Mixt-Linear Circles Adjointly Ex-Inscribed Associated to a Triangle

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Abstract

In [1] we introduced the mixt-linear circles adjointly inscribed associated to a triangle, with emphasizes on some of their properties. Also, we've mentioned about mixt-linear circles adjointly ex-inscribed associated to a triangle.

In this article we'll show several basic properties of the mixt-linear circles adjointly ex-inscribed associate to a triangle.

Definition 1

We define a mixt-linear circle adjointly ex-inscribed associated to a triangle, the circle tangent exterior to the circle circumscribed to a triangle in one of the vertexes of the triangle, and tangent to the opposite side of the vertex of that triangle.

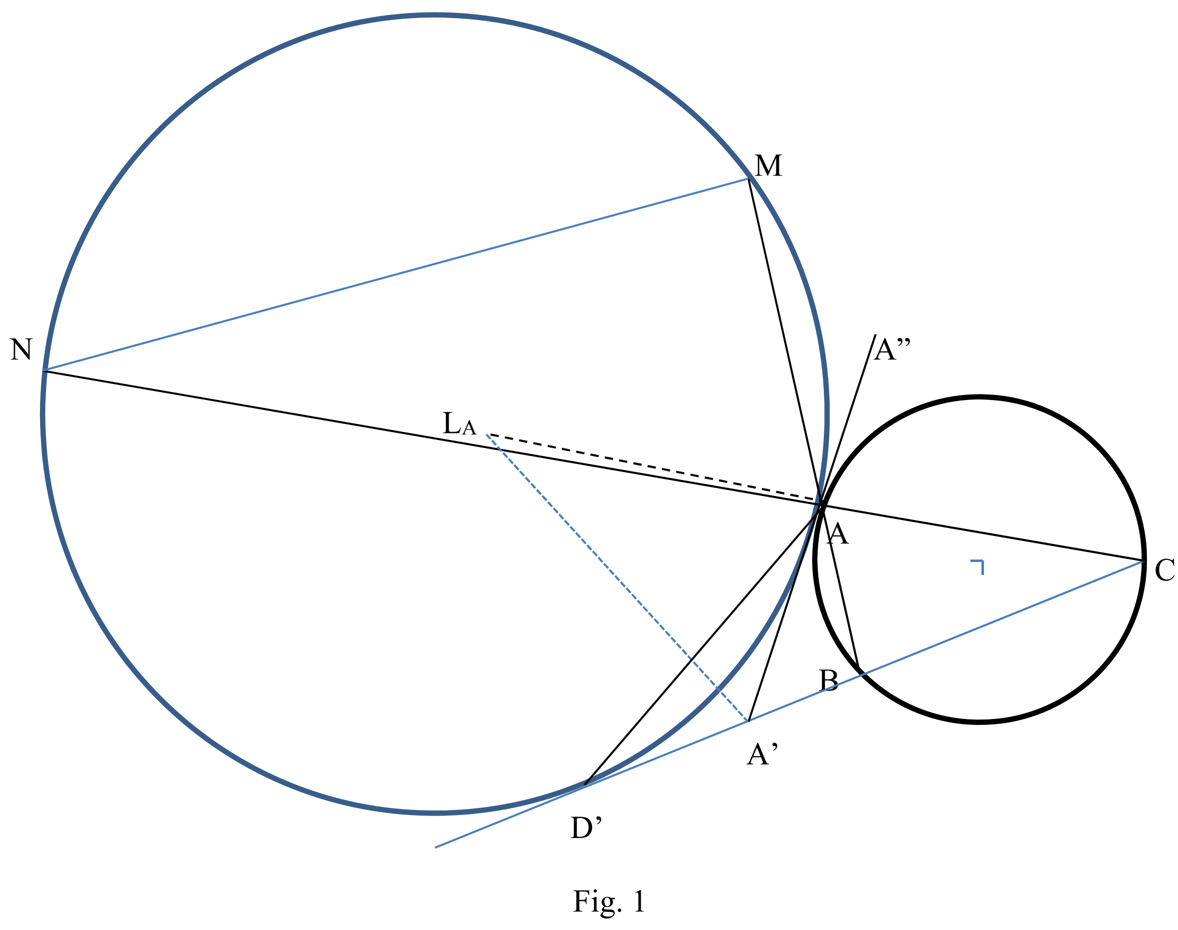

Fig. 1

Observation

In Fig.1 we constructed the mixt-linear circle adjointly ex-inscribed to triangle ABC, which is tangent in A to the circumscribed circle of triangle ABC, and tangent to the side BC. Will call this the A-mixt-linear circle adjointly ex-inscribed to triangle ABC. We note L_A the center of this circle.

Remark

In general, for a triangle exists three mixt-linear circles adjointly ex-inscribed. If the triangle ABC is isosceles with the base BC, then we cannot talk about mixt-linear circles adjointly ex-inscribed associated to the isosceles triangle.

Proposition 1

The tangency point with the side BC of the A-mixt-linear circle adjointly ex-inscribed associated to the triangle is the leg of the of the external bisectrix of the angle BAC

Proof

Let D' the contact point with the side BC of the A-mixt-linear circle adjointly ex-inscribed and let A' the intersection of the tangent in the point A to the circumscribed circle to the triangle ABC with BC (see Fig. 1)

We have

$$m(\sphericalangle AA'B) = \frac{1}{2}\left[m\left(\hat{B}\right) - m\left(\hat{C}\right)\right],$$

(we supposed that $m\left(\hat{B}\right) > m\left(\hat{C}\right)$). The tangents $AA', A'D'$ to the A-mixt-linear circle adjointly ex-inscribed are equal, therefore

$$m(\sphericalangle D'AA') = \frac{1}{4}m\left(\hat{B} - \hat{C}\right).$$

