THE POLAR OF A POINT With Respect TO A CIRCLE

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In this article we establish a connection between the notion of the symmedian of a triangle and the notion of polar of a point in rapport to a circle

We'll prove for beginning two properties of the symmedians.

Lemma 1
If in triangle ABC inscribed in a circle, the tangents to this circle in the points B and C intersect in a point S, then AS is symmedian in the triangle ABC.

Proof
We'll note L the intersection point of the line AS with BC (see fig. 1).

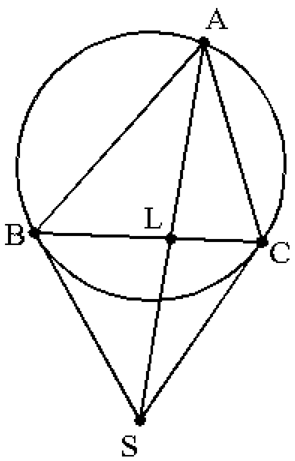

Fig. 1

We have

$$\frac{\text{Aria}\,\Delta ABL}{\text{Aria}\,\Delta ACL} = \frac{BL}{LC} = \frac{\text{Aria}\,\Delta BSL}{\text{Aria}\,\Delta CSL}$$

It result

$$\frac{\text{Aria}\,\Delta ABS}{\text{Aria}\,\Delta ACS} = \frac{BL}{LC} \tag{1}$$

We observe that

$$m(\angle ABS) = m\left(\hat{B}\right) + m\left(\hat{A}\right) \text{ and } m(\angle ACS) = m\left(\hat{C}\right) + m\left(\hat{A}\right)$$

We obtain that

$$\sin(\angle ABS) = \sin C \text{ and } \sin(\angle ACS) = \sin B$$

We have also

$$\frac{\text{Aria}\,\Delta ABS}{\text{Aria}\,\Delta ACS} = \frac{AB \cdot SB \cdot \sin C}{AC \cdot SC \cdot \sin B} = \frac{BL}{LC} \tag{2}$$

From the sinus' theorem it results

$$\frac{\sin C}{\sin B} = \frac{AB}{AC} \tag{3}$$

The relations (2) and lead us to the relation

$$\frac{BL}{LC} = \left(\frac{AB}{AC}\right)^2,$$

which shows that AS is symmedian in the triangle ABC.

Observations

1. The proof is similar if the triangle ABC is obtuse.

2. If ABC is right triangle in A, the tangents in B and C are parallel, and the symmedian from A is the height from A, and, therefore, it is also parallel with the tangents constructed in B and C to the circumscribed circle.

Definition 1

The points A, B, C, D placed, in this order, on a line d form a harmonic division if and only if

$$\frac{AB}{AD} = \frac{CB}{CD}$$

Lemma 2

If in the triangle ABC, AL is the interior symmedian $L \in BC$, and AP is the external median $P \in BC$, then the points P, B, L, C form a harmonic division.

Proof

It is known that the external symmedian AP in the triangle ABC is tangent in A to the circumscribed circle (see fig. 2), also, it can be proved that:

$$\frac{PB}{PC} = \left(\frac{AB}{AC}\right)^2 \tag{1}$$

but

$$\frac{LB}{LC} = \left(\frac{AB}{AC}\right)^2 \tag{2}$$

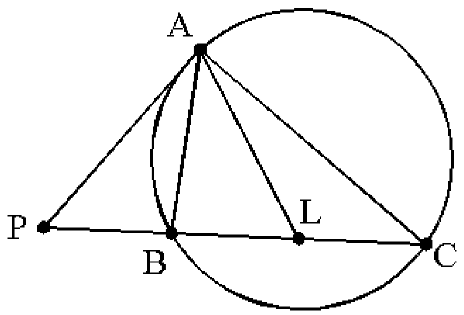

Fig. 2

From the relations (1) and (2) it results

$$\frac{PB}{PC} = \frac{LB}{LC},$$

Which shows that the points P, B, L, C form a harmonic division.

Definition 2

If P is a point exterior to circle $\mathcal{C}(0, r)$ and B, C are the intersection points of the circle with a secant constructed through the point P, we will say about the point $Q \in (BC)$ with the property $\frac{PB}{PC} = \frac{QB}{QC}$ that it is the harmonic conjugate of the point P in rapport to the circle $\mathcal{C}(0, r)$.

Observation

In the same conjunction, the point P is also the conjugate of the point Q in rapport to the circle (see fig. 3).

