic Set**, recurrently going on from a type to the next type, but it becomes more sophisticated and not usable in practice.

14. Example of Symbolic Plithogenic Numbers

The corresponding Symbolic Plithogenic Algebraic Structure is based on the symbolic (or literal) plithogenic components *W, T, O, B, H*, and we get the **plithogenic numbers** (*PN*) of the form:

$$PN = a + bW + cT + dO + eB + fH,$$

where *a, b, c, d, e, f* are real, or complex numbers, or they may belong to a set of a given classical algebraic structure. As a particular example, let $PN_1 = 2 - 3W + 5T - O + 6B - 4H$.

In this example, let's assume that the *prevalence order* is:

$W < T < O < B < H$, where "<" means "less important",

or *W* is less important than *T*, which is less important than *O*, which is less important that B, which is less important than *H*.

The *absorbance law* is defined as follows: the most important absorbs the less important in the multiplication operation, for example $W \cdot T = T$, since *T* absorbs *W* because *T* is more important (bigger) than *W*. Similarly for the other multiplications.

15. Operations with Plithogenic Numbers

Let's consider two plithogenic numbers:

$$PN_1 = a_0 + a_1P_1 + a_2P_2 + ... + a_nP_n$$
$$PN_2 = b_0 + b_1P_1 + b_2P_2 + ... + b_nP_n.$$

15.1. Addition of Plithogenic Numbers

$$PN_1 + PN_2 = (a_0 + b_0) + \sum_{i=1}^{n}(a_i + b_i)P_i$$

15.2. Subtraction of Plithogenic Numbers

$$PN_1 - PN_2 = (a_0 - b_0) + \sum_{i=1}^{n}(a_i - b_i)P_i$$

(SPS, +) is a Symbolic Plithogenic Commutative Group

15.3. Scalar Multiplication of Plithogenic Numbers

$$c \cdot PN_1 = c \cdot (a_0 + a_1P_1 + a_2P_2 + ... + a_nP_n) = c \cdot a_0 + c \cdot a_1P_1 + c \cdot a_2P_2 + ... + c \cdot a_nP_n$$

15.4. Multiplication of and Ppower of Plithogenic Numbers

$PN_1 \cdot PN_2 = (a_0 + a_1P_1 + a_2P_2 + ... + a_nP_n) \cdot (b_0 + b_1P_1 + b_2P_2 + ... + b_nP_n)$

and then one multiplies them, term by term $(a_iP_i) \cdot (a_jP_j) = a_i \cdot a_j \cdot P_{\max\{i,j\}}$, where $\cdot$ is the classical multiplication, as in classical algebra, using the above multiplication of symbolic plithogenic components.

As particular case: $0 \cdot P_i = 0$.

$(SPS, +, \cdot)$ is a Symbolic Plithogenic Commutative Ring, with the plithogenic unitary element: $1_\bullet \equiv 1 + 0 \cdot P_1 + 0 \cdot P_2 + ... + 0 \cdot P_n$.

The symbolic plithogenic components P_i's are not inversible, therefore the elements of SPS are non-inversible (except the plithogenic unitary element 1_*).

$(PN_1)^m = \underbrace{PN_1 \cdot PN_1 \cdot ... \cdot PN_1}_{m-times}$ for integer $m \geq 1$;

The negative power of a plithogenic number $(PN_1)^{-m}$ does not exist.

15.5. Alternative Multiplication of Plithogenic Numbers

$PN_1 \otimes PN_2 = a_0 \cdot b_0 + a_1 \cdot b_1 \cdot P_1 + a_2 \cdot b_2 \cdot P_2 + ... + a_n \cdot b_n \cdot P_n$.

$(SPS, +, \otimes)$, is a Symbolic Plithogenic Commutative Ring, with the unitary element: $1_\otimes \equiv 1 + 1 \cdot P_1 + 1 \cdot P_2 + ... + 1 \cdot P_n$.

The plithogenic numbers that have coefficients equal to zero do not have an inverse, for example: $2 + 3P_1 - 5P_3 = 2 + 3P_1 + 0P_2 - 5P_3$ is not inversible.

