

A FUNCTION IN THE NUMBER THEORY

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Abstract:

In this paper I shall construct a function¹ η having the following properties:

- (1) $\forall n \in \mathbb{Z}, n \neq 0, (\eta(n))! = M n$ (multiple of n).
- (2) $\eta(n)$ is the smallest natural number satisfying property (1).

MSC: 11A25, 11B34.

Introduction:

We consider:

$\mathbb{N} = \{0, 1, 2, 3, \dots\}$ and $\mathbb{N}^* = \{1, 2, 3, \dots\}$.

Lemma 1. $\forall k, p \in \mathbb{N}^*, p \neq 1, k$ is uniquely written

in the form: $k = t_1 a_{n(1)}^{(p)} + \dots + t_l a_{n(l)}^{(p)}$ where

$$a_{n(i)}^{(p)} = \frac{p^{n(i)} - 1}{p - 1}, \quad i = \overline{1, l}, \quad n_1 > n_2 > \dots > n_l > 0 \text{ and } 1 \leq t_j \leq p - 1, j = \overline{1, l - l}, \quad 1 \leq t_l \leq p, n_i, t_i \in \mathbb{N},$$

$$i = \overline{1, l}, l \in \mathbb{N}^*.$$

Proof.

The string $(a_n^{(p)})_{n \in \mathbb{N}}$ consists of strictly increasing infinite natural numbers and

$$a_{n+1}^{(p)} - 1 = p * a_n^{(p)}, \alpha n \in \mathbb{N}^*, p \text{ is fixed,}$$

$$a_1^{(p)} = 1, a_2^{(p)} = 1 + p, a_3^{(p)} = 1 + p + p^2, \dots \dots \text{ Therefore:}$$

$$\mathbb{N}^* = \bigcup_{n \in \mathbb{N}^*} ([a_n^{(p)}, a_{n+1}^{(p)}] \cap \mathbb{N}^*) \text{ where } (a_n^{(p)}, a_{n+1}^{(p)}) \cap (a_{n+1}^{(p)}, a_{n+2}^{(p)}) = \emptyset$$

because $a_n^{(p)} < a_{n+1}^{(p)} < a_{n+2}^{(p)}$.

Let $k \in \mathbb{N}^*, \mathbb{N}^* = \bigcup ((a_n^{(p)}, a_{n+1}^{(p)}) \cap \mathbb{N}^*),$

therefore $\exists! n_1 \in \mathbb{N}^* : k \in (a_{n(1)}^{(p)}, a_{n(1)+1}^{(p)})$, therefore k is uniquely written under the form

$$k = \left(\frac{k}{a_{n_1}^{(p)}} \right) a_{n(1)}^{(p)} + r_1 \text{ (integer division theorem).}$$

¹ This function has been called the Smarandache function. Over one hundred articles, notes, problems and a dozen of books have been written about it.

We note

$$k = \left(\frac{k}{a^{(p)}_{n_1}} \right) = t_1 \rightarrow k = t_1 a_{n(1)}^{(p)} + r_1, r_1 < a_{n(1)}^{(p)} .$$

If $r_1 = 0$, as $a_{n(1)}^{(p)} \leq k \leq a_{n(1)+1}^{(p)} - 1 \rightarrow 1 \leq t_1 \leq p$ and Lemma 1 is proved.

If $r_1 \neq 0$, then $\exists ! n_2 \in \mathbb{N}^* : r_1 \in [a_{n(2)}^{(p)}, a_{n(2)+1}^{(p)})$;

$a_{n(1)}^{(p)} > r_1$ involves $n_1 > n_2$, $r_1 \neq 0$ and $a_{n(1)}^{(p)} \leq k \leq a_{n(1)+1}^{(p)} - 1$ involves $1 \leq t_1 \leq p - 1$ because we have

$$t_1 \leq (a_{n(1)+1}^{(p)} - 1 - r_1) : a_n^{(p)} < p_1 .$$

The procedure continues similarly. After a finite number of steps l , we achieve $r_l = 0$, as $k = \text{finite}$, $k \in \mathbb{N}^*$ and $k > r_1 > r_2 > \dots > r_l = 0$ and between 0 and k there is only a finite number of distinct natural numbers.

