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K-Divisibility and *K*-Strong Divisibility Sequences

In Florentin Smarandache: "Collected Papers", vol. II. Chisinau (Moldova): Universitatea de Stat din Moldova, 1997.

K-Divisibility and K-Strong Divisibility Sequences

A sequence of rational integers g is called a **divisibility sequence** if and only if

$$n|m \Rightarrow g(n)|g(m)$$

for all positive integers n, m. [See [3] and [4]]

Also, g is called a **strong divisibility sequence** if ans only if

$$(g(n), g(m)) = g((n, m))$$

for all positive integers n, m. [See [1], [2], [3], [4] and [5]]

Of course, it is easy to show that the results of the Smarndache function $S(n)$ is niether a divibility or a strong divisibility sequence because $4|20$ but $S(4) = 4$ does not divide $5 = S(20)$, and $(S(4), S(20)) = (4, 5) = 1 \neq 4 = S(4) = S((4, 20))$.

a) However, is there an infinite subsequence of integers $M = \{m_1, m_2, \ldots\}$ such that S is a divisibility sequence on M?

b) If $P\{p_1, p_2, \ldots\}$ is the set of prime numbers, the S is not a strong divisibility sequence on P, because for $i \neq j$ we have

$$(S(p_i), S(p_j)) = (p_i, p_j) = 1 \neq 0 = S(1) = S((p_i, p_j)).$$

And the same question can be asked about P as was asked in part (a).

We introduce the following two notions, which are generalizations of a "divisibility sequence" and "sttrong divisibility sequrnce" respectively.

1) A k-divisibility sequence, where $l \geq 1$ is an integer, is defined in the following way:

If $n|m \Rightarrow g(n)|g(m) \Rightarrow g(g(n))|g(g(m)) \Rightarrow \ldots \Rightarrow \underbrace{g(\ldots(g(n))\ldots)}_{k \text{ times}} | \underbrace{g(\ldots(g(m))\ldots)}_{k \text{ times}}$ for all positive integers n, m.

For example, $g(n) = n!$ is a k-divisibility sequence.

Also: any constant sequence is a k-divisibility sequence.

2) A k-strong divisibility sequence, where $k \geq 1$ is an integer, is defined in the following way:

If $(g(n_1), g(n_2), \ldots, g(n_k)) = g((n_1, n_2, \ldots, n_k))$ for all positive integers $n_1, n_2, \ldots, n_k$.

For example, $g(n) = 2n$ is a k-strong divisibility sequence, because $(2n_1, 2n_2, \ldots, 2n_k) =$ $= 2 * (n_1, n_2, \ldots, n_k) = g((n_1, n_2, \ldots, n_k))$.

Remarks: If g is a divisibility sequence and we apply its definition k-times, we get that g is a k-divisibility sequence for any $k \geq 1$. The converse is also true. If g is k-strong divisibility sequence, $k \geq 2$, then g is a strong divisibility sequence. This can be seen by taking the definition of a k-strong divisibility sequence and replacing n by n_1 and all $n_2, \ldots, n_k$ by m to obtain $(g(n), g(m), \ldots, g(m)) = g((n, m, \ldots, m))$ or $(g(n), g(m)) = g((n, m))$.

The converse is also true, as

$$(n_1, n_2, \ldots, n_k) = ((\ldots((n_1, n_2), n_3), \ldots), n_k).$$

Therefore, we found that:

a) The divisibility sequence notion is equivalent to a k-divisibility sequence, or a generalization of a notion id equivalent to itself.

Is there any paradox or dilemma?

b) The strong divisibility sequence is equivalent to the k-strong divisibility sequence notion. As before, a generalization of a notion is equivalent to itself.

Again, is there any paradox or dilemma?

References

[1] Kimberling C., "Strong Divisibility Sequences With Nonzero Initial Term", The Fibonacci Quaterly, Vol. 16 (1978): pp. 541-544.

[2] Kimberling C., "Strong Divisibility Sequences and Some Conjectures", The Fibonacci Quaterly, Vol. 17 (1979): pp. 13-17.

[3] Ward M., "Note on Divisibility Sequences", Bulletin of the American Mathematical Society, Vol. 38 (1937): pp. 725-732.

[4] Ward M., "A Note on Divisibility Sequences", Bulletin of the American Mathematical Society, Vol. 45 (1939): pp. 334-336.