Advances and Applications of DSmT for Information Fusion

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Outline

Introduction

Part 1 : Fusion based on belief functions in DST

- Dempster-Shafer Theory (DST)
- Rules of combinations and limitations of DST

Part 2 : Fusion based on belief functions in DSmT

- Dezert-Smarandache Theory (DSmT)
- Modeling, fusion and conditioning for quantitative beliefs
- Extension to qualitative beliefs
- Fusion of sources with different importance

Part 3 : Probabilistic Transformations

Part 4 : Multicriteria Decision Making using DSmT

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Conclusions & References

Main theories dealing with uncertainty

Probability Theory (Blaise Pascal 1634 to Kolmogorov 1933): objective (# of favorable cases / # of possible cases) assuming uniform distribution,Frequencies of occurrence drawn from statistical data, or subjective (De Finetti's betting approach interpreting P(.) as degree of belief)

Possibility Theory (Zadeh 1978) : based on fuzzy sets (1965) of mutual exclusive values. Zadeh interprets fuzzy sets as possibility distributions.

Belief Function Theory : introduced by Shafer in 1976

Imprecise Probabilities (Walley 1991): deals with probability intervals

Why belief functions ?

Probabilities do not account for partial knowledge since it deals generally with information drawn from generic knowledge based either on population of items, laws of physics, common sense, ...

Probabilities capture only one aspect of uncertain information (the randomness, i.e. the variability through repeated measurements). Probability can't distinguish between uncertainty due to variability and uncertainty due to the lack of knowledge.

Beliefs often are related with singular event and are not necessarily related with statistical data and generic knowledge.They are related with singular evidence. Belief functions are well adapted for dealing with partial knowledge contrariwise to probabilities.

Variability: Precisely observed random observations

Incompleteness/non specificity: missing/partial information

Introduction: What is DSmT in short ?

DSmT (Dezert-Smarandache Theory) started in end of 2001 as a natural extension to Dempster-Shafer Theory (DST) which :

1 - proposes a new mathematical framework for quantitative or qualitative information fusion

2 - incorporates any kinds of model (free, hybrid DSm models and/or Shafer's model) for taking into account any integrity constraints of the fusion problem

3 - combines uncertain, high conflicting and imprecise sources of evidence with new rules of combination and overcomes limitations of the Dempster's rule

4 - is adapted to static or dynamic fusion applications represented in terms of belief functions based on the same general unified formalism

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Dempster-Shafer Theory (DST) - 1976

We are concerned with the true value of some quantity or hypothesis θ taking its possible values in Θ.

Working with subsets as propositions: $\mathcal{P}_\theta(A) \triangleq$ *The true value of θ is in a subset A of Θ.*

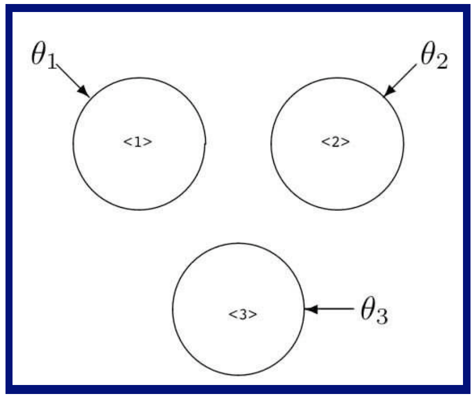
Operations	Subsets	Propositions
Intersection/conjunction	$A \cap B$	$\mathcal{P}_\theta(A) \wedge \mathcal{P}_\theta(B)$
Union/disjunction	$A \cup B$	$\mathcal{P}_\theta(A) \vee \mathcal{P}_\theta(B)$
Inclusion/implication	$A \subset B$	$\mathcal{P}_\theta(A) \Rightarrow \mathcal{P}_\theta(B)$
Complementation/negation	$A = c_\Theta(B)$	$\mathcal{P}_\theta(A) = \neg\mathcal{P}_\theta(B)$

Frame of discernment: $\Theta = \{\theta_i, i = 1, \ldots, n\}$ Finite set of exhaustive and exclusive elements

Shafer's model : Close world assumption + exclusivity (implicit refinement done)

Power set : $\mathcal{P}(\Theta) \triangleq 2^\Theta$ $\quad |\mathcal{P}(\Theta)| = 2^{|\Theta|}$

Example : $\Theta = \{\theta_1, \theta_2, \theta_3\} \Rightarrow$



$$2^\Theta = \{\emptyset, \theta_1, \theta_2, \theta_3, \theta_1 \cup \theta_2, \theta_1 \cup \theta_3, \theta_2 \cup \theta_3, \theta_1 \cup \theta_2 \cup \theta_3\} \qquad |2^\Theta| = 2^3 = 8$$

Belief functions in DST

Basic belief assignment (bba)/mass

$$m(.) : 2^{\Theta} \rightarrow [0,1]$$

$$m(\emptyset) = 0 \quad \text{and} \quad \sum_{A \in 2^{\Theta}} m(A) = 1$$

A is a focal element iff m(A)>0

Core of m(.) = set of focal elements

Belief of A

$$\text{Bel}(A) = \sum_{B \in 2^{\Theta}, B \subseteq A} m(B)$$

Total mass of information implying the occurence of A

Plausibility of A

$$\text{Pl}(A) = \sum_{B \in 2^{\Theta}, B \cap A \neq \emptyset} m(B)$$

Total mass of information consistent with A

In general, $0 \leq \text{Bel}(A) \leq \text{Pl}(A) \leq 1$

Vacuous belief Assignment (VBA)

(represents ignorant source)

$$\forall A \neq \Theta, m_v(A) = 0 \text{ and } m_v(\Theta) = 1 \Longrightarrow \forall A \neq \Theta, \text{Bel}(A) = 0 \quad \text{Bel}(\Theta) = 1$$

Bayesian belief assignment :

focal elements are singletons of the power set

$$m(.) = Bel(.) = Pl(.) = P(.)$$

Dempster's rule of combination

Fusion of 2 independent equally reliable sources with bba's m_1 and m_2

(DS) $$m(\emptyset) = 0 \quad \text{and} \quad \forall A \neq \emptyset, m(A) = \frac{1}{1-k_{12}} \sum_{\substack{X,Y \in 2^\Theta \\ X \cap Y = A}} m_1(X) m_2(Y)$$

Degre of (total) conflict $$k_{12} = \sum_{\substack{X,Y \in 2^\Theta \\ X \cap Y = \emptyset}} m_1(X) m_2(Y)$$

Example: $\Theta = \{\theta_1, \theta_2\}$

$m_1(\theta_1) = 0.1 \quad m_1(\theta_2) = 0.2 \quad m_1(\theta_1 \cup \theta_2) = 0.7$

$m_2(\theta_1) = 0.3 \quad m_2(\theta_2) = 0.2 \quad m_2(\theta_1 \cup \theta_2) = 0.5$

$k_{12} = m_1(\theta_1)m_2(\theta_2) + m_1(\theta_2)m_2(\theta_1)$

$k_{12} = 0.1 \cdot 0.2 + 0.2 \cdot 0.3 = 0.02 + 0.06 = 0.08$

$$m(\theta_1) = [m_1(\theta_1)m_2(\theta_1)+m_1(\theta_1)m_2(\theta_1\cup\theta_2)+m_2(\theta_1)m_1(\theta_1\cup\theta_2)]/(1-k_{12}) = 0.29/0.92 \approx 0.316$$

$$m(\theta_2) = [m_1(\theta_2)m_2(\theta_2)+m_1(\theta_2)m_2(\theta_1\cup\theta_2)+m_2(\theta_2)m_1(\theta_1\cup\theta_2)]/(1-k_{12}) = 0.28/0.92 \approx 0.304$$

$$m(\theta_1 \cup \theta_2) = m_1(\theta_1 \cup \theta_2)m_2(\theta_1 \cup \theta_2)/(1-k_{12}) = 0.35/0.92 \approx 0.380$$

Advantages and drawbacks of DS rule

Advantages

- Commutativity and associativity
- Extension for $N > 2$ sources
- Neutrality of VBA
- Coherence with Bayes' rule when $m(.) \equiv P(.)$

Drawbacks

- (DS) is not defined when conflict is 1
- (DS) provides questionable results when k_{12} increases
- No way to trust (DS) result beforehand
- Justification/necessity of working with Shafer's model ?