Thus:

k is uniquely written: $k = t_1 a_{n(1)}^{(p)} + r_1, 1 \leq t_1 \leq p - 1$,

r_1 is uniquely written: $r_1 = t_2 * a_{n(2)}^{(p)} + r_2, n_2 < n_1$,

$$1 \leq t_2 \leq p-1,$$

r_{l-1} is uniquely written: $r_{l-1} = t_l * a_{n(l)}^{(p)} + r_l$, and $r_l = 0$,

$$n_l < n_{l-1}, 1 \leq t_l \leq p,$$

thus k is uniquely written under the form

$$k = t_1 a_{n(1)}^{(p)} + \dots + t_l a_{n(l)}^{(p)}$$

with $n_1 > n_2 > \dots > n_l > 0$, because $n_i \in \mathbb{N}^*, 1 \leq t_j \leq p-1, j = 1, l-1, 1 \leq t_l \leq p, l \geq 1$.

Let $k \in \mathbb{N}^*, k = t_1 a_{n(1)}^{(p)} + \dots + t_l a_{n(l)}^{(p)}$ with

$$a_{n(i)}^{(p)} = \frac{p^{n_i} - 1}{p - 1},$$

$i = \overline{1, l}, l \geq 1, n_i, t_i \in \mathbb{N}^*, i = \overline{1, l}, n_1 > n_2 > \dots > n_l > 0$

$1 \leq t_j \leq p - 1, j = \overline{1, l-1}, 1 \leq t_l \leq p$.

I construct the function $\eta_p, p = \text{prime} > 0, \eta_p: \mathbb{N}^* \rightarrow \mathbb{N}$ thus:

$$\forall n \in \mathbb{N}^* \eta_p(a_n^{(p)}) = p^n,$$

$$\eta_p(t_1 a_{n(1)}^{(p)} + \dots + t_l a_{n(l)}^{(p)}) = t_1 \eta_p(a_{n(1)}^{(p)}) + \dots + t_l \eta_p(a_{n(l)}^{(p)}).$$

NOTE 1. The function η_p is well defined for each natural number.

Proof

LEMMA 2. $\forall k \in \mathbb{N}^*$, k is uniquely written as $k = t_1 a_{n(1)}^{(p)} + \dots + t_l a_{n(l)}^{(p)}$ with the conditions from Lemma

1, thus $\exists ! t_1 p^{n(1)} + \dots + t_l p^{n(l)} = \eta_p(t_1 a_{n(1)}^{(p)} + \dots + t_l a_{n(l)}^{(p)})$ and $t_1 p^{n(1)} + \dots + t_l p^{n(l)} \in \mathbb{N}^*$.

LEMMA 3. $\forall k \in \mathbb{N}^*$, $\forall p \in \mathbb{N}$, $p = \text{prime}$ then $k = t_1 a_{n(1)}^{(p)} + \dots + t_l a_{n(l)}^{(p)}$ with the conditions from Lemma 2 thus $\eta_p(k) = t_1 p^{n(1)} + \dots + t_l p^{n(l)}$

It is known that

$$\left(\frac{a_1 + \dots + a_n}{b} \right) \geq \left(\frac{a_1}{b} \right) + \dots + \left(\frac{a_n}{b} \right) \quad \forall a_i, b \in \mathbb{N}^* \text{ where through } [\alpha] \text{ we}$$

have written the integer side of the number α . I shall prove that p 's powers sum from the natural numbers which make up the result factors

