FLORENTIN SMARANDACHE

On Solving General Linear Equations in The Set of Natural Numbers

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ON SOLVING GENERAL LINEAR EQUATIONS IN THE SET OF NATURAL NUMBERS

The utility of this article is that it establishes if the number of the natural solutions of a general linear equation is limited or not. We will show also a method of solving, using integer numbers, the equation $ax - by = c$ (which represents a generalization of lemmas 1 and 2 of [4]), an example of solving a linear equation with 3 unknowns in N, and some considerations on solving, using natural numbers, equations with n unknowns.

Let's consider the equation:

(1) $\sum_{i=1}^{n} a_i x_i = b$ with all $a_i, b \in \mathbb{Z}$, $a_i \neq 0$, and the greatest common factor $(a_1, ..., a_n) = \text{d}$.

Lemma 1: The equation (1) admits at least a solution in the set of integers, if d divides b.

This result is classic.

In (1), one does not diminish the generality by considering $(a_1, ..., a_n) = 1$, because in the case when $d \neq 1$, one divides the equation by this number; if the division is not an integer, then the equation does not admit natural solutions.

It is obvious that each homogeneous linear equation admits solutions in $\mathbb{N}$: at least the banal solution!

PROPERTIES ON THE NUMBER OF NATURAL SOLUTIONS OF A GENERAL LINEAR EQUATION

We will introduce the following definition:

Definition 1: The equation (1) has variations of sign if there are at least two coefficients a_i, a_j with $1 \leq i, j \leq n$, such that sign($a_i \cdot a_j$) = -1

Lemma 2: An equation (1) which has sign variations admits an infinity of natural solutions (generalization of lemma 1 of [4]).

Proof: From the hypothesis of the lemma it results that the equation has h no null positive terms, $1 \leq h \leq n$, and $k = n - h$ non null negative terms. We have $1 \leq k \leq n$; it is supposed that the first h terms are positive and the following k terms are negative (if not, we rearrange the terms).

We can then write:

$$\sum_{t=1}^{h} a_t x_t - \sum_{j=h+1}^{n} a_j' x_j = b \text{ where } a_j' = -a_j > 0 .$$

Let's consider $0 < M = [a_1, ..., a_n]$ the least common multiple, and $c_i = |M / a_i|$, $i \in \{1, 2, ..., n\}$.

Let's also consider $0 < P = [h,k]$ the least common multiple, and $h_1 = P / h$ and $k_1 = P / k$.

Taking $\begin{cases} x_t = h_1 c_t \cdot z + x_t^0, & 1 \le t \le h \\ x_j = k_1 c_j \cdot z + x_j^0, & h+1 \le j \le n \end{cases}$

where $z \in \mathbb{N}$, $z \ge \max\left\{\left[\dfrac{-x_t^0}{h_1 c_t}\right], \left[\dfrac{x_j^0}{k_1 c_j}\right]\right\} + 1$ with [γ] meaning integer part of γ, i.e. the greatest integer less than or equal to γ, and x_i^0, $i \in \{1,2,...,n\}$, a particular integer solution (which exists according to lemma 1), we obtain an infinity of solutions in the set of natural numbers for the equation (1).

Lemma 3:

a) An equation (1) which does not have variations of sign has at maximum a limited number of natural solutions.

b) In this case, for $b \ne 0$, constant, the equation has the maximum number of solutions if and only if all $a_i = 1$ for $i \in \{1,2,...,n\}$.

Proof: (see also [6]).

a) One considers all $a_i > 0$ (otherwise, multiply the equation by -1).

If $b < 0$, it is obvious that the equation does not have any solution (in $\mathbb{N}$).

If $b = 0$, the equation admits only the trivial solution.

If $b > 0$, then each unknown x_i takes positive integer values between 0 and $b / a_i = d_i$ (finite), and not necessarily all these values. Thus the maximum number of solutions is lower or equal to: $\prod_{i=1}^{n}(1+d_i)$, which is finite.

b) For $b \ne 0$, constant, $\prod_{i=1}^{n}(1+d_i)$ is maximum if and only if d_i are maximum, i.e. iff $a_i = 1$ for all i, where $i = \{1,2,...,n\}$.

Theorem 1. The equation (1) admits an infinity of natural solutions if and only if it has variations of sign.

This naturally follows from the previous results.

<u>Method of solving.</u>

Theorem 2. Let's consider the equation with integer coefficients $ax - by = c$, where a and $b > 0$ and $(a,b) = 1$. Then the general solution in natural numbers of this equation is:

$\begin{cases} x = bk + x_0 \\ y = ak + y_0 \end{cases}$ where (x_0, y_0) is a particular integer solution of the equation,

and $k \ge \max\{[-x_0 / b], [-y_0 / a]\}$ is an integer parameter (generalization of lemma 2 of [4]).

Proof: It results from [1] that the general integer solution of the equation is $\begin{cases} x = bk + x_0 \\ y = ak + y_0 \end{cases}$ where (x_0, y_0) is a particular integer solution of the equation and $k \in \mathbb{Z}$. Since x and y are natural integers, it is necessary for us to impose conditions for k such that x ≥ 0 and y ≥ 0, from which it results the theorem.

WE CONCLUDE!

To solve in the set of natural numbers a linear equation with n unknowns we will use the previous results in the following way:

a) If the equation does not have variations of sign, because it has a limited number of natural solutions, the solving is made by tests (see also [6])

b) If it has variations of sign and if b is divisible by d, then it admits an infinity of natural solutions. One finds its general integer solution (see [2], [5]);

$x_i = \sum_{j=1}^{n-1} \alpha_{ij} k_j + \beta_i$, $1 \le i \le n$ where all the $\alpha_{ij}, \beta_i \in \mathbb{Z}$ and the k_j are integer parameters.

By applying the restriction $x_i \ge 0$ for i from $\{1, 2, ..., n\}$, one finds the conditions which must be satisfied by the integer parameters k_j for all j of $\{1, 2, ..., n-1\}$. (c)

The case $n = 2$ and $n = 3$ can be done by this method, but when n is bigger, the condition (c) become more and more difficult to find.

Example: Solve in $\mathbb{N}$ the equation $3x - 7y + 2z = -18$.

Solution: In $\mathbb{Z}$ one obtains the general integer solution:

$$\begin{cases} x = k_1 \\ y = k_1 + 2k_2 \\ z = 2k_1 + 7k_2 - 9 \end{cases} \quad \text{with } k_1 \text{ and } k_2 \text{ in } \mathbb{Z}.$$

From the conditions (c) result the inequalities $x \ge 0,\ y \ge 0,\ z \ge 0$. It results that $k_1 \ge 0$ and also:

$k_2 \ge [-k_1 / 2] + 1$ if $-k_1 / 2 \notin Z$, or $k_2 \ge -k_1 / 2$ if $-k_1 / 2 \in Z$;

and $k_2 \ge [(9 - 2k_1) / 7] + 1$ if $(9-2k_1)/7 \notin Z$, or $k_2 \ge (9-2k_1) / 7$ if $(9-2k_1) / 7 \in Z$;

that is $k_2 \ge [(2 - 2k_1) / 7] + 2$ if $(2-2k_1)/7 \notin Z$, or $k_2 \ge (2-2k_1)/7 + 1$ if $(2-2k_1) / 7 \in Z$.

With these conditions on k_1 and k_2 we have the general solution in natural numbers of the equation.

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