

## A Generalization of Certain Remarkable Points of the Triangle Geometry

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In this article we prove a theorem that will generalize the concurrence theorems that are leading to the Franke’s point, Kariya’s point, and to other remarkable points from the triangle geometry.

### Theorem 1:

Let  $P(\alpha, \beta, \gamma)$  and  $A', B', C'$  its projections on the sides  $BC$ ,  $CA$  respectively  $AB$  of the triangle  $ABC$ .

We consider the points  $A'', B'', C''$  such that  $\overline{PA''} = k\overline{PA'}$ ,  $\overline{PB''} = k\overline{PB'}$ ,  $\overline{PC''} = k\overline{PC'}$ , where  $k \in R^*$ . Also we suppose that  $AA', BB', CC'$  are concurrent. Then the lines  $AA'', BB'', CC''$  are concurrent if and only if are satisfied simultaneously the following conditions:

$$\alpha\beta c \left( \frac{\beta}{b} \cos A - \frac{\alpha}{a} \cos B \right) + \beta\gamma a \left( \frac{\gamma}{c} \cos B - \frac{\beta}{b} \cos C \right) + \gamma ab \left( \frac{\alpha}{a} \cos C - \frac{\gamma}{c} \cos A \right) = 0$$

$$\frac{\alpha^2}{a^2} \cos A \left( \frac{\gamma}{c} \cos B - \frac{\beta}{b} \cos C \right) + \frac{\beta^2}{b^2} \cos B \left( \frac{\alpha}{a} \cos C - \frac{\gamma}{c} \cos A \right) + \frac{\gamma^2}{c^2} \cos C \left( \frac{\beta}{b} \cos A - \frac{\alpha}{a} \cos B \right) = 0$$

### Proof:

We find that

$$A'' \left( 0, \frac{\alpha}{2a^2} (a^2 + b^2 - c^2) + \beta, \frac{\alpha}{2a^2} (a^2 - b^2 + c^2) + \gamma \right)$$

$$\overline{PA''} = k\overline{PA'} = k \left[ -\alpha \vec{r}_A + \frac{\alpha}{2a^2} (a^2 + b^2 - c^2) \vec{r}_B + \frac{\alpha}{2a^2} (a^2 - b^2 + c^2) \vec{r}_C \right]$$

$$\overline{PA''} = (\alpha'' - \alpha) \vec{r}_A + (\beta'' - \beta) \vec{r}_B + (\gamma'' - \gamma) \vec{r}_C$$

We have:

$$\begin{cases} \alpha'' - \alpha = -k\alpha \\ \beta'' - \beta = \frac{k\alpha}{2a^2} (a^2 + b^2 - c^2), \\ \gamma'' - \gamma = \frac{k\alpha}{2a^2} (a^2 - b^2 + c^2) \end{cases}$$

Therefore:

$$\begin{cases} \alpha'' = (1-k)\alpha \\ \beta'' = \frac{k\alpha}{2a^2}(a^2 + b^2 - c^2) + \beta \\ \gamma'' = \frac{k\alpha}{2a^2}(a^2 - b^2 + c^2) + \gamma \end{cases}$$

Hence:

$$A'' \left( (1-k)\alpha, \frac{k\alpha}{2a^2}(a^2 + b^2 - c^2) + \beta, \frac{k\alpha}{2a^2}(a^2 - b^2 + c^2) + \gamma \right)$$

Similarly:

$$B' \left( -\frac{\beta}{2b^2}(-a^2 - b^2 + c^2) + \alpha, 0, -\frac{\beta}{2b^2}(a^2 - b^2 - c^2) + \gamma \right)$$

$$B'' \left( -\frac{k\beta}{2b^2}(-a^2 - b^2 + c^2) + \alpha, (1-k)\beta, -\frac{k\beta}{2b^2}(a^2 - b^2 - c^2) + \gamma \right)$$

$$C' \left( -\frac{\gamma}{2c^2}(-a^2 + b^2 - c^2) + \alpha, -\frac{\gamma}{2c^2}(a^2 - b^2 - c^2) + \beta, 0 \right)$$

$$C'' \left( -\frac{k\gamma}{2c^2}(-a^2 + b^2 - c^2) + \alpha, -\frac{k\gamma}{2c^2}(a^2 - b^2 - c^2) + \beta, (1-k)\gamma \right)$$