$(t_1 p^{n(1)} + \dots + t_l p^{n(l)}) ! \text{ is } \geq k;$

$$\left(\frac{t_1 p^{n(1)} + \dots + t_l p^{n(l)}}{p} \right) \geq \left(\frac{t_1 p^{n(1)}}{p} \right) + \dots + \left(\frac{t_l p^{n(l)}}{p} \right) =$$

$$t_1 p^{n(1)-1} + \dots + t_l p^{n(l)-1}$$

$$\left(\frac{t_1 p^{n(1)} + \dots + t_l p^{n(l)}}{p^n} \right) \geq \left(\frac{t_1 p^{n(1)}}{p^{n(l)}} \right) + \dots + \left(\frac{t_l p^{n(l)}}{p^{n(l)}} \right) =$$

$$t_1 p^{n(1)-n(l)} + \dots + t_l p^0$$

$$\left(\frac{t_1 p^{n(1)} + \dots + t_l p^{n(l)}}{p^{n(1)}} \right) \geq \left(\frac{t_1 p^{n(1)}}{p^{n(1)}} \right) + \dots + \left(\frac{t_l p^{n(l)}}{p^{n(1)}} \right) =$$

$$t_1 p^0 + \dots + \frac{t_l p^{n(l)}}{p^{n(1)}} .$$

Adding $\rightarrow p$'s powers the sum is $\geq t_1(p^{n(1)-1} + \dots + p^0) + \dots + t_l(p^{n(l)-1} + \dots + p^0) =$

$$t_1 a_{n(1)}^{(p)} + \dots + t_l a_{n(l)}^{(p)} = k.$$

Theorem 1. The function η_p , $p = \text{prime}$, defined previously, has the following properties:

$$(1) \exists k \in \mathbb{N}^*, (n_p(k))! = M p^k.$$

(2) $n_p(k)$ is the smallest number with the property (1).

Proof

(1) Results from Lemma 3.

$$(2) \forall k \in \mathbb{N}^*, p \geq 2 \text{ one has } k = t_1 a_{n(1)}^{(p)} + \dots + t_l a_{n(l)}^{(p)}$$

(by Lemma 2) is uniquely written, where:

$$n_i, t_i \in \mathbb{N}^*, n_1 > n_2 > \dots > n_l > 0,$$

$$a_{n(i)}^{(p)} = \frac{p^{n(i)} - 1}{p - 1} \in \mathbb{N}^*,$$

$$i = \overline{1, l}, 1 \leq t_j \leq p - 1, j = \overline{1, l - 1}, 1 < t_l < p.$$

$$\rightarrow \eta_p(k) = t_1 p^{n(1)} + \dots + t_l p^{n(l)}. \text{ I note: } z = t_1 p^{n(1)} + \dots + t_l p^{n(l)}.$$

Let us prove that z is the smallest natural number with the property (1). I suppose by the method of reductio ad absurdum that $\exists \gamma \in \mathbb{N}, \gamma < z$:

$$\gamma! = M p^k;$$

$$\gamma < z \rightarrow \gamma \leq z - 1 \rightarrow (z-1)! = M p^k.$$

$$z - 1 = z = t_1 p^{n(1)} + \dots + t_l p^{n(l)} - 1; n_1 > n_2 > \dots > n_l \geq 1 \text{ and}$$

$$n_j \in \mathbb{N}, j = \overline{1, l};$$

$$\left(\frac{z-1}{p} \right) = t_1 p^{n(1)-1} + \dots + t_{l-1} p^{n(l-1)-1} + t_l p^{n(l)-1} - 1 \text{ as } \left(\frac{-1}{p} \right) = -1 \text{ because } p \geq 2,$$

$$\left(\frac{z-1}{p^{n(l)}} \right) = t_1 p^{n(1)-n(l)} + \dots + t_{l-1} p^{n(l-1)-n(l)} + t_l p^0 - 1 \text{ as } \left(\frac{-1}{p^{n(l)}} \right) = -1$$

