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Regarding the First Droz-Farny's Circle

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In this article, we define the **first Droz-Farny's circle**, we establish a connection between it and a **concyclicity theorem**, then we generalize this theorem, leading to the **generalization of Droz-Farny's circle**. The first theorem of this article was enunciated by J. Steiner and it was proven by Droz-Farny (*Mathésis*, 1901).

1st Theorem.

Let ABC be a triangle, H its orthocenter and A_1, B_1, C_1 the means of sides $(BC), (CA), (AB)$.

If a circle, having its center H, intersects B_1C_1 in P_1, Q_1 ; C_1A_1 in P_2, Q_2 and A_1B_1 in P_3, Q_3 , then $AP_1 = AQ_1 = BP_2 = BQ_2 = CP_3 = CQ_3$.

Proof.

Naturally, $HP_1 = HQ_1, B_1C_1 \parallel BC, AH \perp BC$.

It follows that $AH \perp B_1C_1$.

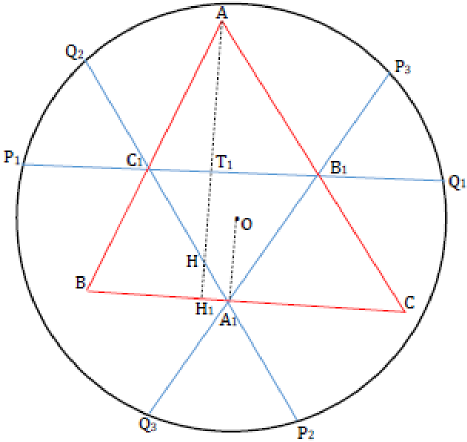

Figure 1.

Therefore, AH is the mediator of segment P_1Q_1; similarly, BH and CH are the mediators of segments P_2Q_2 and P_3Q_3.

Let T_1 be the intersection of lines AH and B_1C_1 (see *Figure 1*); we have $Q_1A^2 - Q_1H^2 = T_1A^2 - T_1H^2$. We denote $R_H = HP_1$. It follows that $Q_1A^2 = R_H^2 + (T_1A + T_1H)(T_1A - T_1H) = R_H^2 + AH \cdot (T_1A - T_1H)$.

However, $T_1A = T_1H_1$, where H_1 is the projection of A on BC; we find that $Q_1A^2 = R_H^2 + AH \cdot HH_1$.

It is known that the symmetric of orthocenter H towards BC belongs to the circle of circumscribed triangle ABC.

Denoting this point by H_1', we have $AH \cdot HH_1' = R^2 - OH^2$ (the power of point H towards the circumscribed circle).

We obtain that $AH \cdot HH_1 = \frac{1}{2} \cdot (R^2 - OH^2)$, and therefore $AQ_1^2 = R_H^2 + \frac{1}{2} \cdot (R^2 - OH^2)$, where O is the center of the circumscribed triangle ABC.

Similarly, we find $BQ_2^2 = CQ_3^2 = R_H^2 + \frac{1}{2}(R^2 - OH^2)$, therefore $AQ_1 = BQ_2 = CQ_3$.

Remarks.

a. The proof adjusts analogously for the obtuse triangle.

b. *1st Theorem* can be equivalently formulated in this way:

2nd Theorem.

If we draw three congruent circles centered in a given triangle's vertices, the intersection points of these circles with the sides of the median triangle of given triangle (middle lines) are six points situated on a circle having its center in triangle's orthocenter.

If we denote by ρ the radius of three congruent circles having A, B, C as their centers, we get:

$$R_H^2 = \rho^2 + \frac{1}{2}(OH^2 - R^2).$$

However, in a triangle, $OH^2 = 9R^2 - (a^2 + b^2 + c^2)$, R being the radius of the circumscribed circle; it follows that:

$$R_H^2 = \rho^2 + 4R^2 - \frac{1}{2}(a^2 + b^2 + c^2).$$

Remark.

A special case is the one in which $\rho = R$, where we find that $R_H^2 = R_1^2 = 5R^2 - \frac{1}{2}(a^2 + b^2 + c^2) = \frac{1}{2}(R^2 + OH^2)$.

Definition.

The circle $\mathcal{C}(H, R_1)$, where:

$$R_1 = \sqrt{5R^2 - \frac{1}{2}(a^2 + b^2 + c^2)},$$

is called the first Droz-Farny's circle of the triangle ABC.

Remark.

Another way to build the first Droz-Farny's circle is offered by the following theorem, which, according to [1], was the subject of a problem proposed in 1910 by V. Thébault in the *Journal de Mathématiques Elementaire*.

3rd Theorem.

The circles centered on the feet of a triangle's altitudes passing through the center of the circle circumscribed to the triangle cut the triangle's sides in six concyclical points.

Proof.