Because  $AA'$ ,  $BB'$ ,  $CC'$  are concurrent, we have:

$$\frac{-\frac{\alpha}{2a^2}(-a^2 - b^2 + c^2) + \beta}{-\frac{\alpha}{2a^2}(-a^2 + b^2 - c^2) + \gamma} \cdot \frac{-\frac{\beta}{2b^2}(-a^2 - b^2 - c^2) + \gamma}{-\frac{\beta}{2b^2}(-a^2 - b^2 + c^2) + \alpha} \cdot \frac{-\frac{\gamma}{2c^2}(-a^2 + b^2 - c^2) + \alpha}{-\frac{\gamma}{2c^2}(a^2 - b^2 - c^2) + \beta} = 1$$

We note

$$M = \frac{\alpha}{2a^2}(a^2 + b^2 - c^2) = \frac{\alpha}{a} \cdot b \cos C$$

$$N = \frac{\alpha}{2a^2}(a^2 - b^2 + c^2) = \frac{\alpha}{a} \cdot c \cos B$$

$$P = \frac{\beta}{2b^2}(-a^2 + b^2 + c^2) = \frac{\beta}{b} \cdot c \cos A$$

$$Q = \frac{\beta}{2b^2}(a^2 + b^2 - c^2) = \frac{\beta}{b} \cdot a \cos C$$

$$R = \frac{\gamma}{2c^2}(a^2 - b^2 + c^2) = \frac{\gamma}{c} \cdot a \cos B$$

$$S = \frac{\gamma}{2c^2}(-a^2 + b^2 + c^2) = \frac{\gamma}{c} \cdot a \cos A$$

The precedent relation becomes

$$\frac{M + \beta}{N + \gamma} \cdot \frac{P + \gamma}{Q + \alpha} \cdot \frac{R + \alpha}{S + \beta} = 1$$

The coefficients  $M, N, P, Q, R, S$  verify the following relations:

$$M + N = \alpha$$

$$P + Q = \beta$$

$$R + S = \gamma$$

$$\frac{M}{Q} = \frac{\alpha}{\beta} \cdot \frac{b^2}{a^2} = \frac{\frac{\alpha}{a^2}}{\frac{\beta}{b^2}}$$

$$\frac{P}{S} = \frac{\beta}{\gamma} \cdot \frac{c^2}{b^2} = \frac{\frac{\beta}{b^2}}{\frac{\gamma}{c^2}}$$

$$\frac{R}{N} = \frac{\gamma}{\alpha} \cdot \frac{a^2}{c^2} = \frac{\frac{\gamma}{c^2}}{\frac{\alpha}{a^2}}$$

Therefore  $\frac{M}{Q} \cdot \frac{P}{S} \cdot \frac{R}{N} = 1$

$$(M + \beta)(P + \gamma)(R + \alpha) = \alpha\beta\gamma + \alpha\beta P + \beta\gamma R + \gamma\alpha M + \alpha MP + \beta PR + \gamma RM + MPR$$

$$(N + \gamma)(Q + \alpha)(S + \beta) = \alpha\beta\gamma + \alpha\beta N + \beta\gamma Q + \gamma\alpha S + \alpha NS + \beta NQ + \gamma QS + NQS .$$

We deduct that:

$$\alpha\beta P + \beta\gamma R + \gamma\alpha M + \alpha MP + \beta PR + \gamma RM = \alpha\beta N + \beta\gamma Q + \gamma\alpha S + \alpha NS + \beta NQ + \gamma QS + NQS \quad (1)$$

We apply the theorem:

Given the points  $Q_i(a_i, b_i, c_i)$ ,  $i = \overline{1,3}$  in the plane of the triangle  $ABC$ , the lines

$AQ_1, BQ_2, CQ_3$  are concurrent if and only if  $\frac{b_1}{c_1} \cdot \frac{c_2}{a_2} \cdot \frac{a_3}{b_3} = 1$ .

For the lines  $AA'', BB'', CC''$  we obtain

$$\frac{kM + \beta}{kN + \gamma} \cdot \frac{kP + \alpha}{kS + \beta} \cdot \frac{kR + \alpha}{kS + \beta} = 1 .$$

It result that

$$\begin{aligned} & k^2(\alpha\beta P + \beta\gamma R + \gamma\alpha M) + k(\alpha MP + \beta PR + \gamma RM) = \\ & = k^2(\alpha\beta N + \beta\gamma Q + \gamma\alpha S) + k(\alpha NS + \beta NQ + \gamma QS) \end{aligned} \quad (2)$$

