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UNIFORM AND PARTIALLY UNIFORM REDISTRIBUTION RULES

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This paper introduces two new fusion rules for combining quantitative basic belief assignments. These rules although very simple have not been proposed in literature so far and could serve as useful alternatives because of their low computation cost with respect to the recent advanced Proportional Conflict Redistribution rules developed in the DSmT framework.

Keywords: Uniform redistribution rule; partially uniform redistribution rule; belief functions; Dezert-Smarandache Theory (DSmT); information fusion.

1. Introduction

Since the development of DSmT (Dezert-Smarandache Theory[1,2]) in 2002, a new look for information fusion in the framework of belief has been proposed which covers many aspects related to the fusion of uncertain and conflicting beliefs. Mainly, the fusion of quantitative or qualitative belief functions of highly uncertain and conflicting sources of evidence with theoretical advances in belief conditioning rules. Shafer's milestone book[3] introducing the concept of belief functions and Dempster's rule of combination of beliefs has been the important step towards non probabilistic reasoning approaches, aside Zadeh's fuzzy logic.[4,5] Since Shafer's seminal work, many alternatives have been proposed to circumvent limitations of Dempster's rule pointed out first by Zadeh[6] (see also Sentz & Ferson's paper[7] and authors book[2] Vol. 2 for a review). The Proportional Conflict Redistribution rule number 5 (PCR5[2]) is one of the most efficient alternative to Dempster's rule which can be used both in Dempster-Shafer Theory (DST) as well as in DSmT. The simple idea behind PCR5 is to redistribute every partial conflict only onto propositions which are truly involved in the partial conflict and proportionally to the corresponding belief mass assignment of each source generating this conflict. Although very efficient and appealing, the PCR5 rule suffers of its relative complexity in implementation and in some cases, it is required to use simpler (but less precise) rule of combination which requires only a low complexity. For this purpose, we herein present two new cheap alternatives for combination of basic belief assignments (bba's): the Uniform Redistribution Rule (URR) and the Partially Uniform Redistribution Rule (PURR). In the sequel, we assume the reader familiar with the basics of DSmT, mainly with the definition and notation of hyper-power set D^Θ and also bba's defined over hyper-power set. Basics of DSmT can be found in chapters 1 of authors books[1,8] which are freely downloadable on internet. Therefore we just recall very briefly in this section the main ideas and specificities of DSmT. Many detailed examples can be easily found in the three volumes devoted to DSmT[1,2,8] and so we do not need to include them in this paper. Preliminary ideas on URR have shortly appeared in an International Workshop,[9] and summarized in the chapter 1 of our third book,[8] and an approach sharing similar idea was also introduced by Lefevre, Colot, Vannoorenberghe and De Brucq.[10]

1.1. *Basics of DSmT*

The basis of DSmT is the refutation of the principle of the third excluded middle and Shafer's model, since for a wide class of fusion problems the intrinsic nature of hypotheses can be only vague and imprecise in such a way that precise refinement is just impossible to obtain in reality so that the exclusive elements θ_i cannot be properly identified and precisely separated. Many problems involving fuzzy continuous and relative concepts described in natural language and having no absolute interpretation like tallness/smallness, pleasure/pain, cold/hot, Sorites paradoxes, etc, enter in this category. DSmT starts with the notion of *free DSm model*, denoted $\mathcal{M}^f(\Theta)$, and considers Θ only as a frame of exhaustive elements θ_i, $i = 1, \ldots, n$ which can potentially overlap. This model is *free* because no other assumption is done on the hypotheses, but the weak exhaustivity constraint which can always be satisfied according the closure principle explained in authors book,[1] Vol. 1. No other constraint is involved in the free DSm model. When the free DSm model holds, the commutative and associative classical DSm rule of combination, denoted DSmC, corresponding to the conjunctive consensus defined on the free Dedekind's lattice is performed.

Depending on the intrinsic nature of the elements of the fusion problem under consideration, it can however happen that the free model does not fit the reality because some subsets of Θ can contain elements known to be truly exclusive but also truly non existing at all at a given time (specially when working on dynamic fusion problem where the frame Θ varies with time with the revision of the knowledge available). These integrity constraints are then explicitly and formally introduced into the free DSm model $\mathcal{M}^f(\Theta)$ in order to adapt it properly to fit as close as possible with the reality and permit to construct a *hybrid DSm model* $\mathcal{M}(\Theta)$ on which the combination will be efficiently performed. Shafer's model, denoted $\mathcal{M}^0(\Theta)$, corresponds to a very specific hybrid DSm model including all possible exclusivity constraints. DST has been developed for working only with $\mathcal{M}^0(\Theta)$ while DSmT has been developed for working with any kind of hybrid model (including Shafer's model and the free DSm model), to manage as efficiently and precisely as possible imprecise, uncertain and potentially highly conflicting sources of evidence while keeping in mind the possible dynamicity of the information fusion problematic. The foundations of DSmT are therefore totally different from those of all existing approaches managing uncertainties, imprecisions and conflicts. DSmT provides a new interesting way to attack the information fusion problematic with a general framework in order to cover a wide variety of problems.

DSmT refutes also the idea that sources of evidence provide their beliefs with the same absolute interpretation of elements of the same frame Θ and the conflict between sources arises not only because of the possible unreliability of sources, but also because of possible different and relative interpretation of Θ, e.g. what is considered as good for somebody can be considered as bad for somebody else. There is some unavoidable subjectivity in the belief assignments provided by the sources of evidence, otherwise it would mean that all bodies of evidence have a same objective and universal interpretation (or measure) of the phenomena under consideration, which unfortunately rarely occurs in reality, but when basic belief assignments (bba's) are based on some *objective probabilities* transformations. But in this last case, probability theory can handle properly and efficiently the information, and DST, as well as DSmT, becomes useless. If we now get out of the probabilistic background argumentation for the construction of bba, we claim that in most of cases, the sources of evidence provide their beliefs about elements of the frame of the fusion problem only based on their own limited knowledge and experience without reference to the (inaccessible) absolute truth of the space of possibilities.