[Zadeh 1979, Yager 1983, Dubois&Prade 1986, Pearl 1988, Voorbraak 1991, Walley 1996, Fixsen&Mahler 1997]

Several origins of the problem

1 Different reliability of the sources (statistical criteria), but sources can be equally reliable.

2 Limited knowledge or experience of sources/experts. Sources have their own interpretation of elements of the frame - subjectivity and biasness is possible.

3 The final interest of experts can also be different when they report their assessment on a given problem ...

Infinite classes of counter-examples for (DS)

Class #1 : Trivial

If every column contains at least one zero, (DS) is not defined

$\Theta = \{\theta_1, \ldots, \theta_n\}, n \geq 2$

	θ_1	θ_2	$\ldots$	θ_n
Source 1	$m_{s_1}(\theta_1)$	$m_{s_1}(\theta_2)$	$\ldots$	$m_{s_1}(\theta_n)$
Source 2	$m_{s_2}(\theta_1)$	$m_{s_2}(\theta_2)$	$\ldots$	$m_{s_2}(\theta_n)$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
Source k	$m_{s_k}(\theta_1)$	$m_{s_k}(\theta_2)$	$\ldots$	$m_{s_k}(\theta_n)$

$$\forall i = 1, \ldots, n \qquad m(\theta_i) = [m_{s_1} \oplus m_{s_2} \oplus \ldots \oplus m_{s_n}](\theta_i) = \frac{0}{0}$$

Class #2 : Generalization of Zadeh's example

1. If there exists a column of small positive masses for say for element i

2. If all other columns ≠ i include at least a zero

(DS) provides a counter-intuitive result because it is independent of values of column i and can reflect the minority opinion

$\Theta = \{\theta_1, \ldots, \theta_n\}, n \geq 2$

	θ_1	θ_2	$\ldots$	θ_i	$\ldots$	θ_n
Source 1	$m_{s_1}(\theta_1)$	$m_{s_1}(\theta_2)$	$\ldots$	$\epsilon_{1,i}$	$\ldots$	$m_{s_1}(\theta_n)$
Source 2	$m_{s_2}(\theta_1)$	$m_{s_2}(\theta_2)$	$\ldots$	$\epsilon_{2,i}$	$\ldots$	$m_{s_2}(\theta_n)$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
Source k	$m_{s_k}(\theta_1)$	$m_{s_k}(\theta_2)$	$\ldots$	$\epsilon_{k,i}$	$\ldots$	$m_{s_k}(\theta_n)$

$$m(\theta_i) = [m_{s_1} \oplus m_{s_2} \oplus \ldots \oplus m_{s_n}](\theta_i) = 1$$

Infinite classes of counter-examples for (DS)

Class #3 : Smarandache (extension of Zadeh's class to non Bayesian case)

$$\Theta = \{\theta_1, \ldots, \theta_n\}, n \geq 2$$

	θ_1	$\ldots$	θ_n	u_1	$\ldots$	u_p
Source 1	$m_{s_1}(\theta_1)$	$\ldots$	$m_{s_1}(\theta_n)$	$m_{s_1}(u_1)$	$\ldots$	$m_{s_1}(u_p)$
Source 2	$m_{s_2}(\theta_1)$	$\ldots$	$m_{s_2}(\theta_n)$	$m_{s_2}(u_1)$	$\ldots$	$m_{s_2}(u_p)$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
Source k	$m_{s_k}(\theta_1)$	$\ldots$	$m_{s_k}(\theta_n)$	$m_{s_k}(u_1)$	$\ldots$	$m_{s_k}(u_p)$

$u_m, m = 1, \ldots, p$ are disjunctions of elements θ_i, ($i \in \{1, \ldots, n\}$ of the frame Θ.

1. If there is at least one zero in every column $\theta_1, \theta_2, \ldots \theta_n$
2. If there exists one column u_i which contains non zero

Then

$$m(u_i) = [m_{s_1} \oplus m_{s_2} \oplus \ldots \oplus m_{s_n}](u_i) = 1$$

independent of the positive values involved in u_i !!!

Example:

$$\Theta = \{\theta_1, \theta_2, \theta_3, \theta_4\}$$

	θ_1	θ_2	θ_3	θ_4	$\theta_3 \cup \theta_4$
$m_1(.)$	0.99	0	0	0	0.01
$m_2(.)$	0	0.98	0	0	0.02

(DS) result

$$m(\theta_3 \cup \theta_4) = \frac{(0.01 \cdot 0.02)}{(0 + 0 + 0 + 0 + 0.01 \cdot 0.02)} = 1$$

How to circumvent troubles with DS rule ?

Classical solutions

Apply some heuristic/ad hoc thresholding techniques on the level of the conflict to accept (or reject) the fusion result. How to choose the threshold ?

Apply discounting techniques on sources. How to be sure that no problem will occur with DS rule after discounting ? How to discount sources when no statistical data are available ?

Mix the two previous «solutions». How and justification ?

Use other alternative rules. Which one ? Why ?

Main question: **How to prevent troubles in fusion beforehand ?**

Proposal (detailed in part 2)

Switch to a new paragdim to deal with the fusion of vague, uncertain, imprecise, highly conflicting quantitative and qualitative information fusion for static or dynamic problematics.

Main alternatives to DS rule

Assumption: Shafer's model

Disjunctive rule: $m_{Disj}(A) = \sum_{\substack{B,C\in 2^\Theta \\ B\cup C=A}} m_1(B)m_2(C)$

Yager's rule: [Yager 1983] **(Y)**

$$\begin{cases} m_Y(\emptyset) = 0 \\ m_Y(A) = \sum_{\substack{X,Y\in 2^\Theta \\ X\cap Y=A}} m_1(X)m_2(Y) & \forall A \in 2^\Theta, A \neq \emptyset, A \neq \Theta \\ m_Y(\Theta) = m_1(\Theta)m_2(\Theta) + \sum_{\substack{X,Y\in 2^\Theta \\ X\cap Y=\emptyset}} m_1(X)m_2(Y) & \text{when} \quad A = \Theta \end{cases}$$

Dubois & Prade's (hybrid) rule: [Dubois & Prade 1988]

(DP)

$$\begin{cases} m_{DP}(\emptyset) = 0 \\ m_{DP}(A) = \sum_{\substack{X,Y\in 2^\Theta \\ X\cap Y=A \\ X\cap Y\neq\emptyset}} m_1(X)m_2(Y) + \sum_{\substack{X,Y\in 2^\Theta \\ X\cup Y=A \\ X\cap Y=\emptyset}} m_1(X)m_2(Y) & \forall A \neq \emptyset \end{cases}$$

Adaptive Combination Rule (ACR): [Florea 2005]

A weighted balance between conjunctive and disjunctive rules depending on the total conflict.

Assumption: Open-world

Smets' rule: [Smets 1994] It is the non-normalized version of Dempster's rule (keep conflicting mass on empty set at credal level when combining).