15.6. Division of Symbolic Plithogenic Components

$$\frac{P_i}{P_j} = \begin{cases} x_0 + x_1P_1 + x_2P_2 + ... + x_jP_j + P_i & x_0 + x_1 + ... + x_j = 0 \quad i > j \\ x_0 + x_1P_1 + x_2P_2 + ... + x_iP_i & x_0 + x_1 + ... + x_i = 1 \quad i = j \\ \phi & i < j \end{cases}$$

where all coefficients $x_0, x_1, x_2, ...x_i, ..., x_j, ... \in SPS$.

There are j-tuple infinities of quotients when i > j,
also i-tuple infinities of quotients when i = j,
and no quotient (indeterminate division) when i < j.
Therefore, the operation of division d(. , .) of symbolic plithogenic components

$d(P_i, P_j) : \{P_1, P_2, ..., P_n\}^2 \to SPS$

is a NeutroOperation, because:

it is well-defined (inner-defined) for no elements, since one never gets a single quotient, or $d(P_i, P_j) \notin SPS$;

it is indeterminate (cannot be calculated) for some elements (when $P_i < P_j$) with $d(P_i, P_j)$ being indeterminate;

and outer-defined (when $P_i = P_j$ and $P_i > P_j$) with $d(P_i, P_j) \notin SPS$

but $d(P_i, P_j) \in P(SPS)$ the powerset of *SPS*.

15.6.1. Example 1 of Division of Symbolic Plithogenic Components

$i > j$ [j-tuple infinities of quotients]

Let's divide P_5 by P_2.

$$\frac{P_5}{P_2} = x$$

where $x = x_0 + x_1P_1 + x_2P_2 + ... + x_nP_n \in SPS$.

$P_5 = x \cdot P_2$

Since the multiplication $x \cdot P_2$ should not exceed P_5 we take $n = 5$ into the formula of x, or

$$x \cdot P_2 = (x_0 + x_1P_1 + x_2P_2 + x_3P_3 + x_4P_4 + x_5P_5) \cdot P_2$$
$$= x_0P_2 + x_1P_1P_2 + x_2P_2P_2 + x_3P_3P_2 + x_4P_4P_2 + x_5P_5P_2$$
$$= (x_0 + x_1 + x_2)P_2 + x_3P_3 + x_4P_4 + x_5P_5$$
$$\equiv P_5 \equiv 0 + 0P_1 + 0P_2 + 0P_3 + 0P_4 + 1P_5$$

Therefore, $x_5 = 1$, $x_4 = 0$, $x_3 = 0$, $x_0 + x_1 + x_2 = 0$.

Whence, $\frac{P_5}{P_2} = x_0 + x_1P_1 + x_2P_2 + P_5$, with $x_0 + x_1 + x_2 = 0$, and the coefficients $x_0, x_1, x_2 \in SPS$ [2-tuple infinities of quotients].

15.6.2. Example 2 of Division of Symbolic Plithogenic Components

$i = j$ [i-tuple infinities of quotients]

$\frac{P_3}{P_3} = x$ or

$$P_3 = P_3 \cdot x = P_3 \cdot (x_0 + x_1P_1 + x_2P_2 + x_3P_3) = x_0P_3 + x_1P_1P_3 + x_2P_2P_3 + x_3P_3P_3$$
$$= x_0P_3 + x_1P_3 + x_2P_3 + x_3P_3 = (x_0 + x_1 + x_2 + x_3)P_3 \equiv 1 \cdot P_3$$

whence $x_0 + x_1 + x_2 + x_3 = 1$.

Thus, $\frac{P_3}{P_3} = x_0 + x_1P_1 + x_2P_2 + x_3P_3$,

where $x_0 + x_1 + x_2 + x_3 = 1$, and the coefficients $x_0, x_1, x_2, x_3 \in SPS$

15.6.3. Example 3 of Division of Symbolic Plithogenic Components

$i < j$ [indeterminate, no quotient]

$\frac{P_2}{P_4} = x$ or $P_2 = P_4 \cdot x \geq P_4 > P_2$ or $P_2 > P_2$, which is impossible.