1.2. *The power set, hyper-power set and super-power set*

In DSmT, we take very care about the model associated with the set Θ of hypotheses where the solution of the problem is assumed to belong to. In particular, the three main sets (power set, hyper-power set and super-power set) can be used depending on their ability to fit adequately with the nature of hypotheses. In the following, we assume that $\Theta = \{\theta_1, \ldots, \theta_n\}$ is a finite set (called frame) of n exhaustive elements[a]. If $\Theta = \{\theta_1, \ldots, \theta_n\}$ is a priori not closed (Θ is said to be an open world/frame), one can always include in it a closure element, say θ_{n+1} in such away that we can work with a new closed world/frame $\{\theta_1, \ldots, \theta_n, \theta_{n+1}\}$. So without loss of generality, we will always assume that we work in a closed world by considering the frame Θ as

[a]We do not assume here that elements θ_i are necessary exclusive, unless specified. There is no restriction on θ_i but the exhaustivity.

a finite set of exhaustive elements. Before introducing the power set, the hyper-power set and the super-power set it is necessary to recall that subsets are regarded as propositions in Dempster-Shafer Theory (see Chapter 2 of milestone Shafer's book[3]) and we adopt the same approach in DSmT.

- **Subsets as propositions**: Glenn Shafer in pages 35–37 of Shafer's book[3] considers the subsets as propositions in the case we are concerned with the true value of some quantity θ taking its possible values in Θ. Then the propositions $\mathcal{P}_\theta(A)$ of interest are those of the form[b]:

$$\mathcal{P}_\theta(A) \triangleq \textit{The true value of } \theta \textit{ is in a subset } A \textit{ of} \Theta$$

Any proposition $\mathcal{P}_\theta(A)$ is thus in one-to-one correspondence with the subset A of Θ. Such correspondence is very useful since it translates the logical notions of conjunction $\wedge$, disjunction $\vee$, implication $\Rightarrow$ and negation $\neg$ into the set-theoretic notions of intersection $\cap$, union $\cup$, inclusion $\subset$ and complementation $c(.)$. Indeed, if $\mathcal{P}_\theta(A)$ and $\mathcal{P}_\theta(B)$ are two propositions corresponding to subsets A and B of Θ, then the conjunction $\mathcal{P}_\theta(A) \wedge \mathcal{P}_\theta(B)$ corresponds to the intersection $A \cap B$ and the disjunction $\mathcal{P}_\theta(A) \vee \mathcal{P}_\theta(B)$ corresponds to the union $A \cup B$. A is a subset of B if and only if $\mathcal{P}_\theta(A) \Rightarrow \mathcal{P}_\theta(B)$ and A is the set-theoretic complement of B with respect to Θ (written $A = c_\Theta(B)$) if and only if $\mathcal{P}_\theta(A) = \neg\mathcal{P}_\theta(B)$. In other words, the following equivalences are then used between the operations on the subsets and on the propositions:

Table 1. Correspondence between operations on subsets and on propositions.

Operations	Subsets	Propositions
Intersection/conjunction	$A \cap B$	$\mathcal{P}_\theta(A) \wedge \mathcal{P}_\theta(B)$
Union/disjunction	$A \cup B$	$\mathcal{P}_\theta(A) \vee \mathcal{P}_\theta(B)$
Inclusion/implication	$A \subset B$	$\mathcal{P}_\theta(A) \Rightarrow \mathcal{P}_\theta(B)$
Complementation/negation	$A = c_\Theta(B)$	$\mathcal{P}_\theta(A) = \neg\mathcal{P}_\theta(B)$

- **Canonical form of a proposition**: In DSmT we consider all propositions/sets in a canonical form. We take the disjunctive normal form, which is a disjunction of conjunctions, and it is unique in Boolean algebra and simplest. For example, $X = A \cap B \cap (A \cup B \cup C)$ it is not in a canonical form, but we simplify the formula and $X = A \cap B$ is in a canonical form.

- **The power set**: $2^\Theta \triangleq (\Theta, \cup)$

Aside Dempster's rule of combination, the power set is one of the corner stones of Dempster-Shafer Theory (DST) since the basic belief assignments to combine

[b]We use the symbol $\triangleq$ to mean *equals by definition*; the right-hand side of the equation is the definition of the left-hand side.

are defined on the power set of the frame Θ. In mathematics, given a set Θ, the power set of Θ, written 2^Θ, is the set of all subsets of Θ. In ZermeloFraenkel set theory with the axiom of choice (ZFC), the existence of the power set of any set is postulated by the axiom of power set. In other words, Θ generates the power set 2^Θ with the $\cup$ (union) operator only.

More precisely, the power set 2^Θ is defined as the set of all composite propositions/subsets built from elements of Θ with $\cup$ operator such that:

(1) $\emptyset, \theta_1, \ldots, \theta_n \in 2^\Theta$.
(2) If $A, B \in 2^\Theta$, then $A \cup B \in 2^\Theta$.
(3) No other elements belong to 2^Θ, except those obtained by using rules 1 and 2.

- **The hyper-power set**: $D^\Theta \triangleq (\Theta, \cup, \cap)$

One of the cornerstones of DSmT is the free Dedekind's lattice[11] denoted as *hyper-power set* in DSmT framework. Let $\Theta = \{\theta_1, \ldots, \theta_n\}$ be a finite set (called frame) of n exhaustive elements. The hyper-power set D^Θ is defined as the set of all composite propositions/subsets built from elements of Θ with $\cup$ and $\cap$ operators such that:

(1) $\emptyset, \theta_1, \ldots, \theta_n \in D^\Theta$.
(2) If $A, B \in D^\Theta$, then $A \cap B \in D^\Theta$ and $A \cup B \in D^\Theta$.
(3) No other elements belong to D^Θ, except those obtained by using rules 1 or 2.

