From a problem of geometrical construction to the Carnot circles

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In this article we'll give solution to a problem of geometrical construction and we'll show the connection between this problem and the theorem relative to Carnot's circles.

Let ABC a given random triangle. Using only a compass and a measuring line, construct a point M in the interior of this triangle such that the circumscribed circles to the triangles MAB and MAC are congruent.

Construction

We'll start by assuming, as in many situations when we have geometrical constructions, that the construction problem is resolved.

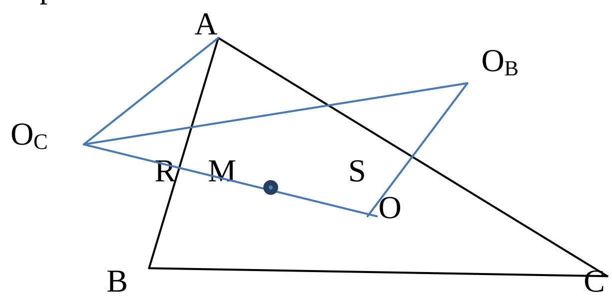

Fig. 1

Let M a point in the interior of the triangle ABC such that the circumscribed circles to the triangles MAB and MAC are congruent.

We'll note O_C and O_B the centers of these triangles, these are the intersections between the mediator of the segments AB and AC. The quadrilateral AO_CMO_B is a rhomb (therefore M is the symmetrical of the point A in rapport to O_BO_C (see Fig. 1).

A. **Step by step construction**

We'll construct the mediators of the segments AB and AC, let R,S their intersection points with AB respectively AC. (We suppose that $AB < AC$, therefore $AR < AS$.). With the compass in A and with the radius larger than AS we construct a circle which intersects OR in O_C and $O_{C'}$ respectively OS in O_B and $O_{B'}$ - O being the circumscribed circle to the triangle ABC.

Now we construct the symmetric of the point A in rapport to O_CO_B; this will be the point M, and if we construct the symmetric of the point A in rapport to $O_{C'}O_{B'}$ we obtain the point M'

Lazare Carnot (1753 – 1823), French mathematician, mechanical engineer and political personality. (Paris)

B. **Proof of the construction**

Because $AO_C = AO_B$ and M is the symmetric of the point A in rapport of O_CO_B, it results that the quadrilateral AO_CMO_B will be a rhombus, therefore $O_CA = O_CM$ and $O_BA = O_BM$. On the other hand, O_C and O_B being perpendicular points of AB respectively AC, we have $O_CA = O_CB$ and $O_BA = O_BC$, consequently

$$O_CA = O_CM = O_BA = O_BM = O_BC,$$

which shows that the circumscribed circles to the triangles MAB and MAC are congruent.

Similarly, it results that the circumscribed circles to the triangles ABM' and ACM' are congruent, more so, all the circumscribed circles to the triangles $MAB, MAC, M'AB, M'AC$ are congruent.

As it can be in the Fig. 2, the point M' is in the exterior of the triangle ABC

Discussion

We can obtain, using the method of construction shown above, an infinity of pairs of points M and M', such that the circumscribed circles to the triangles $MAB, MAC, M'AB, M'AC$ will be congruent. It seems that the point M' is in the exterior of the triangle ABC

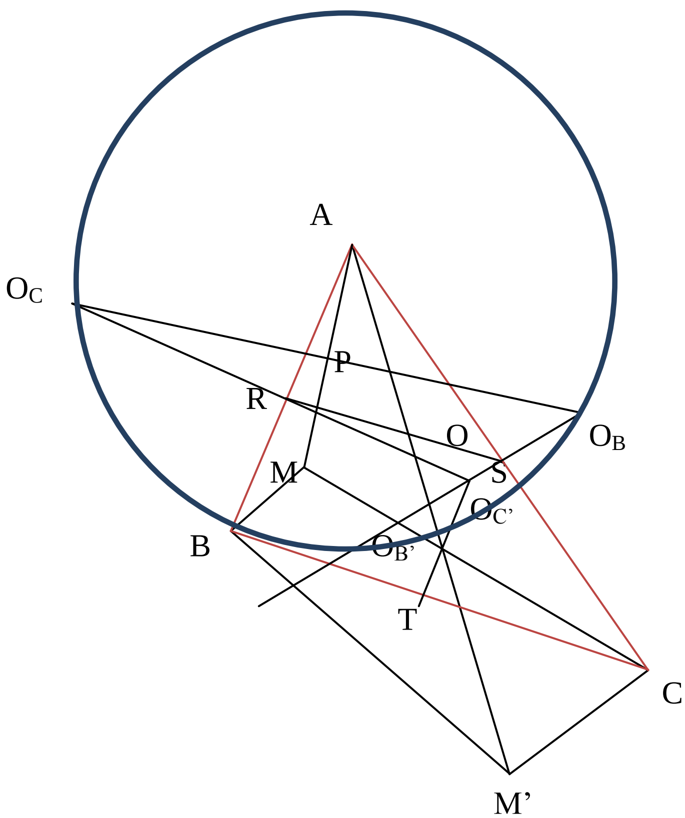

Fig. 2

Observation

The points M from the exterior of the triangle ABC with the property described in the hypothesis are those that belong to the arch BC, which does not contain the vertex A from the circumscribed circle of the triangle ABC.

Now, we'll try to answer to the following:

Questions

1. Can the circumscribed circles to the triangles MAB, MAC with M in the interior of the triangle ABC be congruent with the circumscribed circle of the triangle ABC
2. If yes, then, what can we say about the point M ?

Answers

1. The answer is positive. In this hypothesis we have $OA = AO_B = AO_C$ and it results also that O_C and O_B are the symmetrical of O in rapport to AB respectively AC The point M will be, as we showed, the symmetric of the point A in rapport to O_CO_B.

The point M will be also the orthocenter of the triangle ABC. Indeed, we prove that the symmetric of the point A in rapport to O_CO_B is H which is the orthocenter of the triangle ABC Let RS the middle line of the triangle ABC. We observe that RS is also middle line in the triangle OO_BO_C, therefore O_BO_C is parallel and congruent with BC, therefore it results that M belongs to the height constructed from A in the triangle ABC. We'll note T the middle of BC, and let R the radius of the circumscribed circle to the triangle ABC; we have

$$OT = \sqrt{R^2 - \frac{a^2}{4}}, \text{ where } a = BC.$$

If P is the middle of thesegment AM, we have

$$AP = \sqrt{R^2 - PO_B{}^2} = \sqrt{R^2 - \frac{a^2}{4}}.$$

From the relation $AM = 2 \cdot OT$ it results that M is the orthocenter of the triangle ABC, ($AH = 2OT$).

The answers to the questions 1 and 2 can be grouped in the following form:

Proposition

There is onlyone point in the interior of the triangle ABC such that the circumscribed circles to the triangles MAB, MAC and ABC are congruent. This point is the orthocenter of the triangle ABC.

Remark

From this proposition it practically results that the unique point M from the interior of the right triangle ABC with the property that the circumscribed circles to the triangles

MAB, MAC, MBC are congruent with the circumscribed circle to the triangle is the point H, the triangle's orthocenter.

Definition

If in the triangle ABC, H is the orthocenter, then the circumscribed circles to the triangles HAB, HAC, HBC are called Carnot circles.

We can prove, without difficulty the following:

Theorem

The Carnot circles of a triangle are congruent with the circumscribed circle to the triangle.

References

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