Another proof of a theorem relative to the orthological triangles

Ion Pătraşcu – National College "Fraţii Buzeşti, Craiova, Romania
Florentin Smarandache – New Mexico University, U.S.A.

In [1] we proved, using barycentric coordinates, the following theorem:

Theorem: (generalization of the C. Coşniţă theorem)

If P is a point in the triangle's ABC plane, which is not on the circumscribed triangle, $A'B'C'$ is its pedal triangle and A_1, B_1, C_1 three points such that

$$\overrightarrow{PA'}\cdot\overrightarrow{PA_1} = \overrightarrow{PB'}\cdot\overrightarrow{PB_1} = \overrightarrow{PC'}\cdot\overrightarrow{PC_1} = k, \quad k \in R^*,$$

then the lines AA_1, BB_1, CC_1 are concurrent.

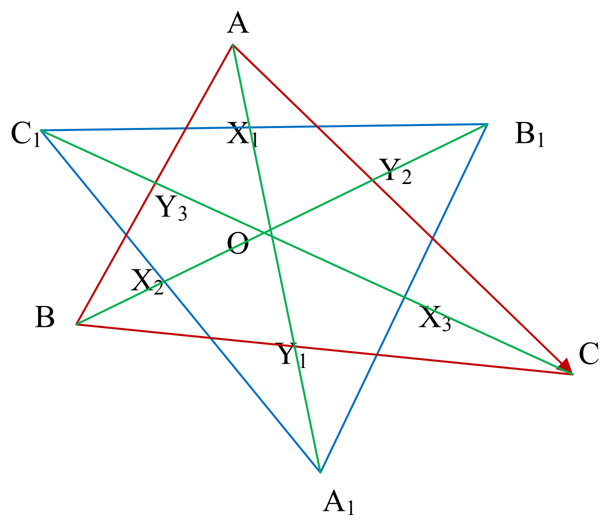
Bellow, will prove, using this theorem, the following:

Theorem

If the triangles ABC and $A_1B_1C_1$ are orthological and their orthological centers coincide, then the lines AA_1, BB_1, CC_1 are concurrent (the triangles ABC and $A_1B_1C_1$ are homological).

Proof:

Let O be the unique orthological center of the triangles ABC and $A_1B_1C_1$ and



$$\{X_1\} = AO \cap B_1C_1$$
$$\{X_2\} = BO \cap A_1C_1$$
$$\{X_3\} = CO \cap A_1B_1$$

We denote

$$\{Y_1\} = OA_1 \cap BC$$
$$\{Y_2\} = OB_1 \cap AC$$
$$\{Y_3\} = OC_1 \cap AB$$

We observe that $\sphericalangle OAY_3 = \sphericalangle OC_1X_1$ (angles with perpendicular sides).

Therefore:

$$\sin OAY_3 = \frac{OY_3}{OA}$$
$$\sin OC_1X_1 = \frac{OX_1}{OC_1},$$

then

$$OX_1 \cdot OA = OY_3 \cdot OC_1 \quad (1)$$

Also

$$\sphericalangle OC_1X_2 = \sphericalangle OBY_3$$

therefore

$$\sin OC_1X_2 = \frac{OX_2}{OC_1}$$
$$\sin OBY_3 = \frac{OY_3}{OB}$$

and consequently:

$$OX_2 \cdot OB = OY_3 \cdot OC_1 \quad (2)$$

Following the same path:

$$\sin OA_1X_2 = \frac{OX_2}{OC_1} = \sin OBY_1 = \frac{OY_1}{OB}$$

from which

$$OX_2 \cdot OB = OA_1 \cdot OY_1 \quad (3)$$

Finally

$$\sin OA_1X_3 = \frac{OX_3}{OA_1} = \sin OCY_1 = \frac{OY_1}{OC}$$

from which:

$$OX_3 \cdot OC = OA_1 \cdot OY_1 \quad (4)$$

The relations (1), (2), (3), (4) lead to

$$OX_1 \cdot OA = OX_2 \cdot OB = OX_3 \cdot OC \quad (5)$$

From (5) using the Coşniţă's generalized theorem, it results that $A_1A,\ B_1B,\ C_1C$ are concurrent.

Observation:

If we denote P the homology center of the triangles ABC and $A_1B_1C_1$ and d is the intersection of their homology axes, them in conformity with the Sondat's theorem, it results that $OP \perp d$.

References:

[1] Ion Pătraşcu – Generalizarea teoremei lui Coşniță – Recreații Matematice, An XII, nr. 2/2010, Iaşi, Romania

[2] Florentin Smarandache – Multispace & Multistructure, Neutrosophic Transdisciliniarity, 100 Collected papers of science, Vol. IV, 800 p., North-European Scientific Publishers, Honka, Finland, 2010.