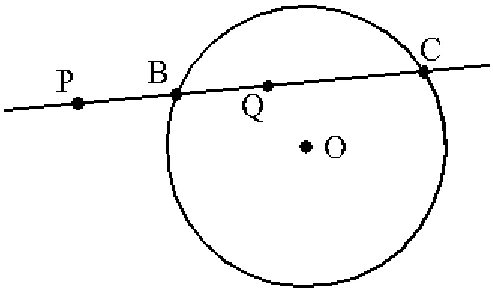

Fig. 3

Definition 3

The set of the harmonic conjugates of a point in rapport with a given circle is called the **polar** of that point in rapport to the circle.

Theorem

The polar of an exterior point to the circle is the circle's cord determined by the points of tangency with the circle of the tangents constructed from that point to the circle.

Proof

Let P an exterior point of the circle $\mathcal{C}(0, r)$ and M, N the intersections of the line PO with the circle (see fig. 4).

We note T and V the tangent points with the circle of the tangents constructed from the point P and let Q be the intersection between MN and TV.

Obviously, the triangle MTN is a right triangle in T, TQ is its height (therefore the interior symmedian, and TP is the exterior symmedian, and therefore the points P, M, Q, N form a harmonic division, (Lemma 2)). Consequently, Q is the harmonic conjugate of P in rapport to the circle and it belongs to the polar of P in rapport to the circle.

We'll prove that (TV) is the polar of P in rapport with the circle. Let $M'N'$ be the intersections of a random secant constructed through the point P with the circle, and X the intersection of the tangents constructed in M' and N' to the circle.

In conformity to Lemma 1, the line XT is for the triangle $M'TN'$ the interior symmedian, also TP is for the same triangle the exterior symmedian.

If we note Q' the intersection point between XT and $M'N'$ it results that the point Q' is the harmonic conjugate of the point P in rapport with the circle, and consequently, the point Q' belongs to the polar P in rapport to the circle.

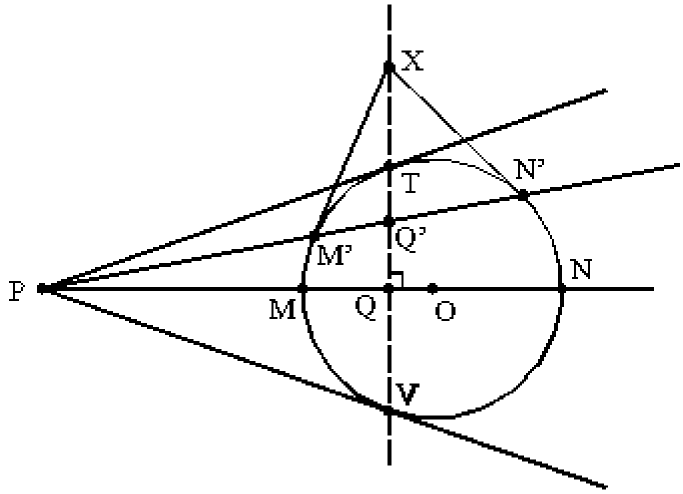

Fig. 4

For the triangle $VM'N'$, according to Lemma 1, the line VX is the interior symmedian and VP is for the same triangle the external symmedian. It will result, according to Lemma 2, that if $\{Q''\} = VX \cap M'N'$, the point Q" is the harmonic conjugate of the point P in rapport to the circle. Because the harmonic conjugate of a point in rapport with a circle is a unique point, it results that $Q'=Q''$. Therefore the points V, T, X are collinear and the point Q' belongs to the segment (TV).

Reciprocal

If $Q_1 \in (TV)$ and PQ_1 intersect the circle in M_1 and N_1, we much prove that the point Q_1 is the harmonic conjugate of the point P in rapport to the circle.

Let X_1 the intersection point of the tangents constructed from M_1 and N_1 to the circle. In the triangle M_1TN_1 the line X_1T is interior symmedian, and the line TP is exterior symmedian. If $\{Q_1'\} = X_1T \cap M_1N_1$ then P, M_1, Q_1', N_1 form a harmonic division.

Similarly, in the triangle M_1VN_1 the line VX_1 is interior symmedian, and VP exterior symmedian. If we note $\{Q_1''\} = VX_1 \cap M_1N_1$, it results that the point Q_1'' is the harmonic conjugate of the point P in rapport to M_1 and N_1. Therefore, we obtain $Q_1' = Q_1''$. On the other side, X_1, T, Q_1' and V, X_1, Q_1'' are collinear, but $Q_1' = Q_1''$, it result that X_1, T, Q_1', V are collinear, and then $Q_1' = Q_1$, therefore Q_1 is the conjugate of P in rapport with the circle.