Unified formulation of the rules

General Weighted Operator (GWO)

Step1 : Derivation of the TOTAL conflict

$$k_{12} \triangleq \sum_{\substack{X_1, X_2 \in 2^\Theta \\ X_1 \cap X_2 = \emptyset}} m_1(X_1) m_2(X_2)$$

Step2 : Redistribution of the total conflict with given set of weights

$$m(\emptyset) = w_m(\emptyset) \cdot k_{12} \qquad \sum_{X \in 2^\Theta} w_m(X) = 1 \quad \text{et} \quad w_m(X) \in [0,1]$$

(GWO)

$$\forall (X \neq \emptyset) \in 2^\Theta \quad m(X) = [\sum_{\substack{X_1, X_2 \in 2^\Theta \\ X_1 \cap X_2 = X}} m_1(X_1) m_2(X_2)] + w_m(X) k_{12}$$

The GWO formalism includes most of known fusion operators based on the conjunctive consensus (Dempster, Smets, Yager, etc) depending on the choice of weighting factors.

There is an infinity of fusion rules !!!

Reliability Discounting of sources

Consider an unreliable source providing the bba $m(.)$ and having a known reliability factor $\alpha \in [0,1]$.

$\alpha = 1$ means no discounting (full reliability of the source)

$\alpha = 0$ means total discounting (full unreliable/ignorant source)

Discounted bba

$$\begin{cases} m(A) \\ m(\Theta) \end{cases} \rightarrow \begin{cases} m'(A) = \alpha \cdot m(A) & \forall A \neq \Theta \\ m'(\Theta) = (1-\alpha) + \alpha \cdot m(\Theta) \end{cases}$$

This approach makes sense (and has to be used) if one has a good estimation of reliability factor of each source (based on statistical experiment AND ground truth).

A sophisticated method exists [Denoeux et al. 2005,2006] where discounting factor depends on subsets.

Remark : Discounting = conjunctive fusion on {Θ x {Rel, notRel}} and the marginalization on Θ [Haenni 2005]

We are not sure of discounting factors (most of the time we don't have these factors at all !!!). Discounting in such cases appears only as an ad-hoc engineering trick to prevent troubles with (DS) ...

Fundamentally, discounting do not solve the inherent problem of (DS); it's just a mean to increase the mass of belief on the total ignorance.

Reliability Discounting ≠ Importance Discounting **(see end of Part 2)**

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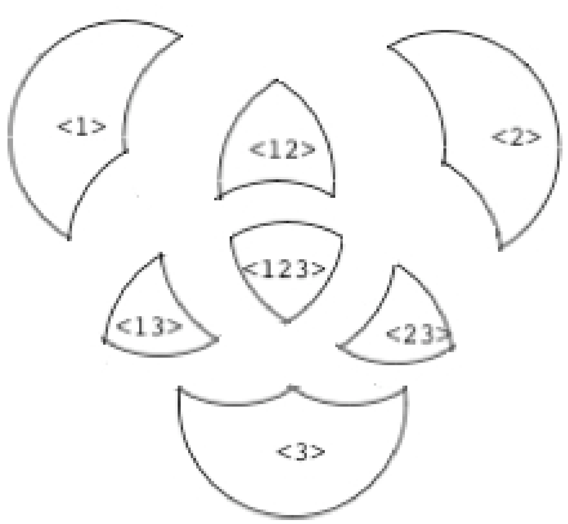
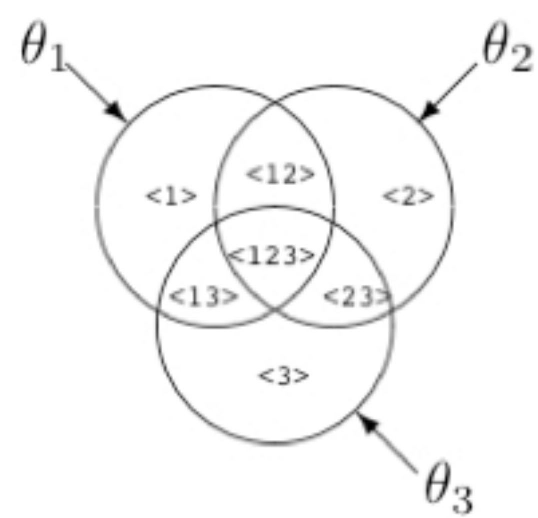
Fusion Spaces

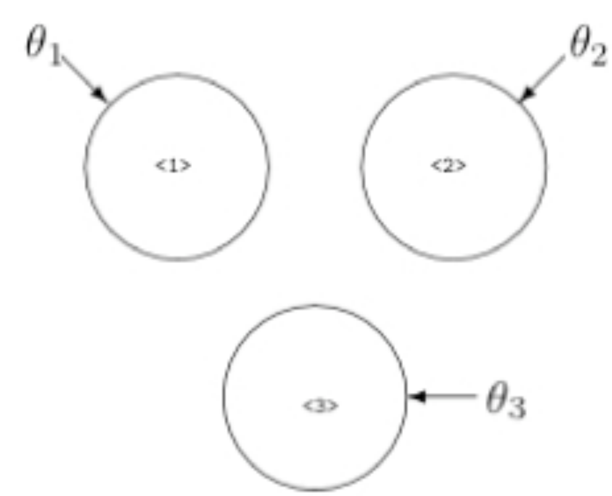
Frame of the problem $\Theta = \{\theta_1, \theta_2, \ldots, \theta_n\}$. Finite set of exhaustive elements (discrete/continuous/fuzzy/relative concepts)

Fusion spaces : Power sets, Hyper-power set (Dedekind's lattice) and Super-power sets

$$|2^{\Theta_{ref}} = S^{\Theta} \triangleq (\Theta, \cup, \cap, c(.))| > |D^{\Theta} = (\Theta, \cup, \cap)| > |2^{\Theta} = (\Theta, \cup)|$$







Super-power set = power set of the refined frame

Hyper-power sets

$\Theta = \{\theta_1, \theta_2, \dots, \theta_n\}$.

How to generate it

1. $\emptyset, \theta_1, \dots, \theta_n \in D^\Theta$
2. $\forall A \in D^\Theta, B \in D^\Theta, (A \cup B) \in D^\Theta, (A \cap B) \in D^\Theta$
3. No other elements belong to D^Θ, except those, obtained by using rules 1 or 2.

Hyper-power set reduces to classical power set for the Shafer's model (when all elements are exclusive)

The cardinality of hyper-power sets follows Dedekind's numbers sequence when the size of the frame increases.

Example for n=3

$\Theta = \{\theta_1, \theta_2, \theta_3\}$ d(n=3)=19

$\alpha_0 \triangleq \emptyset$

$\alpha_1 \triangleq \theta_1 \cap \theta_2 \cap \theta_3$

$\alpha_2 \triangleq \theta_1 \cap \theta_2$

$\alpha_3 \triangleq \theta_1 \cap \theta_3$

$\alpha_4 \triangleq \theta_2 \cap \theta_3$

$\alpha_5 \triangleq (\theta_1 \cup \theta_2) \cap \theta_3$

$\alpha_6 \triangleq (\theta_1 \cup \theta_3) \cap \theta_2$

$\alpha_7 \triangleq (\theta_2 \cup \theta_3) \cap \theta_1$

$\alpha_8 \triangleq (\theta_1 \cap \theta_2) \cup (\theta_1 \cap \theta_3) \cup (\theta_2 \cap \theta_3)$