For relation (1) to imply relation (2) it is necessary that

$$\alpha\beta P + \beta\gamma R + \gamma\alpha M = \alpha\beta N + \beta\gamma Q + \gamma\alpha S$$

and

$$\alpha NS + \beta NQ + \gamma QS = \alpha MP + \beta PR + \gamma RM$$

or

$$\begin{cases} \alpha\beta c \left( \frac{\beta}{b} \cos A - \frac{\alpha}{a} \cos B \right) + \beta\gamma a \left( \frac{\gamma}{c} \cos B - \frac{\beta}{b} \cos C \right) + \gamma\alpha b \left( \frac{\alpha}{a} \cos C - \frac{\gamma}{c} \cos A \right) = 0 \\ \frac{\alpha^2}{a^2} \cos A \left( \frac{\gamma}{c} \cos B - \frac{\beta}{b} \cos C \right) + \frac{\beta^2}{b^2} \cos B \left( \frac{\gamma}{c} \cos B - \frac{\beta}{b} \cos C \right) + \frac{\gamma^2}{c^2} \cos C \left( \frac{\beta}{b} \cos A - \frac{\alpha}{a} \cos B \right) = 0 \end{cases}$$

As an open problem, we need to determine the set of the points from the plane of the triangle  $ABC$  that verify the precedent relations.

We will show that the points  $I$  and  $O$  verify these relations, proving two theorems that lead to Kariya's point and Franke's point.

**Theorem 2** (Kariya -1904)

Let  $I$  be the center of the circumscribe circle to triangle  $ABC$  and  $A', B', C'$  its projections on the sides  $BC, CA, AB$ . We consider the points  $A'', B'', C''$  such that:

$$\overrightarrow{IA''} = k\overrightarrow{IA'}, \overrightarrow{IB''} = k\overrightarrow{IB'}, \overrightarrow{IC''} = k\overrightarrow{IC'}, k \in R^*.$$

Then  $AA'', BB'', CC''$  are concurrent (the Kariya's point)

**Proof:**

The barycentric coordinates of the point  $I$  are  $I \left( \frac{a}{2p}, \frac{b}{2p}, \frac{c}{2p} \right)$ .

Evidently:

$$abc(\cos A - \cos B) + abc(\cos B - \cos C) + abc(\cos C - \cos A) = 0$$

and

$$\cos A(\cos B - \cos C) + \cos B(\cos C - \cos A) + \cos C(\cos A - \cos B) = 0.$$

In conclusion  $AA'', BB'', CC''$  are concurrent.

**Theorem 3** (de Boutin - 1890)

Let  $O$  be the center of the circumscribed circle to the triangle  $ABC$  and  $A', B', C'$  its projections on the sides  $BC, CA, AB$ . Consider the points  $A'', B'', C''$  such that  $\frac{OA'}{OA''} = \frac{OB'}{OB''} = \frac{OC'}{OC''} = k, k \in R^*$ . Then the lines  $AA'', BB'', CC''$  are concurrent (The point of Franke - 1904).

**Proof:**

$$O \left( \frac{R^2}{2S} \sin 2A, \frac{R^2}{2S} \sin 2B, \frac{R^2}{2S} \sin 2C \right), P = N, \text{ because } \frac{\sin 2B \cos A}{\sin B} - \frac{\sin 2A \cos B}{\sin A} = 0.$$

Similarly we find that  $R = Q$  and  $M = S$ .

Also  $\alpha MP = \alpha NS, \beta PR = \beta NQ, \gamma RM = \gamma QS$ . It is also verified the second relation from the theorem hypothesis. Therefore the lines  $AA'', BB'', CC''$  are concurrent in a point called the Franke's point.

**Remark 1:**

It is possible to prove that the Franke's points belong to Euler's line of the triangle  $ABC$ .

**Theorem 4:**

Let  $I_a$  be the center of the circumscribed circle to the triangle  $ABC$  (tangent to the side  $BC$ ) and  $A', B', C'$  its projections on the sites  $BC, CA, AB$ . We consider the points  $A'', B'', C''$  such that  $\overline{IA''} = k\overline{IA'}$ ,  $\overline{IB''} = k\overline{IB'}$ ,  $\overline{IC''} = k\overline{IC'}$ ,  $k \in R^*$ . Then the lines  $AA'', BB'', CC''$  are concurrent.

**Proof**

$$I_a \left( \frac{-a}{2(p-a)}, \frac{b}{2(p-a)}, \frac{c}{2(p-a)} \right);$$

The first condition becomes:

$$-abc(\cos A + \cos B) + abc(\cos B - \cos C) - abc(-\cos C - \cos A) = 0, \quad \text{and} \quad \text{the}$$

second condition:

$$\cos A(\cos B - \cos C) + \cos B(-\cos C - \cos A) + \cos C(\cos A + \cos B) = 0$$

Is also verified.

From this theorem it results that the lines  $AA'', BB'', CC''$  are concurrent.

**Observation 1:**

Similarly, this theorem is proven for the case of  $I_b$  and  $I_c$  as centers of the ex-inscribed circles.

**References**

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