# Quasi-Isogonal Cevians

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In this article we will introduce the quasi-isogonal Cevians and we'll emphasize on triangles in which the height and the median are quasi-isogonal Cevians.

For beginning we'll recall:

**Definition 1**

In a triangle $ABC$ the Cevians $AD$, $AE$ are called isogonal if these are symmetric in rapport to the angle $A$ bisector.

**Observation**

In figure 1, are represented the isogonal Cevians $AD$, $AE$

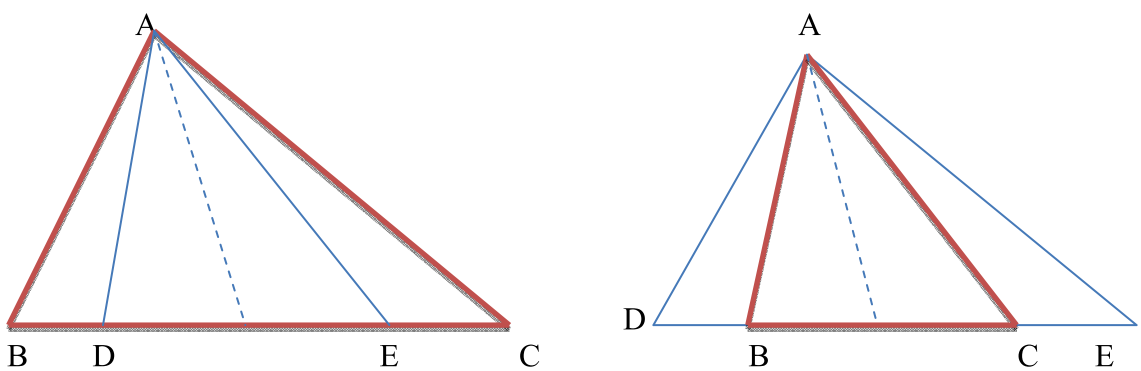


Fig. 1. Isogonal Cevians

**Proposition 1.**

In a triangle $ABC$, the height $AD$ and the radius $AO$ of the circumscribed circle are isogonal Cevians.

**Definition 2.**

We call the Cevians $AD$, $AE$ in the triangle $ABC$ quasi-isogonal if the point $B$ is between the points $D$ and $E$, the point $E$ is between the points $B$ and $C$, and $\sphericalangle DAB \equiv \sphericalangle EAC$.

**Observation**

In figure 2 we represented the quasi-isogonal Cevians $AD, AE$.

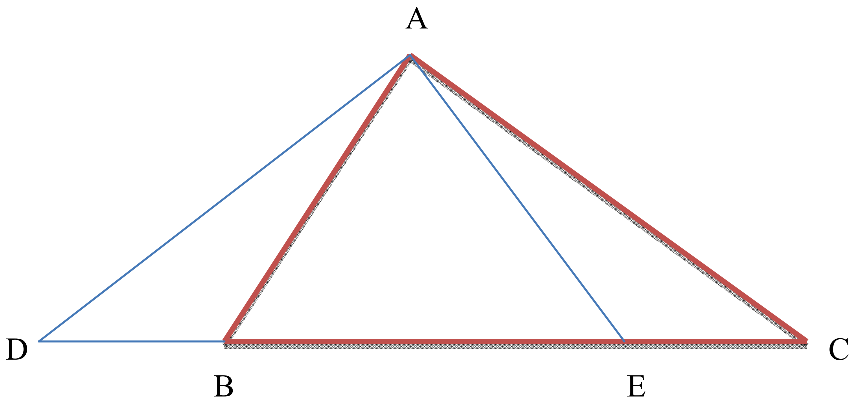

Fig. 2 quasi-isogonal Cevians

**Proposition 2**
There are triangles in which the height and the median are quasi-isogonal Cevians.

**Proof**
It is clear that if we look for triangles $ABC$ for which the height and the median from the point $A$ are quasi isogonal, then these must be obtuse-angled triangle. We'll consider such a case in which $m(\sphericalangle A) > 90°$ (see figure 3).