This multiplication, P_4 times any of $1, P_1, P_2, \ldots, P_n$, will give a result that is greater than or equal to P_4 according to the absorbing law.

This division is undefined (indeterminate).

15.7. Division of Symbolic Plithogenic Numbers

Let consider two symbolic plithogenic numbers as below:

$PN_3 = a_0 + a_1P_1 + a_2P_2 + ... + a_rP_r$ and $PN_4 = b_0 + b_1P_1 + b_2P_2 + ... + b_sP_s$

where $r, s \leq n$ are positive integers, and the leading coefficients (the coefficients of the highest/largest symbolic plithogenic components P_r and respectively P_s) are nonnull, $a_r \neq 0, b_s \neq 0$. The division is also based on the absorbance law.

$$\frac{PN_r}{PN_s} = \frac{a_0 + a_1P_1 + a_2P_2 + ... + a_rP_r}{b_0 + b_1P_1 + b_2P_2 + ... + b_sP_s} = x$$

We need to find $x \in SPS$.

$$a_0 + a_1P_1 + a_2P_2 + ... + a_rP_r \equiv x \cdot (b_0 + b_1P_1 + b_2P_2 + ... + b_sP_s)$$

We are focusing first on the division of their leading symbolic plithogenic components: $\frac{P_r}{P_s}$ as we did on the previous section. Of course the leading coefficients $a_r \neq 0, b_s \neq 0$.

$\frac{PN_r}{PN_s} = \begin{cases} none, one, many & r \geq s \\ \phi & r < s \end{cases}$ This is also a NeutroOperation since one has indeterminate cases.

For $r \geq s$ there may be: none, one, or many (including infinitely many) quotients.

For r < s no quotient. Indeterminacy.

We prove these through several examples:

15.7.1. Example 1 (no quotient)

$\frac{P_1+1}{P_1} = ?$

$\frac{P_1+1}{P_1} = x = (x_0 + x_1 P_1)$, we need to solve for x (actually for x_0 and x_1).

$P_1 + 1 = (x_0 + x_1 P_1) \cdot P_1 = x_0 P_1 + x_1 P_1 P_1 = x_0 P_1 + x_1 P_1 = (x_0 + x_1) P_1$

We may set $x_0 + x_1 = 1$, but we are not able to catch the free coefficient 1 from the left-hand side, i.e.

$P_1 + 1 \neq P_1$

15.7.2. Example 3 (one quotient only)

$\frac{P_1+2}{P_1+1} = ?$

$\frac{P_1+2}{P_1+1} = x = x_0 + x_1 P_1$

whence

$P_1 + 2 = (x_0 + x_1 P_1) \cdot (P_1 + 1) = x_0 P_1 + x_1 P_1 P_1 + x_0 + x_1 P_1$

$= x_0 P_1 + x_1 P_1 + x_0 P_1 + x_1 P_1 = x_0 + (x_0 + 2x_1) P_1$

we get

$x_0 = 2, x_0 + 2x_1 = 1,$ then $x_1 = -0.5$.

There is only one quotient (solution): $x_0 + x_1 P_1 = 2 - 0.5P_1 = -0.5P_1 + 2$.

Let's check the result:

$(P_1 + 1) \cdot (-0.5P_1 + 2) = -0.5P_1 P_1 + 2P_1 - 0.5P_1 + 2 = -0.5P_1 + 2P_1 - 0.5P_1 + 2$

$= P_1 + 2.$

15.7.3. Example 3 (double infinities of quotients (solutions))

$\frac{5P_3}{P_3 - 2P_2} = ?$

$\frac{5P_3}{P_3 - 2P_2} = x = x_0 + x_1 P_1 + x_2 P_2 + x_3 P_3.$

We need to find the coefficients x_0, x_1, x_2, x_3.