Therefore by convention, we write $D^\Theta = (\Theta, \cup, \cap)$ which means that Θ generates D^Θ under operators $\cup$ and $\cap$. The dual (obtained by switching $\cup$ and $\cap$ in expressions) of D^Θ is itself. There are elements in D^Θ which are self-dual (dual to themselves), for example α_8 for the case when $n = 3$ in the following example. The cardinality of D^Θ is majored by 2^{2^n} when the cardinality of Θ equals n, i.e. $|\Theta| = n$. The generation of hyper-power set D^Θ is closely related with the famous Dedekind's problem[11,12] on enumerating the set of isotone Boolean functions. The generation of the hyper-power set is presented in authors book,[1] Vol. 1. Since for any given finite set Θ, $|D^\Theta| \geq |2^\Theta|$ we call D^Θ the *hyper-power set* of Θ.

The cardinality of hyper-power set D^Θ for $n \geq 1$ follows the sequence of Dedekind's numbers,[13] i.e. 1,2,5,19,167, 7580,7828353,... and analytical expression of Dedekind's numbers has been obtained by Tombak[14] (see authors book[1] (Vol. 1) for details on generation and ordering of D^Θ). Interesting investigations on the programming of the generation of hyper-power sets for engineering applications have been done in Chapter 15 of authors book[2] (Vol.2) and also in Vol. 3.[8]

Shafer's model of a frame: More generally, when all the elements of a given frame Θ are known (or are assumed to be) truly exclusive, then the hyper-power set D^Θ reduces to the classical power set 2^Θ. Therefore, working on power set 2^Θ as Glenn Shafer has proposed in his Mathematical Theory of Evidence[3]) is equivalent to work on hyper-power set D^Θ with the assumption that all elements of the frame are exclusive. This is what we call *Shafer's model of the frame* Θ, written $\mathcal{M}^0(\Theta)$, even if such model/assumption has not been clearly stated explicitly by Shafer himself in his milestone book.

- **The super-power set**: $S^\Theta \triangleq (\Theta, \cup, \cap, c(.))$

The notion of super-power set has been introduced by Smarandache in the Chapter 8 of authors book.[2] It corresponds actually to the theoretical construction of the power set of the minimal[c] refined frame Θ^{ref} of Θ. Θ generates S^Θ under operators $\cup$, $\cap$ and complementation $c(.)$. $S^\Theta = (\Theta, \cup, \cap, c(.))$ is a Boolean algebra with respect to the union, intersection and complementation. Therefore working with the super-power set is equivalent to work with a minimal theoretical refined frame Θ^{ref} satisfying Shafer's model. More precisely, S^Θ is defined as the set of all composite propositions/subsets built from elements of Θ with $\cup$, $\cap$ and $c(.)$ operators such that:

(1) $\emptyset, \theta_1, \ldots, \theta_n \in S^\Theta$.
(2) If $A, B \in S^\Theta$, then $A \cap B \in S^\Theta$, $A \cup B \in S^\Theta$.
(3) If $A \in S^\Theta$, then $c(A) \in S^\Theta$.
(4) No other elements belong to S^Θ, except those obtained by using rules 1, 2 and 3.

As already reported in a previous authors' paper,[15] a similar generalization has been previously used in 1993 by Guan and Bell[16] for the Dempster-Shafer rule using propositions in sequential logic and reintroduced in 1994 by Paris in his book,[17] page 4.

A one-to-one correspondence between the elements of S^Θ and $2^{\Theta^{ref}}$ can be defined for any cardinality $|\Theta| \geq 2$ of the frame Θ and thus one can consider S^Θ as the mathematical construction of the power set $2^{\Theta^{ref}}$ of the minimal refinement of the frame Θ. Of course, when Θ already satisfies Shafer's model, the hyper-power set and the super-power set coincide with the classical power set of Θ. It is worth to note that even if we have a mathematical tool to built the minimal refined frame satisfying Shafer's model, it doesn't mean necessary that one must work with this super-power set in general in real applications because most of the times the elements/granules of S^Θ have no clear physical meaning, not to mention the drastic increase of the complexity (see Table 2) since one has $2^\Theta \subseteq D^\Theta \subseteq S^\Theta$ and

$$|2^\Theta| = 2^{|\Theta|} < |D^\Theta| < |S^\Theta| = 2^{|\Theta^{ref}|} = 2^{2^{|\Theta|}-1} \qquad (1)$$

In summary, DSmT offers truly the possibility to build and to work on refined frames and to deal with the complement whenever necessary, but in most of applications either the frame Θ is already built/chosen to satisfy Shafer's model or

[c]The minimality refers here to the cardinality of the refined frames.

Table 2. Cardinalities of 2^Θ, D^Θ and S^Θ.

$\lvert\Theta\rvert = n$	$\lvert 2^\Theta\rvert = 2^n$	$\lvert D^\Theta\rvert$	$\lvert S^\Theta\rvert = \lvert 2^{\Theta_{ref}}\rvert = 2^{2^n-1}$
2	4	5	$2^3 = 8$
3	8	19	$2^7 = 128$
4	16	167	$2^{15} = 32768$
5	32	7580	$2^{31} = 2147483648$

the refined granules have no clear physical meaning which finally prevent to be considered/assessed individually so that working on the hyper-power set is usually sufficient for dealing with uncertain imprecise (quantitative or qualitative) and highly conflicting sources of evidences. Working with S^Θ is actually very similar to working with 2^Θ in the sense that in both cases we work with classical power sets; the only difference is that when working with S^Θ we have implicitly switched from the original frame Θ representation to a minimal refinement Θ^{ref} representation. Therefore, working with hyper-power set rather than (super-) power set which has already been the basis for the development of DST is a true specificity and novelty of DSmT. But as already mentioned, DSmT can easily deal with belief functions defined on 2^Θ or S^Θ similarly as those defined on D^Θ if the user prefers for his/her own reasons.

Generic notation: In the sequel, we use the generic notation G^Θ for denoting the sets (power set, hyper-power set and super-power set) on which the belief functions are defined.