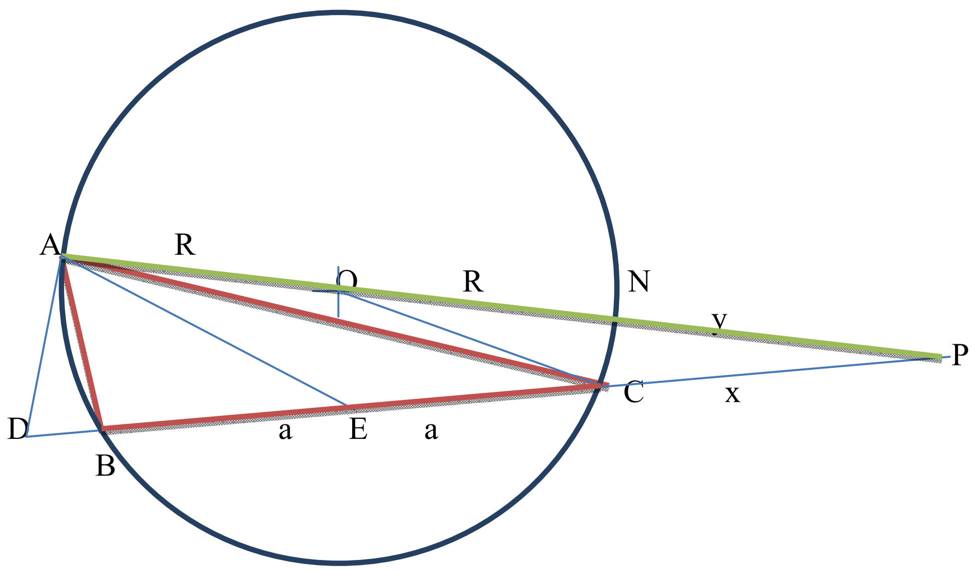

Fig. 3

Let $O$ the center of the circumscribed triangle, we note with $N$ the diametric point of $A$ and with $P$ the intersection of the line $AO$ with $BC$.

We consider known the radius $R$ of the circle and $BC = 2a$, $a < R$ and we try to construct the triangle $ABC$ in which the height $AD$ and the median $AE$ are quasi isogonal Cevians; therefore $\sphericalangle DAB \equiv \sphericalangle EAC$. This triangle can be constructed if we find the lengths $PC$ and $PN$ in function of $a$ and $R$. We note $PC = x,\ PN = y$.

We consider the power of the point $P$ in function of the circle $\mathcal{C}(O,R)$. It results that

$$x\cdot(x+2a)=y\cdot(y+2R) \qquad (1)$$

From the Property 1 we have that $\sphericalangle DAB \equiv \sphericalangle OAC$. On the other side $\sphericalangle OAC \equiv \sphericalangle OCA$ and $AD, AE$ are quasi isogonal, we obtain that $OC \parallel AE$.

The Thales' theorem implies that:

$$\frac{x}{a}=\frac{y+R}{R} \qquad (2)$$

Substituting $x$ from (2) in (1) we obtain the equation:

$$\left(a^2-R^2\right)y^2-2R\left(R^2-2a^2\right)y+3a^2R^2=0 \qquad (3)$$

The discriminant of this equation is:

$$\Delta=4R^2\left(R^4-a^2R^2+a^4\right)$$

Evidently $\Delta>0$, therefore the equation has two real solutions.

Because the product of the solutions is $\dfrac{3a^2R^2}{a^2-R^2}$ and it is negative we obtain that one of solutions is strictly positive. For this positive value of $y$ we find the value of $x$, consequently we can construct the point $P$, then the point $N$ and at the intersection of the line $PN$ we find $A$ and therefore the triangle $ABC$ is constructed.

For example, if we consider $R=\sqrt{2}$ and $a=1$, we obtain the triangle $ABC$ in which $AB=\sqrt{2}$, $BC=2$ and $AC=1+\sqrt{3}$.

We leave to our readers to verify that the height and the median from the point $A$ are quasi isogonal.