

Two Triangles with the Same Orthocenter and a Vectorial Proof of Stevanovic's Theorem

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Abstract. In this article we'll emphasize on two triangles and provide a vectorial proof of the fact that these triangles have the same orthocenter. This proof will, further allow us to develop a vectorial proof of the Stevanovic's theorem relative to the orthocenter of the Fuhrmann's triangle.

Lemma 1

Let ABC an acute angle triangle, H its orthocenter, and A',B',C' the symmetrical points of H in rapport to the sides BC,CA,AB .

We denote by X,Y,Z the symmetrical points of A,B,C in rapport to $B'C',C'A',A'B'$. The orthocenter of the triangle XYZ is H .

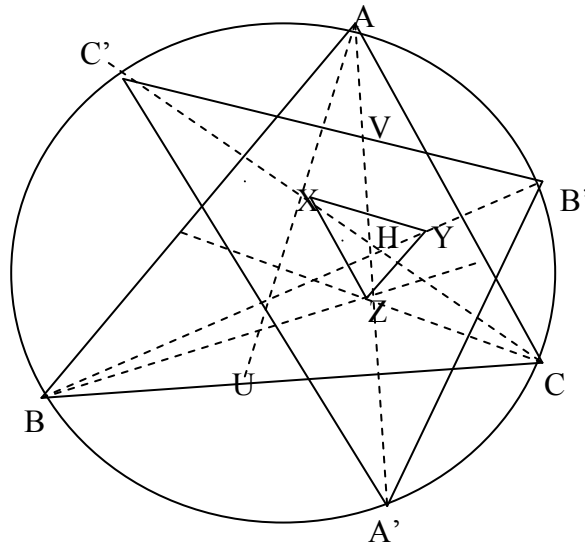


Fig. 1

Proof

We will prove that $XH \perp YZ$, by showing that $\overrightarrow{XH} \cdot \overrightarrow{YZ} = 0$.
 We have (see Fig.1)

$$\begin{aligned} \overrightarrow{VH} &= \overrightarrow{AH} - \overrightarrow{AX} \\ \overrightarrow{BC} &= \overrightarrow{BY} + \overrightarrow{YZ} + \overrightarrow{ZC}, \end{aligned}$$

from here

$$\overrightarrow{YZ} = \overrightarrow{BC} - \overrightarrow{BY} - \overrightarrow{ZC}$$

Because Y is the symmetric of B in rapport to $A'C'$ and Z is the symmetric of C in rapport to $A'B'$, the parallelogram's rule gives us that:

$$\begin{aligned}\overrightarrow{BY} &= \overrightarrow{BC'} + \overrightarrow{BA'} \\ \overrightarrow{CZ} &= \overrightarrow{CB'} + \overrightarrow{CA'}.\end{aligned}$$

Therefore

$$\overrightarrow{YZ} = \overrightarrow{BC} - (\overrightarrow{BC'} + \overrightarrow{BA'}) + \overrightarrow{B'C} + \overrightarrow{A'C}$$

But

$$\begin{aligned}\overrightarrow{BC'} &= \overrightarrow{BH} + \overrightarrow{HC'} \\ \overrightarrow{BA'} &= \overrightarrow{BH} + \overrightarrow{HA'} \\ \overrightarrow{CB'} &= \overrightarrow{CH} + \overrightarrow{HB'} \\ \overrightarrow{CA'} &= \overrightarrow{CH} + \overrightarrow{HA'}\end{aligned}$$

By substituting these relations in the \overrightarrow{YZ} , we find:

$$\overrightarrow{YZ} = \overrightarrow{BC} + \overrightarrow{C'B'}$$

We compute

$$\overrightarrow{XH} \cdot \overrightarrow{YZ} = (\overrightarrow{AH} - \overrightarrow{AX}) \cdot (\overrightarrow{BC} + \overrightarrow{C'B'}) = \overrightarrow{AX} \cdot \overrightarrow{BC} + \overrightarrow{AH} \cdot \overrightarrow{C'B'} - \overrightarrow{AX} \cdot \overrightarrow{BC} - \overrightarrow{AX} \cdot \overrightarrow{C'B'}$$

Because

$$AH \perp BC$$

we have

$$\overrightarrow{AH} \cdot \overrightarrow{BC} = 0,$$

also

$$AX \perp B'C'$$

and therefore

$$\overrightarrow{AX} \cdot \overrightarrow{B'C'} = 0.$$

We need to prove also that

$$\overrightarrow{XH} \cdot \overrightarrow{YZ} = \overrightarrow{AH} \cdot \overrightarrow{C'B'} - \overrightarrow{AX} \cdot \overrightarrow{BC}$$

We note:

$$\{U\} = AX \cap BC \text{ and } \{V\} = AH \cap B'C'$$

$$\overrightarrow{AX} \cdot \overrightarrow{BC} = AX \cdot BC \cdot \cos \angle(A, BC) = AX \cdot BC \cdot \cos(\angle AUC)$$

$$\overrightarrow{AH} \cdot \overrightarrow{C'B'} = AH \cdot C'B' \cdot \cos \angle(AH, C'B') = AH \cdot C'A' \cdot \cos(\angle AVC')$$