1.3. *Notion of free and hybrid DSm models*

Free DSm model: The elements θ_i, $i = 1, \ldots, n$ of Θ constitute the finite set of hypotheses/concepts characterizing the fusion problem under consideration. When there is no constraint on the elements of the frame, we call this model the *free DSm model*, written $\mathcal{M}^f(\Theta)$. This free DSm model allows to deal directly with fuzzy concepts which depict a continuous and relative intrinsic nature and which cannot be precisely refined into finer disjoint information granules having an absolute interpretation because of the unreachable universal truth. In such case, the use of the hyper-power set D^Θ (without integrity constraints) is particularly well adapted for defining the belief functions one wants to combine.

Shafer's model: In some fusion problems involving discrete concepts, all the elements θ_i, $i = 1, \ldots, n$ of Θ can be truly exclusive. In such case, all the exclusivity constraints on θ_i, $i = 1, \ldots, n$ have to be included in the previous model to characterize properly the true nature of the fusion problem and to fit it with the reality. By doing this, the hyper-power set D^Θ as well as the super-power set S^Θ reduce naturally to the classical power set 2^Θ and this constitutes what we have called

Shafer's model, denoted $\mathcal{M}^0(\Theta)$. Shafer's model corresponds actually to the most restricted hybrid DSm model.

Hybrid DSm models: Between the class of fusion problems corresponding to the free DSm model $\mathcal{M}^f(\Theta)$ and the class of fusion problems corresponding to Shafer's model $\mathcal{M}^0(\Theta)$, there exists another wide class of hybrid fusion problems involving in Θ both fuzzy continuous concepts and discrete hypotheses. In such (hybrid) class, some exclusivity constraints and possibly some non-existential constraints (especially when working on dynamic[d] fusion) have to be taken into account. Each hybrid fusion problem of this class will then be characterized by a proper hybrid DSm model denoted $\mathcal{M}(\Theta)$ with $\mathcal{M}(\Theta) \neq \mathcal{M}^f(\Theta)$ and $\mathcal{M}(\Theta) \neq \mathcal{M}^0(\Theta)$.

In any fusion problems, we consider as primordial at the very beginning and before combining information expressed as belief functions to define clearly the proper frame Θ of the given problem and to choose explicitly its corresponding model one wants to work with. Once this is done, the second important point is to select the proper set 2^Θ, D^Θ or S^Θ on which the belief functions will be defined. The third point concerns the choice of an efficient rule of combination of belief functions and finally the criteria adopted for decision-making.

In the sequel, we focus our presentation mainly on hyper-power set D^Θ (unless specified) since it is the most interesting new aspect of DSmT for readers already familiar with DST framework, but a fortiori we can work similarly on classical power set 2^Θ if Shafer's model holds, and even on $2^{\Theta^{ref}}$ (the power set of the minimal refined frame) whenever one wants to use it and if possible.

1.4. *Generalized belief functions*

From a general frame Θ, we define a map $m(.) : G^\Theta \to [0,1]$ associated to a given body of evidence $\mathcal{B}$ as $m(\emptyset) = 0$ and

$$\sum_{A \in G^\Theta} m(A) = 1 \tag{2}$$

$m(A)$ is called the *generalized* basic belief assignment/mass (bba) of A.

The *generalized* belief and plausibility functions are defined in almost the same manner as within DST, i.e.

$$Bel(A) = \sum_{B \subseteq A, B \in G^\Theta} m(B) \tag{3}$$

$$Pl(A) = \sum_{B \cap A \neq \emptyset, B \in G^\Theta} m(B) \tag{4}$$

[d]i.e. when the frame Θ and/or the model $\mathcal{M}$ is changing with time.

We recall that G^Θ is the generic notation for the set on which the gbba is defined (G^Θ can be 2^Θ, D^Θ or even S^Θ depending on the model chosen for Θ). These definitions are compatible with the definitions of the classical belief functions in DST framework when $G^\Theta = 2^\Theta$ for fusion problems where Shafer's model $\mathcal{M}^0(\Theta)$ holds. We still have $\forall A \in G^\Theta$, $Bel(A) \leq Pl(A)$. Note that when working with the free DSm model $\mathcal{M}^f(\Theta)$, one has always $Pl(A) = 1\ \forall A \neq \emptyset \in (G^\Theta = D^\Theta)$ which is normal.

1.5. *Fusion rules of combination of DSmT*

1.5.1. *The classic DSm rule of combination*

When the free DSm model $\mathcal{M}^f(\Theta)$ holds for the fusion problem under consideration, the classic DSm rule of combination $m_{\mathcal{M}^f(\Theta)} \equiv m(.) \triangleq [m_1 \oplus m_2](.)$ of two independent[e] sources of evidences $\mathcal{B}_1$ and $\mathcal{B}_2$ over the same frame Θ with belief functions $Bel_1(.)$ and $Bel_2(.)$ associated with bba's $m_1(.)$ and $m_2(.)$ corresponds to the conjunctive consensus of the sources. It is given by authors book,[1] Vol. 1:

$$\forall C \in D^\Theta, \qquad m_{\mathcal{M}^f(\Theta)}(C) \equiv m(C) = \sum_{A,B \in D^\Theta,\ A \cap B = C} m_1(A) m_2(B) \tag{5}$$

Since D^Θ is closed under $\cup$ and $\cap$ set operators, this new rule of combination guarantees that $m(.)$ is a proper generalized belief assignment, i.e. $m(.) : D^\Theta \to [0,1]$. This rule of combination is commutative and associative and can always be used for the fusion of sources involving fuzzy concepts when free DSm model holds for the problem under consideration. This rule has been extended for $s > 2$ sources in authors book,[1] Vol. 1.