Because

$$m(\sphericalangle A'AB) = \frac{1}{2}m\left(\hat{C}\right)$$

we obtain that

$$m(\sphericalangle D'AB) = \frac{1}{2}\left[m\left(\hat{B}\right) + m\left(\hat{C}\right)\right]$$

This relation shows that D' is the leg of the external bisectrix of the angle BAC.

Proposition 2

The A-mixt-linear circle adjointly ex-inscribed to triangle ABC intersects the sides AB, AC, respectively, in two points of a cord which is parallel to BC.

Proof

We'll note with M, N the intersection points with AB respectively AC of the A-mixt-linear circle adjointly ex-inscribed. We have $\sphericalangle BCA \equiv \sphericalangle BAA'$ and $\sphericalangle A'AB \equiv \sphericalangle A''AM$ (see Fig.1).

Because $\sphericalangle A''AM = \sphericalangle ANM$, we obtain $\sphericalangle ANM \equiv \sphericalangle ACB$ which implies that MN is parallel to BC.

Proposition 3

The radius R_A of the A-mixt-linear circle adjointly ex-inscribed to triangle ABC is given by the following formula

$$R_A = \frac{4(p-b)(p-c)R}{(b-c)^2}$$

Proof

The sinus theorem in the triangle AMN implies

$$R_A = \frac{MN}{2\sin A}$$

We observe that the triangles AMN and ABC are similar; it results that

$$\frac{MN}{a} = \frac{AM}{c}.$$

Considering the power of the point B in rapport to the A-mixt-linear circle adjointly ex-inscribed of triangle ABC, we obtain

$$BA \cdot BM = BD'^2.$$

From the theorem of the external bisectrix we have $\frac{D'B}{D'C} = \frac{c}{b}$ from which we retain $D'B = \frac{ac}{b-c}$. We obtain then $BM = \frac{a^2c}{(b-c)^2}$, therefore

$$AM = \frac{c(a-b+c)(a+b-c)}{(b-c)^2} = \frac{4c(p-b)(p-c)}{(b-c)^2}$$

and

$$MN = \frac{4a(p-b)(p-c)}{(b-c)^2}$$

From the sinus theorem applied in the triangle ABC results that $\frac{a}{2\sin A} = R$ and we obtain that

$$R_A = \frac{4(p-b)(p-c)R}{(b-c)^2}.$$

Remark

If we note $P \in L_A A' \cap AD'$ and $AD' = l_a'$ (the length of the exterior bisectrix constructed from A) in triangle $L_A PA'$, we find

$$R_A = \frac{l_a'}{2\sin\frac{B-C}{2}}.$$

We'll remind here several results needed for the remaining of this presentation.