$\alpha_9 \triangleq \theta_1$

$\alpha_{10} \triangleq \theta_2$

$\alpha_{11} \triangleq \theta_3$

$\alpha_{12} \triangleq (\theta_1 \cap \theta_2) \cup \theta_3$

$\alpha_{13} \triangleq (\theta_1 \cap \theta_3) \cup \theta_2$

$\alpha_{14} \triangleq (\theta_2 \cap \theta_3) \cup \theta_1$

$\alpha_{15} \triangleq \theta_1 \cup \theta_2$

$\alpha_{16} \triangleq \theta_1 \cup \theta_3$

$\alpha_{17} \triangleq \theta_2 \cup \theta_3$

$\alpha_{18} \triangleq \theta_1 \cup \theta_2 \cup \theta_3$

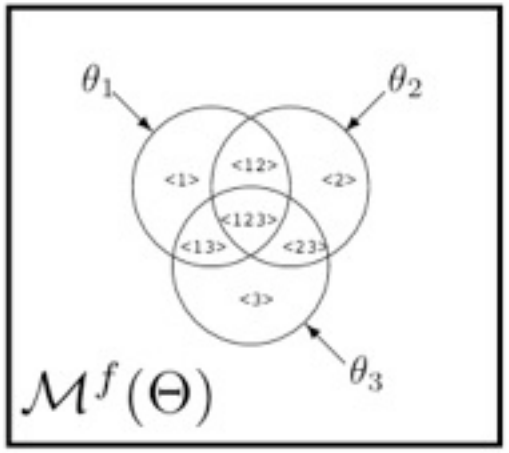
DSmT basics : DSm Models

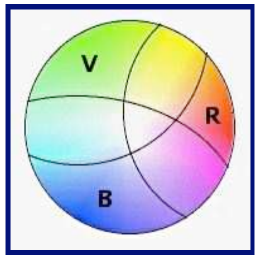
The granularity of the model of the frame characterizes the intrinsic nature (discrete/continuous,precise/vague,absolute/relative, etc) of the concepts involved in the fusion process.

Parts have vague boundaries

Free DSm model

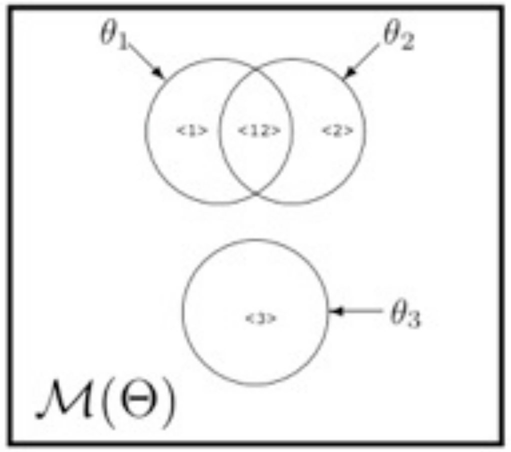
Elements of the frame are vague and potentially overlapping. Free = no constraint on elements. Useful to manipulate continuous concepts having relative interpretation (where ultimate refinement is inaccessible)





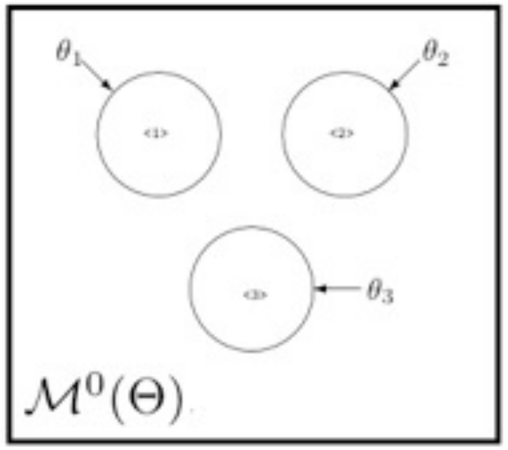
Hybrid DSm model

Some elements of the frame can be exclusive and/or non existing specially for dynamic fusion applications. Hybrid model means introduction of integrity constraints into the free DSm model.



Special hybrid model: Shafer's model

All exhaustive elements of the frame are known to be truly exclusive (i.e. a refinement is accessible)



Constraints are represented by the characteristic non-emptiness function Φ(A) for all A in hyper-power set: Φ(A)=1 if A non-empty or 0 otherwise.

Parts have precise boundaries

Generalized (quantitative) belief functions

Generalized basic belief assignment (gbba)

$$m(.) : G^{\Theta} \rightarrow [0,1] \quad \text{with} \quad m(\emptyset) = 0 \quad \text{and} \quad \sum_{A \in G^{\Theta}} m(A) = 1$$

where G^{Θ} is the fusion space (i.e. 2^{Θ}, D^{Θ}, or $S^{\Theta} = 2^{\Theta_{refined}}$)

Generalized belief function

$$\text{Bel}(A) = \sum_{\substack{B \subseteq A \\ B \in G^{\Theta}}} m(B)$$

Generalized plausibility function

$$\text{Pl}(A) = \sum_{\substack{B \cap A \neq \emptyset \\ B \in G^{\Theta}}} m(B)$$

Question: How to combine efficiently belief functions generated by several sources of evidence ?

$$[m_1 \oplus \ldots \oplus m_s](X)$$

Generalized bba (example)

Let's consider the simple frame $\Theta = \{A, B\}$, then depending on the model we choose for G^Θ, one will deal with:

- G^Θ as Θ (Bayesian bba):

$$m(A) + m(B) = 1$$

- G^Θ as the power set 2^Θ and therefore:

$$m(A) + m(B) + m(A \cup B) = 1$$

- G^Θ as the hyper-power set D^Θ and therefore:

$$m(A) + m(B) + m(A \cup B) + m(A \cap B) = 1$$

- G^Θ as the super-power set S^Θ and therefore:

$$m(A) + m(B) + m(A \cup B) + m(A \cap B) \\ + m(c(A)) + m(c(B)) + m(c(A) \cup c(B)) = 1$$

Fusion based on belief functions

Decision level

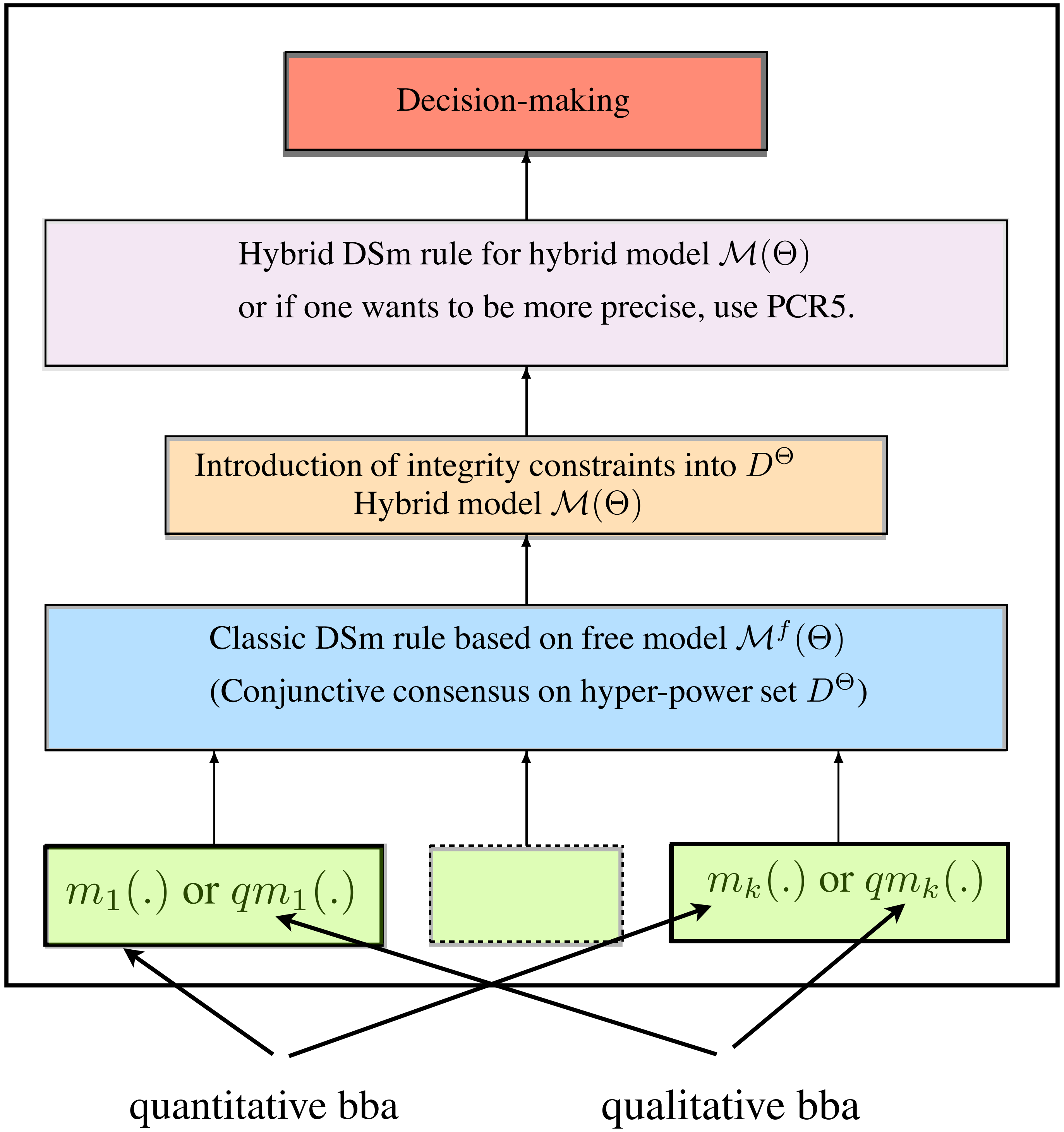
Fusion level
(DSmH/PCR5)