According to Table 2, this classic DSm rule of combination looks very expensive in terms of computations and memory size due to the huge number of elements in D^Θ when the cardinality of Θ increases. This remark is however valid only if the cores (the set of focal elements of gbba) $\mathcal{K}_1(m_1)$ and $\mathcal{K}_2(m_2)$ coincide with D^Θ, i.e. when $m_1(A) > 0$ and $m_2(A) > 0$ for all $A \neq \emptyset \in D^\Theta$. Fortunately, it is important to note here that in most of the practical applications the sizes of $\mathcal{K}_1(m_1)$ and $\mathcal{K}_2(m_2)$ are much smaller than $|D^\Theta|$ because bodies of evidence generally allocate their basic belief assignments only over a subset of the hyper-power set. This makes things easier for the implementation of the classic DSm rule Eq. (5). The DSm rule is actually very easy to implement. It suffices for each focal element of $\mathcal{K}_1(m_1)$ to multiply it with the focal elements of $\mathcal{K}_2(m_2)$ and then to pool all combinations which are equivalent under the algebra of sets. While very costly in term on memory storage in the worst case (i.e. when all $m(A) > 0$, $A \in D^\Theta$ or $A \in 2^{\Theta^{ref}}$), the DSm

[e]While independence is a difficult concept to define in all theories managing epistemic uncertainty, we follow here the interpretation of Smets,[18,19] p. 285 and consider that two sources of evidence are independent (i.e distinct and noninteracting) if each leaves one totally ignorant about the particular value the other will take.

rule however requires much smaller memory storage than when working with S^Θ, i.e. working with a minimal refined frame satisfying Shafer's model.

In most fusion applications only a small subset of elements of D^Θ have a non null basic belief mass because all the commitments are just usually impossible to obtain precisely when the dimension of the problem increases. Thus, it is not necessary to generate and keep in memory all elements of D^Θ (or eventually S^Θ) but only those which have a positive belief mass. However there is a real technical challenge on how to manage efficiently all elements of the hyper-power set. This problem is obviously much more difficult when trying to work on a refined frame of discernment Θ^{ref} if one really prefers to use Dempster-Shafer theory and apply Dempster's rule of combination. It is important to keep in mind that the ultimate and minimal refined frame consisting in exhaustive and exclusive finite set of refined exclusive hypotheses is just impossible to justify and to define precisely for all problems dealing with fuzzy and ill-defined continuous concepts. A discussion on refinement with an example has be included in authors book,[1] Vol. 1.

1.5.2. *The hybrid DSm rule of combination*

When the free DSm model $\mathcal{M}^f(\Theta)$ does not hold due to the true nature of the fusion problem under consideration which requires to take into account some known integrity constraints, one has to work with a proper hybrid DSm model $\mathcal{M}(\Theta) \neq \mathcal{M}^f(\Theta)$. In such case, the hybrid DSm rule (DSmH) of combination based on the chosen hybrid DSm model $\mathcal{M}(\Theta)$ for $k \geq 2$ independent sources of information is defined for all $A \in D^\Theta$ as:[1]

$$m_{DSmH}(A) = m_{\mathcal{M}(\Theta)}(A) \triangleq \phi(A)\Big[S_1(A) + S_2(A) + S_3(A)\Big] \tag{6}$$

where all sets involved in formulas are in the canonical form and $\phi(A)$ is the *characteristic non-emptiness function* of a set A, i.e. $\phi(A) = 1$ if $A \notin \emptyset_{ext}$ and $\phi(A) = 0$ otherwise, where $\emptyset_{ext} \triangleq \{\emptyset_\mathcal{M}, \emptyset\}$ is the *extended* empty set. $\emptyset_\mathcal{M}$ is the set of all elements of D^Θ which have been forced to be empty through the constraints of the model $\mathcal{M}$ and $\emptyset$ is the classical/universal empty set. $S_1(A) \equiv m_{\mathcal{M}^f(\theta)}(A)$, $S_2(A)$, $S_3(A)$ are defined by

$$S_1(A) \triangleq \sum_{\substack{X_1,X_2,\ldots,X_k \in D^\Theta \\ X_1 \cap X_2 \cap \ldots \cap X_k = A}} \prod_{i=1}^{k} m_i(X_i) \tag{7}$$

$$S_2(A) \triangleq \sum_{\substack{X_1,X_2,\ldots,X_k \in \emptyset_{ext} \\ [\mathcal{U}=A] \vee [(\mathcal{U} \in \emptyset_{ext}) \wedge (A = I_t)]}} \prod_{i=1}^{k} m_i(X_i) \tag{8}$$

$$S_3(A) \triangleq \sum_{\substack{X_1,X_2,\ldots,X_k \in D^\Theta \\ X_1 \cup X_2 \cup \ldots \cup X_k = A \\ X_1 \cap X_2 \cap \ldots \cap X_k \in \emptyset_{ext}}} \prod_{i=1}^{k} m_i(X_i) \tag{9}$$

with $\mathcal{U} \triangleq u(X_1) \cup u(X_2) \cup \ldots \cup u(X_k)$ where $u(X)$ is the union of all θ_i that compose X, $I_t \triangleq \theta_1 \cup \theta_2 \cup \ldots \cup \theta_n$ is the total ignorance.

$S_1(A)$ corresponds to the classic DSm rule for k independent sources based on the free DSm model $\mathcal{M}^f(\Theta)$; $S_2(A)$ represents the mass of all relatively and absolutely empty sets which is transferred to the total or relative ignorances associated with non existential constraints (if any, like in some dynamic problems); $S_3(A)$ transfers the sum of relatively empty sets directly onto the canonical disjunctive form of non-empty sets.

The hybrid DSm rule of combination generalizes the classic DSm rule of combination and is not equivalent to Dempter's rule. It works for any models (the free DSm model, Shafer's model or any other hybrid models) when manipulating *precise* generalized (or eventually classical) basic belief functions. Aside these basic specifities, DSmT offers also new approaches for dealing with imprecise bba's, for combining qualitative belief assignments, for belief conditioning and for approximating a bba to a subjective probability measure as well. This is however out of the scope of this paper and therefore this will no be presented here. We suggest readers interested more in DSmT to download and read authors books Vols. 1–3.[1,2,8]

1.5.3. *Proportional conflict redistribution rule*

Instead of applying a direct transfer of partial conflicts onto partial uncertainties as with DSmH, the idea behind the Proportional Conflict Redistribution (PCR) rule[2,20] is to transfer (total or partial) conflicting masses to non-empty sets involved in the conflicts proportionally with respect to the masses assigned to them by sources as follows:

(1) calculation the conjunctive rule of the belief masses of sources;
(2) calculation the total or partial conflicting masses;
(3) redistribution of the (total or partial) conflicting masses to the non-empty sets involved in the conflicts proportionally with respect to their masses assigned by the sources.