Definition 2

We define an adjointly circle of triangle ABC a circle which contains two vertexes of the triangle and in one of these vertexes is tangent to the respective side.

Theorem 1

The adjointly circles $A\overline{B}, B\overline{C}, C\overline{A}$ have a common point Ω; similarly, the circles $B\overline{A}, C\overline{B}, A\overline{C}$ have a common point Ω'.

The points Ω and Ω' are called the points of Brocard: Ω is the direct point of Brocard and Ω' is called the retrograde point.

The points Ω and Ω' are conjugate isogonal

$$\sphericalangle \Omega AB = \sphericalangle \Omega BC = \sphericalangle \Omega CA = \omega$$
$$\sphericalangle \Omega' AC = \sphericalangle \Omega' CB = \sphericalangle \Omega' BA = \omega$$

(see Fig. 2).

The angle ω is called the Brocard angle. More information can be found in [3].

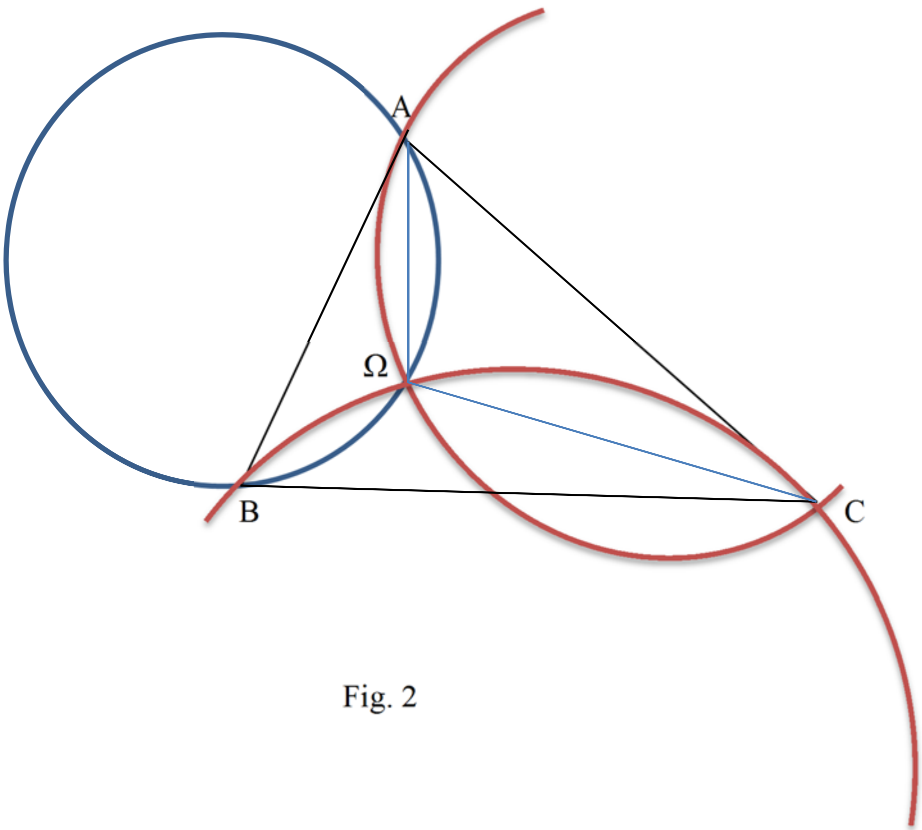

Fig. 2

Proposition 4

In triangle ABC in which D' is the leg of the external bisectrix of the angle BAC, the A-mixt-linear circle adjointly ex-inscribed to triangle ABC is an adjointly circle of triangles $AD'B, AD'C$.