Integrity level

Intermediate level
(DSmC)

Sources level
(+ discounting)

Static scheme (all sources are combined altogether)



DSm Hybrid rule of combination (DSmH)

For <u>any</u> model, the fusion of k independent equally (otherwise discounting techniques are applied first) reliable sources is done by

(DSmH)

$$m_{\mathcal{M}(\Theta)}(X) \triangleq \phi(X)\Big[S_1(X) + S_2(X) + S_3(X)\Big]$$

hybrid rule means conjunctive mixed with disjunctive

No division is required, DSmH ≠ Dempster's rule

(DSmC)

$$S_1(X) \triangleq \sum_{\substack{X_1,X_2,\ldots,X_s\in D^\Theta\\ X_1\cap X_2\cap\ldots\cap X_s=X}} \prod_{i=1}^{s} m_i(X_i)$$

$$S_2(X) \triangleq \sum_{\substack{X_1,X_2,\ldots,X_s\in \boldsymbol{\emptyset}\\ [\mathcal{U}=X]\vee[(\mathcal{U}\in\boldsymbol{\emptyset})\wedge(X=I_t)]}} \prod_{i=1}^{s} m_i(X_i)$$

$$S_3(A) \triangleq \sum_{\substack{X_1,X_2,\ldots,X_s\in D^\Theta\\ X_1\cup X_2\cup\ldots\cup X_s=A\\ X_1\cap X_2\cap\ldots\cap X_s\in\boldsymbol{\emptyset}}} \prod_{i=1}^{s} m_i(X_i)$$

$I_t \triangleq \theta_1\cup\ldots\cup\theta_n$ is the total ignorance.

$\mathcal{U} \triangleq u(X_1)\cup\ldots\cup u(X_k)$

$u(X)$ is the union of all θ_i that compose X

$\boldsymbol{\emptyset} \triangleq \{\emptyset, \boldsymbol{\emptyset}_{\mathcal{M}}\}$

$\boldsymbol{\emptyset}_{\mathcal{M}}$ = set of propositions forced to be empty in $\mathcal{M}$

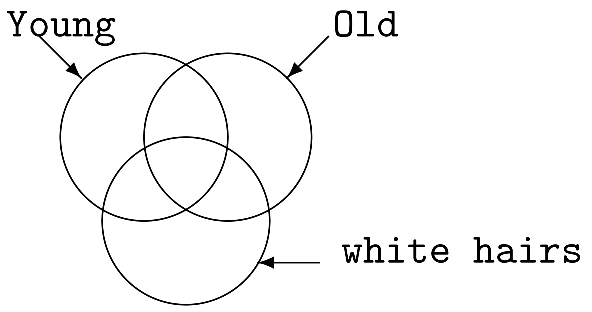
All propositions involved in formulas are expressed in their canonical form (i.e. disjunctive normal form, also known as disjunction of conjunctions in Boolean algebra, which is unique).

Special case : (DSmH) reduces to classic DSm rule (i.e. DSmC) when the free DSm-model is used, i.e. only S1(X) is kept in (DSmH) formula.

Static versus dynamic fusion

Static Fusion : The frame and its model do not change with time

Dynamic Fusion: The frame and/or its model change with time

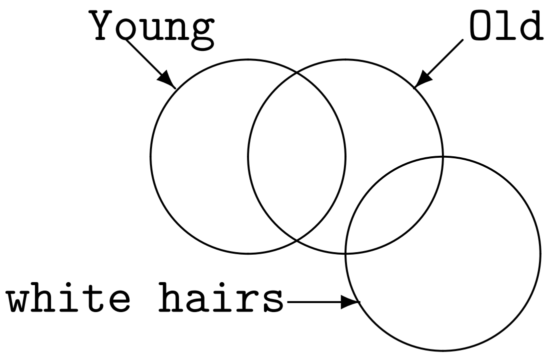


Example of dynamic fusion (testimony problem)

$$\Theta(t_l) \triangleq \{\theta_1 \equiv \text{young}, \theta_2 \equiv \text{old}, \theta_3 \equiv \text{white hairs}\}$$

Reports $\begin{cases} m_1(\theta_1) = 0.5 & m_1(\theta_3) = 0.5 \\ m_2(\theta_2) = 0.5 & m_2(\theta_3) = 0.5 \end{cases}$

$m_{\mathcal{M}^f(\Theta(t_l))}(\theta_1 \cap \theta_2) = 0.25$ $m_{\mathcal{M}^f(\Theta(t_l))}(\theta_1 \cap \theta_3) = 0.25$ $m_{\mathcal{M}^f(\Theta(t_l))}(\theta_2 \cap \theta_3) = 0.25$ $m_{\mathcal{M}^f(\Theta(t_l))}(\theta_3) = 0.25$



If one learns later that young people don't have white hairs, one introduces this integrity constraint in the model, i.e. $\Phi(\theta_1 \cap \theta_3) \overset{\mathcal{M}}{\equiv} 0$

$m_{\mathcal{M}(\Theta(t_{l+1}))}(\theta_3) = 0.25$ $m_{\mathcal{M}(\Theta(t_{l+1}))}(\theta_1 \cap \theta_2) = 0.25$ $m_{\mathcal{M}(\Theta(t_{l+1}))}(\theta_2 \cap \theta_3) = 0.25$ $m_{\mathcal{M}(\Theta(t_{l+1}))}(\theta_1 \cup \theta_3) = 0.25$

Example in Zadeh's class

$$\Theta = \{\theta_1, \theta_2, \theta_3\}$$

Inputs

$m_1(\theta_1) = 1 - e_1 \quad m_1(\theta_2) = 0 \quad m_1(\theta_3) = e_1$

$m_2(\theta_1) = 0 \quad m_2(\theta_2) = 1 - e_2 \quad m_2(\theta_3) = e_2$

If one adopts Shafer's model

$$m(\theta_3) = \frac{e_1 e_2}{(1 - e_1) \cdot 0 + 0 \cdot (1 - e_2) + e_1 e_2} = 1$$

When $0 < e_1 < 1$ and $0 < e_2 < 1$, Dempster's rule provides in this case same result whatever the values of e_1 and e_2 are !!! Dempster's rule is mathematically not defined when $e_1 = e_2 = 0$.
It provides only a coherent and trivial solution when $e_1 = e_2 = 1$.