The way the conflicting mass is redistributed yields actually several versions of PCR rules. These PCR fusion rules work for any degree of conflict, for any DSm models (Shafer's model, free DSm model or any hybrid DSm model) and both in DST and DSmT frameworks for static or dynamical fusion situations. We present below only the most sophisticated proportional conflict redistribution rule[2,20] denoted PCR5. PCR5 rule is what we feel the most efficient PCR fusion rule developed so far. This rule redistributes the partial conflicting mass to the elements involved in the partial conflict, considering the conjunctive normal form of the partial conflict. PCR5 is what we think the most mathematically exact redistribution of conflicting mass to non-empty sets following the logic of the conjunctive rule. It does a better redistribution of the conflicting mass than Dempster's rule since PCR5 goes backwards on the tracks of the conjunctive rule and redistributes the conflicting mass only to the sets involved in the conflict and proportionally to their masses put in the conflict. PCR5 rule is quasi-associative and preserves the neutral impact of the vacuous belief assignment because in any partial conflict, as well in the total conflict (which is a sum of all partial conflicts), the conjunctive normal form of each partial conflict does not include Θ since Θ is a neutral element for intersection (conflict), therefore Θ gets no mass after the redistribution of the conflicting mass. We have proved in our book[2] the continuity property of the fusion result with continuous variations of bba's to combine.

The PCR5 formula for the combination of two sources ($s = 2$) is given by: $m_{PCR5}(\emptyset) = 0$ and $\forall X \in G^{\Theta} \setminus \{\emptyset\}$

$$m_{PCR5}(X) = m_{12}(X) + \sum_{\substack{Y \in G^{\Theta} \setminus \{X\} \\ X \cap Y = \emptyset}} \left[\frac{m_1(X)^2 m_2(Y)}{m_1(X) + m_2(Y)} + \frac{m_2(X)^2 m_1(Y)}{m_2(X) + m_1(Y)} \right] \quad (10)$$

where all sets involved in formulas are in canonical form and where G^{Θ} corresponds to classical power set 2^{Θ} if Shafer's model is used, or to a constrained hyper-power set D^{Θ} if any other hybrid DSm model is used instead, or to the super-power set S^{Θ} if the minimal refinement Θ^{ref} of Θ is used; $m_{12}(X) \equiv m_{\cap}(X)$ corresponds to the conjunctive consensus on X between the $s = 2$ sources and where all denominators are different from zero. If a denominator is zero, that fraction is discarded. A general formula of PCR5 for the fusion of $s > 2$ sources has been proposed in authors book,[2] and a more intuitive PCR formula (denoted PCR6) which provides good results in practice has been proposed by Martin and Osswald[2] (pages 69-88). For two sources ($s = 2$), PCR5 and PCR6 formulas coincide. From implementation point of view, PCR6 it a bit more easier to code than PCR5.

2. Uniform Redistribution Rule

Let's consider a finite and discrete frame of discernment Θ, its hyper-power set G^{Θ} (i.e. Dedekind's lattice) and two quantitative basic belief assignments $m_1(.)$ and $m_2(.)$ defined on G^{Θ} expressed by two independent sources of evidence.

The Uniform Redistribution Rule (URR) consists in redistributing the total conflicting mass k_{12} to all focal elements of G^{Θ} generated by the consensus operator. This way of redistributing mass is very simple and URR is different from Dempster's rule of combination,[3] because Dempster's rule redistributes the total conflict proportionally with respect to the masses resulted from the conjunctive rule of non-empty sets. PCR5 and PCR6[2] do proportional redistributions of partial conflicting masses to the sets involved in the conflict. Here it is the URR formula for two sources: $\forall A \neq \emptyset$, one has

$$m_{12URR}(A) = m_{12}(A) + \frac{1}{n_{12}} \sum_{\substack{X_1, X_2 \in G^\Theta \\ X_1 \cap X_2 = \emptyset}} m_1(X_1) m_2(X_2) \tag{11}$$

where $m_{12}(A)$ is the result of the conjunctive rule applied to belief assignments $m_1(.)$ and $m_2(.)$, and $n_{12} = Card\{Z \in G^\Theta, m_1(Z) \neq 0 \text{ } or \text{ } m_2(Z) \neq 0\}$.

For $s \geq 2$ sources to combine: $\forall A \neq \emptyset$, one has

$$m_{12\ldots sURR}(A) = m_{12\ldots s}(A) + \frac{1}{n_{12\ldots s}} \sum_{\substack{X_1, X_2, \ldots, X_s \in G^\Theta \\ X_1 \cap X_2 \cap \ldots \cap X_s = \emptyset}} \prod_{i=1}^{s} m_i(X_i) \tag{12}$$

where $m_{12\ldots s}(A)$ is the result of the conjunctive rule applied to $m_i(.)$, for all $i \in \{1, 2, \ldots, s\}$ and

$$n_{12\ldots s} = Card\{Z \in G^\Theta, m_1(Z) \neq 0 \text{ } or \text{ } m_2(Z) \neq 0 \text{ } or \text{ } \ldots \text{ } or \text{ } m_s(Z) \neq 0\}$$

As alternative, we can also consider the cardinal of the ensemble of sets whose masses resulted from the conjunctive rule are non-null, i.e. the cardinality of the core of conjunctive consensus:

$$n^c_{12\ldots s} = Card\{Z \in G^\Theta, m_{12\ldots s}(Z) \neq 0\}$$

We denote this modified version of URR as MURR in the sequel.

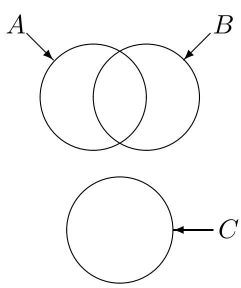

Fig. 1. Hybrid model for $\Theta = \{A, B, C\}$.