We observe that

$$\angle AUC \equiv \angle AVC' \text{ (angles with the sides respectively perpendicular).}$$

The point B' is the symmetric of H in rapport to AC , consequently

$$\angle HAC \equiv \angle CAB',$$

also the point C' is the symmetric of the point H in rapport to AB , and therefore

$$\angle HAB \equiv \angle BAC'.$$

From these last two relations we find that

$$\angle B'AC' = 2\angle A.$$

The sinus theorem applied in the triangles $AB'C'$ and ABC gives:

$$B'C' = 2R \cdot \sin 2A$$

$$BC = 2R \sin A$$

We'll show that

$$AX \cdot BC = AH \cdot C'B',$$

and from here

$$AX \cdot 2R \sin A = AH \cdot 2R \cdot \sin 2A$$

which is equivalent to

$$AX = 2AH \cos A$$

We noticed that

$$\sphericalangle B'AC' = 2A,$$

Because

$$AX \perp B'C',$$

it results that

$$\sphericalangle TAB \equiv \sphericalangle A,$$

we noted $\{T\} = AX \cap B'C'$.

On the other side

$$AC' = AH, \quad AT = \frac{1}{2}AY,$$

and

$$AT = AC' \cos A = AH \cos A,$$

therefore

$$\overrightarrow{XH} \cdot \overrightarrow{YZ} = 0.$$

Similarly, we prove that

$$YH \perp XZ,$$

and therefore H is the orthocenter of triangle XYZ .

Lemma 2

Let ABC a triangle inscribed in a circle, I the intersection of its bisector lines, and A', B', C' the intersections of the circumscribed circle with the bisectors AI, BI, CI respectively.

The orthocenter of the triangle $A'B'C'$ is I .

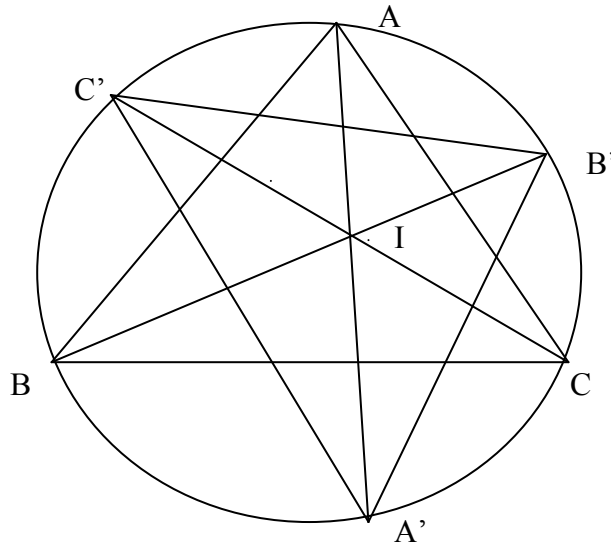


Fig. 2

Proof

We'll prove that $A'I \perp B'C'$.

Let

$$\alpha = m(\widehat{A'C}) = m(\widehat{A'B}),$$

$$\beta = m(\widehat{B'C}) = m(\widehat{B'A})$$

$$\gamma = m(\widehat{C'A}) = m(\widehat{C'B})$$

Then

$$m\angle(A'IC') = \frac{1}{2}(\alpha + \beta + \gamma)$$

Because

$$2(\alpha + \beta + \gamma) = 360^\circ$$

it results

$$m\angle(A'IC') = 90^\circ,$$

therefore

$$A'I \perp B'C'.$$

Similarly, we prove that

$$B'I \perp A'C',$$

and consequently the orthocenter of the triangle $A'B'C'$ is I , the center of the circumscribed circle of the triangle ABC .

Definition

Let ABC a triangle inscribed in a circle with the center in O and A', B', C' the middle of the arcs \widehat{BC} , \widehat{CA} , \widehat{AB} respectively. The triangle XYZ formed by the symmetric of the points A', B', C' respectively in rapport to BC, CA, AB is called the Fuhrmann triangle of the triangle ABC .

Note

In 2002 the mathematician Milorad Stevanovic proved the following theorem:

Theorem (M. Stevanovic)

In an acute angle triangle the orthocenter of the Fuhrmann's triangle coincides with the center of the circle inscribed in the given triangle.

Proof

We note $A'B'C'$ the given triangle and let A, B, C respectively the middle of the arcs $\widehat{B'C'}$, $\widehat{C'A'}$, $\widehat{A'B'}$ (see Fig. 1). The lines AA', BB', CC' being bisectors in the triangle $A'B'C'$ are concurrent in the center of the circle inscribed in this triangle, which will note H , and which, in conformity with Lemma 2 is the orthocenter of the triangle ABC . Let XYZ the Fuhrmann triangle of the triangle $A'B'C'$, in conformity with Lemma 1, the orthocenter of XYZ coincides with H the orthocenter of ABC , therefore with the center of the inscribed circle in the given triangle $A'B'C'$.

References

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