If one adopts free DSm model and DSmC rule

$m(\theta_3) = e_1 e_2 \quad m(\theta_1 \cap \theta_2) = (1 - e_1)(1 - e_2) \quad m(\theta_1 \cap \theta_3) = (1 - e_1)e_2 \quad m(\theta_2 \cap \theta_3) = (1 - e_2)e_1$

If one adopts Shafer's model and DSmH rule

$m(\theta_3) = e_1 e_2 \quad m(\theta_1 \cup \theta_2) = (1 - e_1)(1 - e_2) \quad m(\theta_1 \cup \theta_3) = (1 - e_1)e_2 \quad m(\theta_2 \cup \theta_3) = (1 - e_2)e_1$

(DSmH) provides a more consistent result which depends on e_1 and e_2.
e_1 and e_2 can take any values in [0,1].

Same conclusion is drawn for examples in Smarandache's class.

Robustness of (DS) and (DSmH) w.r.t. imprecision

Frame $\Theta = \{\theta_1, \theta_2, \theta_3\}$

Shafer's model

Inputs

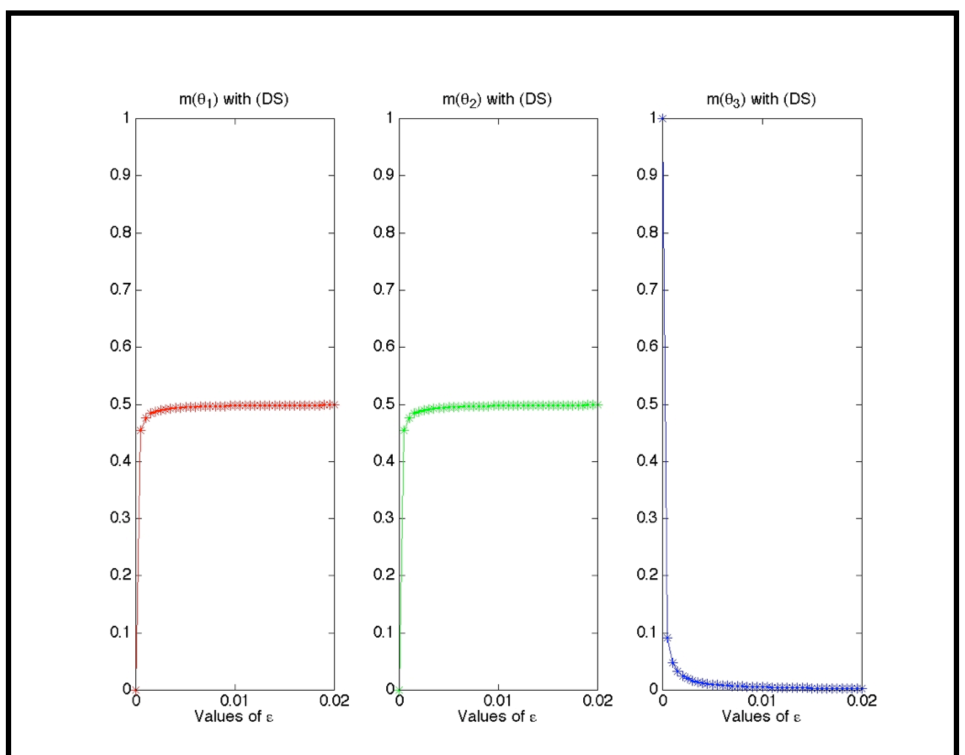
	θ_1	θ_2	θ_3
Source 1	$m_1(\theta_1) = 0.99 - \epsilon$	$m_1(\theta_2) = \epsilon$	$m_1(\theta_3) = 0.01$
Source 2	$m_2(\theta_1) = \epsilon$	$m_2(\theta_2) = 0.99 - \epsilon$	$m_2(\theta_3) = 0.01$

(DS) is not robust

A small variation of ε induces a big variation of (DS) result

For $\epsilon = 0, m(\theta_3) = 1$

$$\text{For } \epsilon = 0.0005, \begin{cases} m(\theta_1) = 0.45410 \\ m(\theta_2) = 0.45410 \\ m(\theta_3) = 0.0918 \end{cases}$$

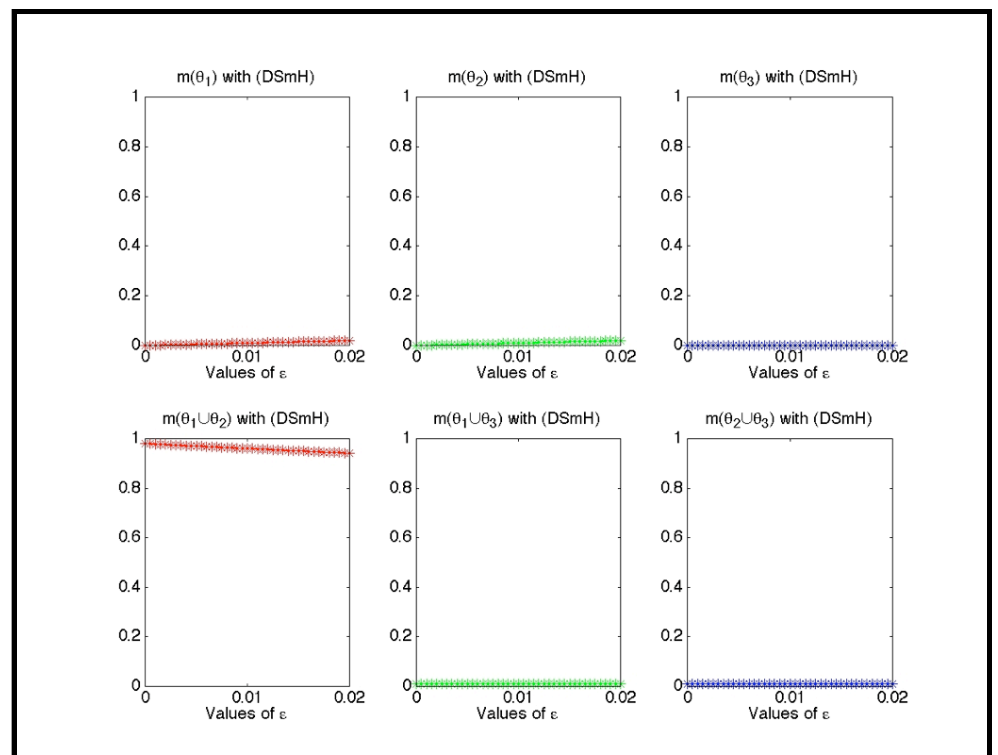


(DSmH) is more robust

A small variation of ε induces a small variation of (DSmH) result.

$$\text{For } \epsilon = 0, \begin{cases} m(\theta_1) = m(\theta_2) = 0 \\ m(\theta_3) = 0.0001 \\ m(\theta_1 \cup \theta_2) = 0.9801 \\ m(\theta_1 \cup \theta_3) = m(\theta_2 \cup \theta_3) = 0.0099 \end{cases}$$

$$\text{For } \epsilon = 0.0005, \begin{cases} m(\theta_1) = m(\theta_2) = 0.0005 \\ m(\theta_3) = 0.0001 \\ m(\theta_1 \cup \theta_2) = 0.9791 \\ m(\theta_1 \cup \theta_3) = m(\theta_2 \cup \theta_3) = 0.0099 \end{cases}$$



Example in Smarandache's class

$$\Theta = \{\theta_1, \theta_2, \theta_3, \theta_4\}$$

Inputs

$$m_1(\theta_1) = 0.99 \quad m_1(\theta_3 \cup \theta_4) = 0.01$$
$$m_2(\theta_2) = 0.98 \quad m_2(\theta_3 \cup \theta_4) = 0.02$$