$5P_3 = (x_0 + x_1 P_1 + x_2 P_2 + x_3 P_3) \cdot (P_3 - 2P_2)$

$= (x_0 + x_1 P_1 + x_2 P_2 + x_3 P_3) \cdot P_3 + (x_0 + x_1 P_1 + x_2 P_2 + x_3 P_3) \cdot (-2P_2)$

$= (x_0 + x_1 + x_2 + x_3) P_3 - (2x_0 + 2x_1 + 2x_2) P_2 - 2x_3 P_3$

$= (x_0 + x_1 + x_2 - x_3) P_3 - (2x_0 + 2x_1 + 2x_2) P_2 \equiv 5P_3 + 0P_2$

Whence we get two equations:

$x_0 + x_1 + x_2 - x_3 = 5$

$2x_0 + 2x_1 + 2x_2 = 0$

Hence $2(x_0 + x_1 + x_2) = 0$, or $x_0 + x_1 + x_2 = 0$.

Replace it into the first equation:

0 – x_3 = 5, then x_3 = -5.

$$\frac{5P_3}{P_3-2P_2}=x_0+x_1P_1+x_2P_2+x_3P_3=x_0+x_1P_1+x_2P_2-5P_3,$$

where $x_0+x_1+x_2=0$.

15.8. **m-th Root of the Plithogenic Number**

$$\sqrt[m]{PN_1}=\sqrt[m]{a_0+a_1P_1+a_2P_2+...+a_nP_n}=x_0+x_1P_1+x_2P_2+...+x_nP_n.$$

We need to find the coefficients $x_0,x_1,x_2,...,x_n$.

Raising to the power *m* both sides, one gets:

$a_0+a_1P_1+a_2P_2+...+a_nP_n=(x_0+x_1P_1+x_2P_2+...+x_nP_n)^m$, where $x_0,x_1,x_2,...,x_n$ are coefficients that we need to find out. After raising to the power *k* the right-hand side, we identify the coefficients two by two.

The *m-root* of a plithogenic number may have: no solution, several solutions, or infinitely many solutions.

Example 1 of *m-th* Root of the Plithogenic Number with real coefficients (several solutions)

$\sqrt{4-3P_1}=?$

$\sqrt{4-3P_1}=x_0+x_1P_1$, where we need to find x_0 and x_1.

Raise both sides to the second power:

$(\sqrt{4-3P_1})^2=(x_0+x_1P_1)^2$ or

$4-3P=(x_0)^2+2x_0x_1P_1+(x_1)^2(P_1)^2=(x_0)^2+2x_0x_1P_1+(x_1)^2P_1$

$=(x_0)^2+[2x_0x_1+(x_1)^2]P_1\equiv 4-3P_1$

Identify the coefficients:

$$\begin{Bmatrix}(x_0)^2=4\\2x_0x_1+(x_1)^2=-3\end{Bmatrix}$$

Whence $x_0=2,-2$ from the first equation. Replaced into the second equation one gets:

$\pm 4x_1+(x_1)^2=-3$, or two quadratic equations $(x_1)^2\pm 4x_1+3=0$ that we need to solve.

For $x_0=2$, $(x_1)^2+4x_1+3=0$, has the solutions $x_1=-1,-3$,

thus $(x_0,x_1)=(2,-1)$ or $(2,-3)$.

For $x_0=-2$, $(x_1)^2-4x_1+3=0$, has the solutions $x_1=1,3$,

thus $(x_0,x_1)=(-2,1)$ or $(-2,3)$.

Final answer:

$\sqrt{4-3P_1}=x_0+x_1P_1=2-P_1,2-3P_1,-2+P_1,-2+3P_1$ (four solutions).

Example 2 of m-th Root of the Plithogenic Number with real coefficients (no solution)

$\sqrt{-4-3P_1}$ has no solution since one gets, in the above calculation $(x_0)^2=-4$, which does not work in the set of real numbers.

15.9. Remark 1

Other operations may be constructed on the Symbolic Plithogenic Set (SPS), giving birth to various symbolic plithogenic algebraic structures.

15.10. Remark 2

All previous operations are valid for the absorbance law and prevalence order defined above. If different law and order are defined by the experts, then different operations and results one gets.