Proposition 5

In a triangle ABC in which D' is the leg of the external bisectrix of the angle BAC, the direct points of Brocard corresponding to triangles $AD'B$, $AD'C$, A, D' are concyclic.

The following theorems show remarkable properties of the mixt-linear circles adjointly ex-inscribed associated to a triangle ABC.

Theorem 2

The triangle $L_A L_B L_C$ determined by the centers of the mixt-linear circles adjointly ex-inscribed to triangle ABC and the tangential triangle $T_a T_b T_c$ corresponding to ABC are orthological. Their orthological centers are O the center of the circumscribed circle to triangle ABC and the radical center of the mixt-linear circles adjointly ex-inscribed associated to triangle ABC.

Proof

The perpendiculars constructed from L_A, L_B, L_C on the corresponding sides of the tangential triangle contain the radiuses $OA,\ OB,\ OC$ respectively of the circumscribed circle.

Consequently, O is the orthological center of triangles $L_A L_B L_C$ and $T_a T_b T_c$.

In accordance to the theorem of orthological triangles and the perpendiculars constructed from T_a, T_b, T_c respectively on the sides of the triangle $L_A L_B L_C$ are concurrent.

The point T_a belongs to the radical axis of the circumscribed circles to triangle ABC and the C-mixt-linear circle adjointly ex-inscribed to triangle ABC (belongs to the common tangent constructed in C to these circles).

On the other side T_a belongs to the radical axis of the B and C-mixt-linear circle adjointly ex-inscribed, which means that the perpendicular constructed from T_a on the $L_B L_C$ centers line passes through the radical center of the mixt-linear circle adjointly ex-inscribed associated to the triangle; which is the second orthological center of the considered triangles.

Proposition 6

The triangle $L_a L_b L_c$ (determined by the centers of the mixt-linear circles adjointly inscribed associated to the triangle ABC) and the triangle $L_A L_B L_C$ (determined by the centers of the mixt-linear circles adjointly ex-inscribed associated to the triangle ABC) are homological. The homological center is the point O, which is the center of the circumscribed circle of triangle ABC.

The proof results from the fact that the points L_A, A, L_a, O are collinear. Also, L_B, B, L_b, O and L_C, C, L_c, O are collinear.

Definition 3

Given three circles of different centers, we define their Apollonius circle as each of the circles simultaneous tangent to three given circles.

Observation

The circumscribed circle to the triangle ABC is the Apollonius circle for the mixt-linear circles adjointly ex-inscribed associated to ABC.

Theorem 3

The Apollonius circle which has in its interior the mixt-linear circles adjointly ex-inscribed to triangle ABC is tangent with them in the points T_1, T_2, T_3 respectively. The lines AT_1, BT_2, CT_3 are concurrent.

Proof

We'll use the D'Alembert theorem: Three circles non-congruent whose centers are not collinear have their six homothetic centers placed on four lines, three on each line.

The vertex A is the homothety inverse center of the circumscribed circle (O) and of the A-mixt-linear circle adjointly ex-inscribed (L_A); T_1 is the direct homothety center of the Apollonius circle which is tangent to the mixt-linear circles adjointly ex-inscribed and of circle (L_A), and J is the center of the direct homothety of the Apollonius circle and of the circumscribed circle (O).

According to D'Alembert theorem, it results that the points A, J, T_1 are collinear. Similarly is shown that the points B, J, T_2 and C, J, T_3 are collinear.

Consequently, J is the concurrency point of the lines AT_1, BT_2, CT_3.

[1] I. Pătraşcu, Cercuri mixtliniare adjunct inscrise associate unui triunghi, Revista Recreatii Matematice , No. 2/2013.

[2] R. A. Johnson, Advanced Euclidean Geometry, Dover Publications Inc., New York, 2007.

[3] F. Smarandache, I. Pătraşcu, The Geometry of Homological Triangles, The Education Publisher Inc., Columbus, Ohio, U.S.A., 2012.