If one adopts Shafer's model

(DS)

$$m(\theta_3 \cup \theta_4) = \frac{(0.01 \cdot 0.02)}{(0 + 0 + 0 + 0 + 0.01 \cdot 0.02)} = 1$$

Other masses are zero.
Counter-intuitive result

If one adopts free DSm model

(DSmC)

$$m(\theta_1 \cap \theta_2) = 0.9702 \quad m(\theta_1 \cap (\theta_3 \cup \theta_4)) = 0.0198 \quad m(\theta_2 \cap (\theta_3 \cup \theta_4)) = 0.0098 \quad m(\theta_3 \cup \theta_4) = 0.0002$$

If one adopts Shafer's model

(DSmH)

$$m(\theta_1 \cup \theta_2) = 0.9702 \quad m(\theta_1 \cup \theta_3 \cup \theta_4) = 0.0198 \quad m(\theta_2 \cup \theta_3 \cup \theta_4) = 0.0098 \quad m(\theta_3 \cup \theta_4) = 0.0002$$

DSmT still provides a coherent result

Testinomy example (dynamic case)

$\Theta = \{\theta_1, \theta_2, \theta_3\}$

set of a priori exclusive and exhaustive suspects

original witnesses reports

$m_1(\theta_1) = 0.1 \quad m_1(\theta_2) = 0.4 \quad m_1(\theta_3) = 0.2 \quad m_1(\theta_1 \cup \theta_2) = 0.3$

$m_2(\theta_1) = 0.5 \quad m_2(\theta_2) = 0.1 \quad m_2(\theta_3) = 0.3 \quad m_2(\theta_1 \cup \theta_2) = 0.1$

New info arrives: The third suspect provides a strong alibi

$\theta_3 \overset{\mathcal{M}}{\equiv} \emptyset$ **(non existential/integrity constraint)**

(DSmC)

$m(\theta_1) = 0.21 \quad m(\theta_2) = 0.11 \quad m(\theta_3) = 0.06 \quad m(\theta_1 \cup \theta_2) = 0.03$

$m(\theta_1 \cap \theta_2) = 0.21 \quad m(\theta_1 \cap \theta_3) = 0.13 \quad m(\theta_2 \cap \theta_3) = 0.14 \quad m(\theta_3 \cap (\theta_1 \cup \theta_2)) = 0.11$

The conflicting mass to transfer is then $k_{12} = 0.06 + 0.21 + 0.13 + 0.14 + 0.11 = 0.65$

(DSmH) $m(\emptyset) = 0 \quad m(\theta_1) = 0.34 \quad m(\theta_2) = 0.25 \quad m(\theta_1 \cup \theta_2) = 0.41$

(S) Smets $m_S(\emptyset) = 0.65 \quad m_S(\theta_1) = 0.21 \quad m_S(\theta_2) = 0.11 \quad m_S(\theta_1 \cup \theta_2) = 0.03$

(Y) Yager $m_Y(\emptyset) = 0 \quad m_Y(\theta_1) = 0.21 \quad m_Y(\theta_2) = 0.11 \quad m_Y(\theta_1 \cup \theta_2) = 0.03 + k_{12} = 0.03 + 0.65 = 0.68$

Testinomy example (dynamic case)

Dempster's rule

(DS) $m_{DS}(\emptyset) = 0 \quad m_{DS}(\theta_1) = \frac{0.21}{1 - 0.65} = 0.60 \quad m_{DS}(\theta_2) = \frac{0.11}{1 - 0.65} \approx 0.314 \quad m_{DS}(\theta_1 \cup \theta_2) = \frac{0.03}{1 - 0.65} \approx 0.086$

Dubois & Prade's rule **(DP)**

$$m_{DP}(\emptyset) = 0$$

$$m_{DP}(\theta_1) = [m_1(\theta_1)m_2(\theta_1) + m_1(\theta_1)m_2(\theta_1 \cup \theta_2) + m_2(\theta_1)m_1(\theta_1 \cup \theta_2)] + [m_1(\theta_1)m_2(\theta_3) + m_2(\theta_1)m_1(\theta_3)] = 0.34$$

$$m_{DP}(\theta_2) = [m_1(\theta_2)m_2(\theta_2) + m_1(\theta_2)m_2(\theta_1 \cup \theta_2) + m_2(\theta_2)m_1(\theta_1 \cup \theta_2)] + [m_1(\theta_2)m_2(\theta_3) + m_2(\theta_2)m_1(\theta_3)] = 0.25$$

$$m_{DP}(\theta_1 \cup \theta_2) = [m_1(\theta_1 \cup \theta_2)m_2(\theta_1 \cup \theta_2)] + [m_1(\theta_1 \cup \theta_2)m_2(\theta_3) + m_2(\theta_1 \cup \theta_2)m_1(\theta_3)] + [m_1(\theta_1)m_2(\theta_2) + m_2(\theta_1)m_1(\theta_2)] = 0.35$$

If one adds the masses up, one gets 0.94 < 1

Dubois & Prade's rule doesn't work for dynamic fusion problems when a singleton or an union of singletons becomes empty.
This problem is fixed by the sum S_2 in DSmH.
When there is no non-existential constraint, DSmH = DP

DSm rules for imprecise beliefs

Operations on sets

Addition $S_1 \boxplus S_2 = S_2 \boxplus S_1 \triangleq \{x \mid x = s_1 + s_2, s_1 \in S_1, s_2 \in S_2\}$

Subtraction $S_1 \boxminus S_2 \triangleq \{x \mid x = s_1 - s_2, s_1 \in S_1, s_2 \in S_2\}$

Multiplication $S_1 \boxdot S_2 \triangleq \{x \mid x = s_1 \cdot s_2, s_1 \in S_1, s_2 \in S_2\}$

Inputs:

Imprecise admissible generalized bba $m^I(.)$ are of the form

$$m^I(A) = [a_1, b_1] \cup \ldots \cup [a_m, b_m] \cup (c_1, d_1) \cup \ldots \cup (c_n, d_n) \cup (e_1, f_1] \cup \ldots \cup (e_p, f_p] \cup [g_1, h_1) \cup \ldots \cup [g_q, h_q) \cup \{A_1, \ldots, A_r\}$$

where all the bounds or elements involved into $m^I(A)$ belong to $[0, 1]$

DSmH for imprecise beliefs

(DSmH-Imp) $$m^I_{\mathcal{M}(\Theta)}(A) \triangleq \phi(A) \boxdot \left[S^I_1(A) \boxplus S^I_2(A) \boxplus S^I_3(A) \right]$$

A simple 2D example

$A \in D^\Theta$	$m^I_1(A)$	$m^I_2(A)$
θ_1	$[0.1, 0.2] \cup \{0.3\}$	$[0.4, 0.5]$
θ_2	$(0.4, 0.6) \cup [0.7, 0.8]$	$[0, 0.4] \cup \{0.5, 0.6\}$

Inputs

(DSmH-Imp)

$A \in D^\Theta$	$m^I_{\mathcal{M}}(A) = [m^I_1 \oplus m^I_2](A)$
θ_1	$[0.04, 0.10] \cup [0.12, 0.15]$
θ_2	$[0, 0.40] \cup [0.42, 0.48]$
$\theta_1 \cap \theta_2 \overset{\mathcal{M}}{\equiv} \emptyset$	0
$\theta_1 \cup \theta_2$	$(0.16, 0.58]$

(DSmC-Imp)

$A \in D^\Theta$	$m^I(A) = [m^I_1 \oplus m^I_2](A)$
θ_1	$[0.04, 0.10] \cup [0.12, 0.15]$
θ_2	$[0, 0.40] \cup [0.42, 0.48]$
$\theta_1 \cap \theta_2$	$(0.16, 0.58]$
$\theta_1 \cup \theta_2$	0

Proportional Conflict Redistribution (PCR)

Why PCR fusion rules ? **To not increase the mass on uncertainties in the fusion**

- **Step 1**: Compute the conjunctive rule $m_{12}(X) = \sum_{\substack{X_1, X_2 \in G^\Theta \\ X_1 \cap X_2 = X}} m_1(X_1) m_2(X_2)$

- **Step 2**: compute all the conflicting masses (partial and/or total).

$$k_{12} = \sum_{\substack{X_1, X_2 \in G^\Theta \\ X_1 \cap X_2 = \emptyset}} m_1(X_1) m_2(X_2)$$

Partial conflicts

- **Step 3**: then proportionally redistribute the conflicting mass (total or partial) to non-empty sets involved in the model according to all integrity constraints.