as $p \geq 2, n_l \geq 1$,

$$\left(\frac{z-1}{p^{n(l)+1}} \right) = t_1 p^{n(1)-n(l)-1} + \dots + t_{l-1} p^{n(l-1)-n(l)-1} + \left(\frac{t_l p^{n(l)} - 1}{p^{n(l)+1}} \right) =$$

$$t_1 p^{n(1)-n(l)-1} + \dots + t_{l-1} p^{n(l-1)-n(l)-1} \text{ because}$$

$$0 < t_l p^{n(l)} - 1 \leq p^* p^{n(l)} - 1 < p^{n(l)+1} \text{ as } t_l < p;$$

$$\left(\frac{z-1}{p^{n(l-1)}} \right) = t_1 p^{n(1)-n(l-1)} + \dots + t_{l-1} p^0 + \left(\frac{t_l p^{n(l)} - 1}{p^{n(l-1)}} \right) =$$

$$t_1 p^{n(1) - n(l-1)} + \dots + t_{l-1} p^0 \text{ as } n_{l-1} > n_l,$$

$$\left(\frac{z - 1}{p^{n(1)}} \right) = t_1 p^0 + \left(\frac{t_2 p^{n(2)} + \dots + t_l p^{n(l)} - 1}{p^{n(1)}} \right) = t_1 p^0.$$

$$\text{Because } 0 < t_2 p^{n(2)} + \dots + t_l p^{n(l)} - 1 \leq (p-1)p^{n(2)} + \dots + (p-1)p^{n(l-1)} + p^* p^{n(l)} - 1 \leq$$

$$(p-1) * \sum_{i=n(l-1)}^{n_2} p_i + p^{n(l)+1} - 1 \leq$$

$$(p-1) \frac{p^{n(2)+1}}{p-1} = p^{n(2)+1} - 1 < p^{n(1)} - 1 < p^{n(1)} \text{ therefore}$$

$$\left(\frac{t_2 p^{n(2)} + \dots + t_l p^{n(l)} - 1}{p^{n(1)}} \right) = 0$$

$$\left(\frac{z - 1}{p^{n(1)+1}} \right) = \left(\frac{t_1 p^{n(1)} + \dots + t_l p^{n(l)} - 1}{p^{n(1)+1}} \right) = 0 \text{ because:}$$

$$0 < t_1 p^{n(1)} + \dots + t_l p^{n(l)} - 1 < p^{n(1)+1} - 1 < p^{n(1)+1} \text{ according to a reasoning similar to the previous one.}$$

Adding one gets p 's powers sum in the natural numbers which make up the product factors $(z-1)!$ is:

$$t_1 (p^{n(1)-1} + \dots + p^0) + \dots + t_{l-1} (p^{n(l-1)-1} + \dots + p^0) + t_l (p^{n(l)-1} + \dots + p^0) \text{ whence}$$

$$1 * n_l = k \text{ or } n_l < k \text{ or } 1 < k \text{ because}$$

$n_l > 1$ one has $(z-1)! \neq M p^k$, this contradicts the supposition made.

Whence $\eta_p(k)$ is the smallest natural number with the property $(\eta_p(k))! = M p^k$.

I construct a new function $\eta: Z \setminus \{0\} \rightarrow \mathbb{N}$ defined as follows:

$$\left\{ \begin{array}{l} \eta(\pm 1) = 0. \\ \alpha n = \varepsilon p_1^{\alpha(1)} \dots p_s^{\alpha(s)} \text{ with } \varepsilon = \pm 1, p_i \text{ prime,} \\ p_i = p_j \text{ for } i \neq j, \alpha_i \geq 1, i = 1, s, \eta(n) = \max_{i=1, \dots, s} \{ \eta(\alpha_i) \}. \end{array} \right.$$

Note 2. η is well defined all over.