16. Particular Cases of Symbolic Plithogenic Algebraic Structures

16.1. Neutrosophic Quadruple Numbers

Let's consider an entity (i.e. a number, an idea, an object, etc.) which is represented by a known part (a) and an unknown part ($bT + cI + dF$).
Numbers of the form

$NQ = a + bT + cI + dF$,

where a, b, c, d are real (or complex) numbers (or intervals, or in general subsets), and
T = truth / membership / probability,
I = indeterminacy / neutrality,
F = false / membership / improbability,
are called Neutrosophic Quadruple (Real respectively Complex) Numbers (or Intervals, or in general Subsets) [1].

"a" is called the known part of NQ,

while "$bT + cI + dF$" is called the unknown part of NQ.

Neutrosophic Quadruple Numbers [1] are particular case of the Plithogenic Numbers, since one takes $n = 3$, and P_1, P_2, P_3 are more general than T, I, and F respectively.

16.2. Refined Neutrosophic Quadruple Numbers

The Refined Neutrosophic Quadruple Numbers [1, 7] have the form:

$$RQN = a + \sum_{j=1}^{p} b_j T_j + \sum_{k=1}^{r} c_k I_k + \sum_{l=1}^{s} d_l F_l$$

where a, all b_i, all c_j, and all d_k are real (or complex) numbers, intervals, or, in general, subsets, while $T_1, T_2, \ldots, T_p$ are refinements of T;
$I_1, I_2, \ldots, I_r$ are refinements of I;
and $F_1, F_2, \ldots, F_s$ are refinements of F,

for integers $p, r, s \geq 0$ and at least one of them be ≥ 2, with $p + r + s = n$.

Refined Neutrosophic Quadruple Numbers are also particular case of the Plithogenic Numbers, since instead of symbolic sub-truths / sub-indeterminacies / sub-falsehoods T_j, I_k, F_l one may use all kinds of symbolic plithogenic components $P_1, P_2, \ldots, P_n$.

All, Neutrosophic Quadruple Numbers and Refined Neutrosophic Quadruple Numbers, together with the Prevalence Order and Absorbance Law, were introduced by Smarandache [1] in 2015.

16.3. (Symbolic) Turiyam Set

Turiyam Set (TS) was introduced by P. K Singh [9] in 2021, who added to the neutrosophic components *T (Truth), I (Indeterminacy), F(Falsehood),* another component Y (called state of *awareness*).

According to him, Turiyam component (Y) means: "Rejection of both acceptation and rejection of attribute at the given time i.e. unknown region (l). It needs Turiyam consciousness to explore it" [9].

Turiyam Set is very similar to Belnap's Logic, based on: True (T), False (F), Unknown (U), and Contradiction (C), where T, F, U, C are taken as symbols, not numbers. Belnap's Logic is a particular case of Refined Neutrosophic Logic [10].

Turiyam Set was defined as:

$TS = \{(a_0, a_1T, a_2F, a_3I, a_4Y), a_i \in A\}$, where A is a given set, or it is the set of a given classical algebraic structure.

The Symbolic Turiyam Numbers have the form:

$$STN = a_0 + a_1T + a_2F + a_3I + a_4Y$$

where $a_i \in A$.

It is clear that Turiyam Set (2021) is a particular case of the Plithogenic Set, because one replaces $n = 4$, and P_1, P_2, P_3, P_4 by T, F, I, Y respectively, since the symbolic plithogenic components may be either independent, or dependent, or partially independent/dependent as we desire.

The operations on *TS* were defined as particular cases to Smarandache's 2015 neutrosophic quadruple numbers and absorbance law [1] and 2019 symbolic plithogenic numbers [5].

Let

$$x = (a_0, a_1T, a_2F, a_3I, a_4Y) = a_0 + a_1T + a_2F + a_3I + a_4Y$$

$$y = (b_0, b_1T, b_2F, b_3I, b_4Y) = b_0 + b_1T + b_2F + b_3I + b_4Y$$

be two STNs, and *c* be a scalar.