3. Example for URR and MURR

Example for URR: Let's consider $\Theta = \{A, B, C\}$ with the DSm hybrid model as shown on the Fig. 1. In this hybrid model $C \cap (A \cup B) = \emptyset$ (therefore $A \cap C = \emptyset$ and $B \cap C = \emptyset$). We consider also the following two belief assignments

$$m_1(A) = 0.4 \qquad m_1(B) = 0.2 \qquad m_1(A \cup B) = 0.4$$
$$m_2(A) = 0.2 \qquad m_2(C) = 0.3 \qquad m_2(A \cup B) = 0.5$$

then the conjunctive operator provides for this DSm hybrid model a consensus on A, B, C, $A \cup B$, and $A \cap B$ with supporting masses

$$m_{12}(A) = 0.36 \quad m_{12}(B) = 0.10 \quad m_{12}(A \cup B) = 0.20 \quad m_{12}(A \cap B) = 0.04$$

and partial conflicts between two sources on $A \cap C$, $B \cap C$ and $C \cap (A \cup B)$ with

$$m_{12}(A \cap C) = 0.12 \quad m_{12}(B \cap C) = 0.06 \quad m_{12}(C \cap (A \cup B)) = 0.12$$

Then with URR, the total conflicting mass

$$m_{12}(A \cap C) + m_{12}(B \cap C) + m_{12}(C \cap (A \cup B)) = 0.12 + 0.06 + 0.12 = 0.30$$

is uniformly (i.e. equally) redistributed to A, B, C and $A \cup B$ because the sources support only these propositions. That is $n_{12} = 4$ and thus $0.30/n_{12} = 0.075$ is added to $m_{12}(A)$, $m_{12}(B)$, $m_{12}(C)$ and $m_{12}(A \cup B)$ with URR. One finally gets:

$$\begin{aligned}
m_{12URR}(A) &= m_{12}(A) + \frac{0.30}{n_{12}} = 0.36 + 0.075 = 0.435 \\
m_{12URR}(B) &= m_{12}(B) + \frac{0.30}{n_{12}} = 0.10 + 0.075 = 0.175 \\
m_{12URR}(C) &= m_{12}(C) + \frac{0.30}{n_{12}} = 0.00 + 0.075 = 0.075 \\
m_{12URR}(A \cup B) &= m_{12}(A \cup B) + \frac{0.30}{n_{12}} = 0.20 + 0.075 = 0.275
\end{aligned}$$

while the others remain the same. That is $m_{12URR}(A \cap B) = 0.04$. Of course, one has also

$$m_{12URR}(A \cap C) = m_{12URR}(B \cap C) = m_{12URR}(C \cap (A \cup B)) = 0$$

Example for MURR: Let's consider the same frame, same model and same bba as in previous example. In this case the total conflicting mass 0.30 is uniformly redistributed to the sets A, B, $A \cup B$, and $A \cap B$ only, i.e. to the sets whose masses, after applying the conjunctive rule to the given sources, are non-zero. Thus $n_{12} = 4$, and $0.30/4 = 0.075$. Hence:

$$\begin{aligned}
m_{12MURR}(A) &= 0.36 + 0.075 = 0.435 \\
m_{12MURR}(B) &= 0.10 + 0.075 = 0.175 \\
m_{12MURR}(A \cup B) &= 0.20 + 0.075 = 0.275 \\
m_{12MURR}(A \cap B) &= 0.04 + 0.075 = 0.115
\end{aligned}$$

4. Partially Uniform Redistribution Rule

It is also possible to do a uniformly partial redistribution, i.e. to uniformly redistribute the conflicting mass only to the sets involved in the conflict. For example, if $m_{12}(A \cap B) = 0.08$ and $A \cap B = \emptyset$, then 0.08 is equally redistributed to A and B only, supposing A and B are both non-empty, so 0.04 assigned to A and 0.04 to B.

$\forall A \neq \emptyset$, one has the Partially Uniform Redistribution Rule (PURR) for two sources

$$m_{12PURR}(A) = m_{12}(A) + \frac{1}{2} \sum_{\substack{X_1, X_2 \in G^\Theta \\ X_1 \cap X_2 = \emptyset,\ X_1 = A \ or \ X_2 = A}} m_1(X_1) m_2(X_2) \tag{14}$$

where $m_{12}(A)$ is the result of the conjunctive rule applied to belief assignments $m_1(.)$ and $m_2(.)$.

For $s \geq 2$ sources to combine: $\forall A \neq \emptyset$, one has

$$m_{12\ldots sPURR}(A) = \frac{1}{s} \sum_{\substack{X_1, X_2, \ldots, X_s \in G^\Theta \\ X_1 \cap X_2 \cap \ldots \cap X_s = \emptyset \\ at\ least\ one\ X_j = A,\ j \in \{1, \ldots, s\}}} Card_A(\{X_1, \ldots, X_s\}) \prod_{i=1}^{s} m_i(X_i) + m_{12\ldots s}(A) \tag{15}$$

where $Card_A(\{X_1, \ldots, X_s\})$ is the number of A's occurring in $\{X_1, X_2, \ldots, X_s\}$.

If $A = \emptyset$, $m_{12PURR}(A) = 0$ and $m_{12\ldots sPURR}(A) = 0$.