Proof

(a) $\forall n \in \mathbb{Z}, n \neq 0, n \neq \pm 1$, n is uniquely written, abstraction of the order of the factors, under the form:

$n = \varepsilon p_1^{\alpha(1)} \dots p_s^{\alpha(s)}$ with $\varepsilon = \pm 1$, where $p_i = \text{prime}, p_i \neq p_j, \alpha_i \geq 1$ (decomposed into prime factors in \mathbb{Z} , which is a factorial ring).

Then $\exists ! \eta(n) = \max_{i=1,s} \{ \eta_{p(i)}(\alpha_i) \}$ as $s = \text{finite}$ and $\eta_{p(i)}(\alpha_i) \in \mathbb{N}^*$

and $\exists \max_{i=1,\dots,s} \{ \eta_{p(i)}(\alpha_i) \}$

(b) $n = \pm 1 \rightarrow E! \eta(n) = 0$.

Theorem 2. The function η previously defined has the following properties:

- (1) $(\eta(n))! = M n, \forall n \in \mathbb{Z} \setminus \{0\}$;
- (2) $\eta(n)$ is the smallest natural number with this property.

Proof

$$(a) \eta(n) = \max_{i=1,\dots,s} \{ \eta_{p(i)}(\alpha_i) \}, n = \varepsilon * p_1^{\alpha(1)} \dots p_s^{\alpha(s)} \quad (n \neq \pm 1),$$

$$(\eta_{p(1)}(\alpha_1))! = M p_1^{\alpha(1)},$$

$$(\eta_{p(s)}(\alpha_s))! = M p_s^{\alpha(s)}.$$

Supposing $\max_{i=1,\dots,s} \{ \eta_{p(i)}(\alpha_i) \} = \eta_{p(i_0)}(\alpha_{i_0}) \rightarrow (\eta_{p(i_0)}(\alpha_{i_0}))! =$

$M p_{i_0}^{\alpha_{i_0}}$, $\eta_{p(i_0)}(\alpha_{i_0}) \in \mathbb{N}^*$ and because $(p_i, p_j) = 1, i \neq j$,

then $(\eta_{p(i_0)}(\alpha_{i_0}))! = M p_j^{\alpha(j)}, \overline{j} = \overline{1}, s$.

Also $(\eta_{p(i_0)}(\alpha_{i_0}))! = M p_1^{\alpha(1)} \dots p_s^{\alpha(s)}$.

(b) $n = \pm 1 \rightarrow \eta(n) = 0; 0! = 1, 1 = M \varepsilon * 1 = M n$.

(2) (a) $n \neq \pm 1 \rightarrow n = p_1^{\alpha(1)} \dots p_s^{\alpha(s)}$ hence $\eta(n) = \max_{i=1,2} \eta_{p(i)}$

Let $\max_{i=1,s} \{ \eta_{p(i)}(\alpha_i) \} = \eta_{p(i_0)}(\alpha_{i_0}), 1 \leq i \leq s$;

$\eta_{p(i_0)}(\alpha_{i_0})$ is the smallest natural number with the property:

$$(\eta_{p_{i_0}}(\alpha_{i_0}))! = M p_{i_0}^{\alpha_{i_0}} \rightarrow \alpha \gamma \in \mathbb{N}, \gamma < \eta_{p_{i_0}}(\alpha_{i_0}) \text{ when } c_w$$

$$\gamma! \neq M p_{i_0}^{\alpha_{i_0}} \text{ then } \gamma! \neq M \varepsilon * p_1 \dots p_i \dots p_s = M n \text{ whence}$$

$\eta_{p_{i_0}}(\alpha_{i_0})$ is the smallest natural number with the property.

(b) $n = \pm 1 \rightarrow \eta(n) = 0$ and it is the smallest natural number $\rightarrow 0$ is the smallest natural number with the property $0! = M(\pm 1)$.

NOTE 3. The functions η_p are increasing, not injective, on $\mathbb{N}^* \rightarrow \{p^k \mid k = 1, 2, 3, \dots\}$ they are surjective.