Then the addition

$$x + y = (a_0 + b_0, (a_1 + b_1)T, (a_2 + b_2)F, (a_3 + b_3)I, (a_4 + b_4)Y)$$

$$= (a_0 + b_0) + (a_1 + b_1)T + (a_2 + b_2)F + (a_3 + b_3)I + (a_4 + b_4)Y$$

The multiplication of the symbolic components T, F, I, Y were more complicated listed in [12], as:

$$T \cdot T = T^2 = T, F \cdot F = F^2 = F, I \cdot I = I^2 = I, Y \cdot Y = Y^2 = Y, T \cdot Y = Y \cdot T = Y,$$

$$T \cdot F = F \cdot T = F, T \cdot I = I \cdot T = I, I \cdot Y = Y \cdot I = I, F \cdot Y = Y \cdot F = Y, F \cdot I = I \cdot F = I.$$

While using the absorbance law (the stronger absorbs the weaker) and the prevalence order T < F < I < Y (as chosen by author Singh [12]) it would have been much simpler.

Multiplication of STNs:

$$x \cdot y = (a_0 + a_1T + a_2F + a_3I + a_4Y) \cdot (b_0 + b_1T + b_2F + b_3I + b_4Y)$$

Then similarly multiply them term by term, taking into consideration the multiplication of symbolic components T, F, I, Y as explained above.

Scalar Multiplication in the similar way:

$$c \cdot x = c \cdot (a_0 + a_1T + a_2F + a_3I + a_4Y) = c \cdot a_0 + c \cdot a_1T + c \cdot a_2F + c \cdot a_3I + c \cdot a_4Y.$$

Consequently, the Symbolic Turiyam Group [11] and Symbolic Turiyan Ring [12], as algebraic structures, are particular cases of the **Symbolic Plithogenic Commutative Group** (defined above in sections 15.1 & 15.2), and respectively **Symbolic Plithogenic Commutative Ring** (defined above in sections 15.4 or 15.5).

17. Practical Application

Since the cases $n = 3$ and 4 of Symbolic Plithogenic Algebraic Structures have been investigated, the reader may try to develop it for the case when $n = 5$, using Hexagonal Plithogenic Numbers (*HPN*), hexa since the dimension of *HPN* is $5 + 1 = 6$ because one has 6 vectors into the base: $1, P_1, P_2, P_3, P_4, P_5$.

$HPN = a_0 + a_1P_1 + a_2P_2 + a_3P_3 + a_4P_4 + a_5P_5$, where all coefficients a_i belong to a given set.

As practical application, for example, assume that the parameters represent various colors, C_1, C_2, C_3, C_4, C_5, then we denote it as:

$$HPN = a_0 + a_1C_1 + a_2C_2 + a_3C_3 + a_4C_4 + a_5C_5.$$

As multiplication law of the symbolic plithogenic components C_i with C_j one adopts a law from the real world. For example, if C_1 = *yellow*, and C_2 = *red*, then it makes sense to consider $C_1 \cdot C_2$ = *pink* (because *yellow* mixed with *red* give *pink*), and so on.

In this practical application, the absorbance law does not work, that's why one designs a new law in order to be able to multiply the components.

18. Open Question

Future possible study for researchers would be to investigate the *infinite-case*, we mean when each element in the plithogenic set (section 2 above) is characterized by infinitely many attributes (parameters), and similarly the symbolic plithogenic numbers (section 3 above) have infinitely many symbolic plithogenic components $P_1, P_2, ..., P_\infty$ and, eventually, their applications.

19. Conclusion

In this paper, the new types of algebraic structures from 2018-2019, called Symbolic Plithogenic Algebraic Structures, were revisited, and afterwards compared to other related structures.

We proved that the Symbolic Plithogenic Numbers are generalizations of Neutrosophic Quadruple Numbers, Refined Neutrosophic Quadruple Numbers, and Symbolic Turiyam Numbers.

Consequently, the Symbolic Plithogenic algebraic structures (semigroup, group, ring, etc.) are generalization of the corresponding algebraic structures built on these particular cases described above. We recalled the Symbolic Plithogenic Group and Ring.

Many examples and practical applications were also revealed.

Any future application may require a special multiplication law of the components and of plithogenic numbers that the experts should design themselves.

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