5. Example for PURR

Let's take back the example of section 3. Based on PURR, $m_{12}(A \cap C) = 0.12$ is redistributed as follows: 0.06 to A and 0.06 to C; $m_{12}(B \cap C) = 0.06$ is redistributed as follows: 0.03 to B and 0.03 to C; and $m_{12}(C \cap (A \cup B)) = 0.12$ is redistributed in this way: 0.06 to C and 0.06 to $A \cup B$. Therefore we finally get

$$m_{12PURR}(A) = m_{12}(A) + \frac{0.12}{2} = 0.36 + 0.06 = 0.42$$

$$m_{12PURR}(B) = m_{12}(B) + \frac{0.06}{2} = 0.10 + 0.03 = 0.13$$

$$m_{12PURR}(C) = m_{12}(C) + \frac{0.12}{2} + \frac{0.06}{2} + \frac{0.12}{2} = 0.15$$

$$m_{12PURR}(A \cup B) = m_{12}(A \cup B) + \frac{0.12}{2} = 0.20 + 0.06 = 0.26$$

while the others remain the same. That is $m_{12PURR}(A \cap B) = 0.04$. Of course, one has also

$$m_{12PURR}(A \cap C) = m_{12PURR}(B \cap C) = m_{12PURR}(C \cap (A \cup B)) = 0$$

6. Neutrality of Vacuous Belief Assignment

Both URR (with MURR included) and PURR are commutative and quasi-associative, and they verify the neutrality of Vacuous Belief Assignment (VBA):

since any bba $m_1(.)$ combined with the VBA defined on any frame $\Theta = \{\theta_1, \dots, \theta_n\}$ by $m_{VBA}(\theta_1 \cup \dots \cup \theta_n) = 1$, using the conjunctive rule, gives $m_1(.)$, so no conflicting mass is needed to transfer.

7. Conclusion

Two new simple rules of combination have been presented in the framework of DSmT which have a lower complexity than PCR5. These rules are very easy to implement but from a theoretical point of view remain less precise in their transfer of conflicting beliefs since they do not take into account the proportional redistribution with respect to the mass of each set involved in the conflict. So we cannot reasonably expect that URR or PURR outperforms PCR5 but they may hopefully appear as good enough in some specific fusion problems when the level of total conflict is not important. PURR does a more refined redistribution that URR and MURR but it requires a little more calculation.

References

1. F. Smarandache and J. Dezert (eds.), Applications and Advances of DSmT for Information Fusion (Collected works), Amer. Res. Press, Rehoboth, 2004, `http://www.gallup.unm.edu/~smarandache/DSmT-book1.pdf`.
2. F. Smarandache and J. Dezert (eds.), Applications and Advances of DSmT for Information Fusion (Collected works), Vol. 2, Amer. Res. Press, Rehoboth, 2006. `http://www.gallup.unm.edu/~smarandache/DSmT-book2.pdf`.
3. G. Shafer, *A Mathematical Theory of Evidence* (Princeton University Press, Princeton, NJ, 1976).
4. L. Zadeh, Concept of a linguistic variable and its application to approximate reasoning, *Information Sciences*, Part 1 & 2, Vol. 8, pp. 199–249, 301–357, 1975, Part 3, Vol. 9 pp. 43–80, 1976.
5. L. Zadeh, A theory of approximate reasoning, *Machine Intelligence* **9** (1979) 149–194.
6. L. Zadeh, On the validity of Dempster's rule of combination, Memo M 79/24, Univ. of California, Berkeley, 1979.
7. K. Sentz and S. Ferson, Combination of Evidence in Dempster-Shafer Theory, SANDIA Tech. Report, SAND2002-0835, 96 pages, April 2002.
8. F. Smarandache and J. Dezert (eds.), Applications and Advances of DSmT for Information Fusion (Collected works), Vol. 3, Amer. Res. Press, Rehoboth, 2009. `http://www.gallup.unm.edu/~smarandache/DSmT-book3.pdf`.
9. F. Smarandache and J. Dezert, Uniform and Partially Uniform Redistribution Rules, short paper version appeared in Advances and Applications of DSmT for Plausible and Paradoxical Reasoning for Information Fusion, International Workshop organized by the Bulgarian IST Center of Competence in 21st Century, December 14, 2006, Bulg. Acad. of Sciences, Sofia, Bulgaria (available on `http://lanl.arxiv.org/abs/cs/0702028`).
10. E. Lefevre, O. Colot, P. Vannoorenberghe and D. de Brucq, A generic frame- work for resolving the conflict in the combination of belief structures, *3rd Int. Conf. Information Fusion*, FUSION'2000, pp. MOD4 11-18, Paris, France, July 10th–13th, 2000.
11. R. Dedekind, Uber Zerlegungen von Zahlen durch ihre größten gemeinsammen Teiler, in Gesammelte Werke, Bd. 1. pp. 103–148, 1897.

12. L. Comtet, Sperner Systems, Sec.7.2 in *Advanced Combinatorics: The Art of Finite and Infinite Expansion* (D. Reidel Publ. Co., 1974), pp. 271–273.
13. N. J. A. Sloane, The On-line Encyclopedia of Integer Sequences 2003, (Sequence No. A014466). `http://www.research.att.com/~njas/sequences/`
14. M. Tombak, A. Isotamm and T. Tamme, On logical method for counting Dedekind numbers, *Lect. Notes on Comp. Sci.*, Vol. 2138 (Springer-Verlag, 2001), pp. 424–427.
15. F. Smarandache, Unification of Fusion Theories (UFT), Presented at NATO Advanced Study Institute, Albena, Bulgaria, 16-27 May 2005 also in International Journal of Applied Mathematics & Statistics, Vol. 2, 1–14, 2004 and on `http://arxiv.org/abs/cs/0409040`.
16. J. W. Guan and D. A. Bell, Generalizing the Dempster-Shafer rule of combination to Boolean algebras, in *Proc. IEEE/ACM Int. Conf. Developing and Managing Projects*, Washington, March, pp. 229–236, 1993.
17. J. B. Paris, *The Uncertain Reasoner's Companion, a Mathematical Perspective* (Cambridge University Press, 1994).
18. Ph. Smets, Combining non distinct evidence, in *Proc. North American Fuzzy Information Processing* (*NAFIP 1986*), New Orleans, LA, 1986.
19. Ph. Smets, E. H. Mamdani, D. Dubois and H. Prade (eds.), *Non-Standard Logics for Automated Reasoning* (Academic Press, 1988).
20. F. Smarandache and J. Dezert, Information fusion based on new proportional conflict redistribution rules, in *Proc. Fusion 2005 Conf.*, Philadelphia, July 26–29, 2005.