We consider ABC an acute triangle and H_1, H_2, H_3 the altitudes' feet. We denote by A_1, A_2; B_1, B_2; C_1, C_2 the intersection points of circles having their centers H_1, H_2, H_3 to BC, CA, AB, respectively.

We calculate HA_2 of the right angled triangle HH_1A_2 (see *Figure 2*). We have $HA_2^2 = HH_1^2 + H_1A_2^2$.

Because $H_1A_2 = H_1O$, it follows that $HA_2^2 = HH_1^2 + H_1O^2$. We denote by O_9 the mean of segment OH; the median theorem in triangle H_1HO leads to $H_1H^2 + H_1O^2 = 2H_1O_9^2 + \frac{1}{2}OH^2$.

It is known that O_9H_1 is the nine-points circle's radius, so $H_1O_9 = \frac{1}{2}R$; we get: $HA_1^2 = \frac{1}{2}(R^2 + OH^2)$; similarly, we find that $HB_1^2 = HC_1^2 = \frac{1}{2}(R^2 + OH^2)$, which shows that the points A_1, A_2; B_1, B_2; C_1, C_2 belong to the first Droz-Farny's circle.

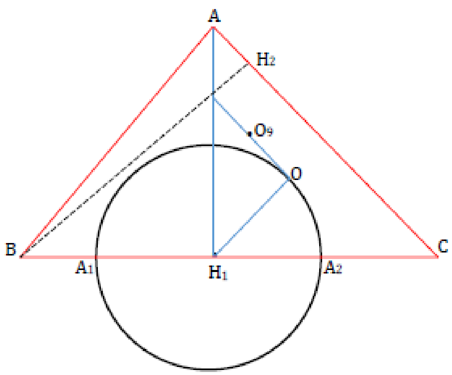

Figure 2.

Remark.

The 2^{nd} and the 3^{rd} theorems show that the first Droz-Farny's circle pass through 12 points, situated two on each side of the triangle and two on each side of the median triangle of the given triangle.

The following theorem generates the 3^{rd} *Theorem*.

4^{th} Theorem.

The circles centered on the feet of altitudes of a given triangle intersect the sides in six concyclical points if and only if their radical center is the center of the circle circumscribed to the given triangle.

Proof.

Let A_1, A_2; B_1, B_2; C_1, C_2 be the points of intersection with the sides of triangle ABC of circles having their centers in altitudes' feet H_1, H_2, H_3.

Suppose the points are concyclical; it follows that their circle's radical center and of circles centered in H_2 and H_3 is the point A (sides AB and AC are radical axes), therefore the perpendicular from A on H_2H_3 is radical axis of centers having their centers H_2 and H_3.

Since H_2H_3 is antiparallel to BC, it is parallel to tangent taken in A to the circle circumscribed to triangle ABC.

Consequently, the radical axis is the perpendicular taken in A on the tangent to the circumscribed circle, therefore it is AO.

Similarly, the other radical axis of circles centered in H_1, H_2 and of circles centered in H_1, H_3 pass through O, therefore O is the radical center of these circles.

Reciprocally.

Let O be the radical center of the circles having their centers in the feet of altitudes. Since AO is perpendicular on H_2H_3, it follows that AO is the radical axis of circles having their centers in H_2, H_3, therefore $AB_1 \cdot AB_2 = AC_1 \cdot AC_2$.

From this relationship, it follows that the points B_1, B_2; C_1, C_2 are concyclic; the circle on which these points are located has its center in the orthocenter H of triangle ABC.

Indeed, the mediators' chords B_1B_2 and C_1C_2 in the two circles are the altitudes from C and B of triangle ABC, therefore $HB_1 = HB_2 = HC_1 = HC_2$.

This reasoning leads to the conclusion that BO is the radical axis of circles having their centers H_1 and H_3, and from here the concyclicality of the points A_1, A_2; C_1, C_2 on a circle having its center in H, therefore $HA_1 = HA_2 = HC_1 = HC_2$. We obtained that $HA_1 = HA_2 = HB_1 = HB_2 = HC_1 = HC_2$, which shows the concyclicality of points $A_1, A_2, B_1, B_2, C_1, C_2$.

Remark.

The circles from the 3^{rd} *Theorem*, passing through *O* and having noncollinear centers, admit *O* as radical center, and therefore the 3^{rd} *Theorem* is a particular case of the 4^{th} *Theorem*.

References.

[1] N. Blaha: *Asupra unor cercuri care au ca centre două puncte inverse* [On some circles that have as centers two inverse points], in "Gazeta Matematica", vol. XXXIII, 1927.

[2] R. Johnson: *Advanced Euclidean Geometry*. New York: Dover Publication, Inc. Mineola, 2004

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[4] F. Smarandache, I. Patrascu: *Geometry of Homological Triangle*. Columbus: The Education Publisher Inc., Ohio, SUA, 2012.