The function η is increasing, it is not injective, it is surjective on $\mathbb{Z} \setminus \{0\} \rightarrow \mathbb{N} \setminus \{1\}$.

CONSEQUENCE. Let $n \in \mathbb{N}^*, n > 4$. Then $n = \text{prime}$ involves $\eta(n) = n$.

Proof

“ \rightarrow ”

$n = \text{prime}$ and $n \geq 5$ then $\eta(n) = \eta_n(1) = n$.

“ \leftarrow ”

Let $\eta(n) = n$ and assume by reduction ad absurdum that $n \neq \text{prime}$. Then

$$(a) \ n = p_1^{\alpha(1)} \dots p_s^{\alpha(s)} \text{ with } s \geq 2, \alpha_i \in \mathbb{N}^*, i = \overline{1, s},$$

$$\eta(n) = \max_{i=1, s} \{ \eta_{p(i)}(\alpha_i) \} = \eta_{p(i)}(\alpha_i) < \alpha_i p_i < n$$

contradicting the assumption.

$$(b) \ n = p_1^{\alpha(1)} \text{ with } \alpha_1 \geq 2 \text{ involves } \eta(n) = \eta_{p(1)}(\alpha_1) \leq p_1 * \alpha_1 < p_1^{\alpha(1)} = n$$

because $\alpha_1 \geq 2$ and $n > 4$, which contradicts the hypothesis.

Application

1. Find the smallest natural number with the property:

$$n! = M(\pm 2^{31} * 3^{27} * 7^{13}).$$

Solution

$$\eta(\pm 2^{31} * 3^{27} * 7^{13}) = \max \{ \eta_2(31), \eta_3(27), \eta_7(13) \}.$$

Let us calculate $\eta_2(31)$; we make the string

$$(a_n^{(2)})_{n \in \mathbb{N}^*} = 1, 3, 7, 15, 31, 63, \dots$$

$$31 = 1 * 31 \rightarrow \eta_2(1 * 31) = 1 * 2^5 = 32.$$

Let's calculate $\eta_3(27)$ by making the string

$$(a_n^{(3)})_{n \in \mathbb{N}^*} = 1, 4, 13, 40, \dots; 27 = 2*13 + 1 \text{ involves } \eta_3(27) = \eta_3(2*13+1*1) =$$

$$2* \eta_3(13) + 1* \eta_3(1) = 2*3^3 + 1*3^1 = 54 + 3 = 57.$$

Let's calculate $\eta_7(13)$; making the string

$$(a_n^{(7)})_{n \in \mathbb{N}^*} = 1, 8, 57, \dots; 13 = 1*8 + 5*1 \rightarrow \eta_7(13) = 1* \eta_7(8) + 5* \eta_7(1) =$$

$$1*7^2 + 5*7^1 = 49 + 35 = 84 \rightarrow \eta(\pm 2^{31} * 3^{27} * 7^{13}) = \max \{ 32, 57, 84 \} = 84 \text{ involves } 84! =$$

$M(\pm 2^{31} * 3^{27} * 7^{13})$ and 84 is the smallest number with this property.

2. What are the numbers n where $n!$ ends with 1000 zeros?

Solution:

$n = 10^{1000}$, $(\eta(n))! = M 10^{1000}$ and it is the smallest number with this property.

$$\eta(10^{1000}) = \eta(2^{1000} * 5^{1000}) = \max \{ \eta_2(1000), \eta_5(1000) \} = \eta_5(1000) =$$

$$\eta_5(1*781 + 1*156 + 2*31 + 1) = 1*5^5 + 1*5^4 + 2*5^3 + 1*5^2 = 4005, 4005 \text{ is the smallest}$$

number with this property. 4006, 4007, 4008, 4009 also satisfy this property, but 4010 does not because $4010! = 4009! * 4010$ which has 1001 zeros.

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