The way the conflicting mass is redistributed yields to several versions of PCR (PCR1-PCR6) which work for any degree of conflict and for any models and both in DST and DSmT and for static or dynamical fusion applications.

PCR rule # 5 (PCR5)

PCR5 transfers the **partial conflicting masses** to the elements involved in the partial conflict **proportionally** to mass $m_1(.)$ and $m_2(.)$ of elements involved in the partial conflict **ONLY**.

$$\forall X \neq \emptyset, X \in G^{\Theta}$$

$$m_{PCR5}(X) = m_{12}(X) + \sum_{\substack{Y \in G^{\Theta}\setminus\{X\} \\ X \cap Y = \emptyset}} [\frac{m_1(X)^2 m_2(Y)}{m_1(X) + m_2(Y)} + \frac{m_2(X)^2 m_1(Y)}{m_2(X) + m_1(Y)}]$$

Extension possible for N>2 sources

Advantage : PCR5 does a more exact redistribution than PCR1- PCR4. PCR5 works on any model and preserves the neutrality of VBA.

A new rule (PCR6), more intuitive than (PCR5) for combining s>2 sources, is proposed by Martin & Osswald in DSmT Book, Vol. 2.

Drawback: PCR5 as most rules (but DS rule) is not associative (quasi-associative only)

TCN Fusion rule (Fuzzified PCR5)

[Tchamova, Dezert, Smarandache 2006, DSmT Book3 Chap 15]

This rule is based on fuzzy T-norm (min for conjunction) and fuzzy T-conorm (max for disjunction) operators.

min T-norm conjunctive consensus

$$m(A) = \sum_{\substack{X,Y \in G^{\Theta} \\ X \cap Y = A}} \min\{m_1(X), m_2(Y)\}$$

Conflicting masses are distributed to all non-empty sets involved in the conflict proportionally with respect to the maximum between the elements of corresponding mass matrix's columns, associated with the given element of G^{Θ}.

$$\tilde{m}_{12TCN}(A) = \sum_{\substack{X,Y \in G^{\Theta} \\ X \cap Y = A}} \min\{m_1(X), m_2(Y)\} + \sum_{\substack{X \in G^{\Theta} \\ X \cap A = \emptyset}} \left(m_1(A) \times \frac{\min\{m_1(A), m_2(X)\}}{\max\{m_1(A), m_2(X)\}} + m_2(A) \times \frac{\min\{m_2(A), m_1(X)\}}{\max\{m_2(A), m_1(X)\}}\right)$$

Normalization $\longrightarrow$

$$m_{TCN}(A) = \frac{\tilde{m}_{TCN}(A)}{\sum_{A \in G^{\theta}} \tilde{m}_{TCN}(A)}$$

Can be extented to N sources;
TCN does not belong to the General Weighted Operator Class;
very easy to implement, satisfying the neutrality of Vacuous Belief Assignment;
commutative, convergent to idempotence, reflecting majority opinion.

Example for PCR5

$\Theta = \{A, B\}$

Shafer's model

Inputs

	A	B	$A \cup B$
$m_1(.)$	0.6	0.3	0.1
$m_2(.)$	0.2	0.3	0.5
$m_{12}(.)$	0.44	0.27	0.05

$$k_{12} = m_{12}(A \cap B) = m_1(A)m_2(B) + m_1(B)m_2(A) = 0.18 + 0.06 = 0.24$$

$m_2(A) = 0.2$ and $m_1(B) = 0.3$ did make an impact on the conflict because $m_2(A)m_1(B) = 0.2 \cdot 0.3 = 0.06$ was added to the conflicting mass. So, A and B are involved in the conflict ($A \cup B$ is not involved), hence only A and B deserve a part of the conflicting mass, $A \cup B$ does not deserve.

Let x_1 be the conflicting mass to be redistributed to A, and y_1 the conflicting mass redistributed to B from the first partial conflicting mass 0.18, and similarly for x_2 and y_2 with partial conflict 0.06; one has:

$$x_1/0.6 = y_1/0.3 = (x_1 + y_1)/(0.6 + 0.3) = 0.18/0.9 = 0.2 \longrightarrow \begin{cases} x_1 = 0.6 \cdot 0.2 = 0.12 \\ y_1 = 0.3 \cdot 0.2 = 0.06 \end{cases}$$

$$x_2/0.2 = y_2/0.3 = (x_2 + y_2)/(0.2 + 0.3) = 0.06/0.5 = 0.12 \longrightarrow \begin{cases} x_2 = 0.2 \cdot 0.12 = 0.024 \\ y_2 = 0.3 \cdot 0.12 = 0.036 \end{cases}$$

With PCR5
$m_{PCR5}(A) = 0.44 + 0.12 + 0.024 = 0.584$
$m_{PCR5}(B) = 0.27 + 0.06 + 0.036 = 0.366$
$m_{PCR5}(A \cup B) = 0.05 + 0 = 0.05$

With DSmH and Dubois & Prade's rules
$m_{DSmH}(A) = m_{DP}(A) = 0.44$
$m_{DSmH}(B) = m_{DP}(B) = 0.27$
$m_{DSmH}(A \cup B) = m_{DP}(A \cup B) = 0.29$

With Dempster's rule
$m_{DS}(A) \approx 0.579$
$m_{DS}(B) \approx 0.355$
$m_{DS}(A \cup B) \approx 0.066$

The mass put on ignorance with PCR5 is the lowest

Note: Example for imp-PCR5 can be found in [DSmT Book 2]

PCR6 versus PCR5

The difference between PCR5 and PCR6 lies in the way the proportional conflict redistribution is done as soon as three or more sources are involved in the fusion (for 2 sources, PCR6=PCR5).

Let's consider $m_1(.)$, $m_2(.)$ and $m_3(.)$, $A \cap B = \emptyset$ for the model of the frame Θ.

$$m_1(A) = 0.6, \quad m_2(B) = 0.3, \quad m_3(B) = 0.1$$

With PCR5:

$$\frac{x_A^{PCR5}}{m_1(A)} = \frac{x_B^{PCR5}}{m_2(B)m_3(B)} = \frac{m_1(A)m_2(B)m_3(B)}{m_1(A) + m_2(B)m_3(B)}$$

$$\frac{x_A^{PCR5}}{0.6} = \frac{x_B^{PCR5}}{0.03} = \frac{0.018}{0.6 + 0.03} \approx 0.02857$$

Therefore, one gets

$$\begin{cases} x_A^{PCR5} = 0.60 \cdot 0.02857 \approx 0.01714 \\ x_B^{PCR5} = 0.03 \cdot 0.02857 \approx 0.00086 \end{cases}$$

With PCR6:

$$\frac{x_A^{PCR6}}{m_1(A)} = \frac{x_{B,2}^{PCR6}}{m_2(B)} = \frac{x_{B,3}^{PCR6}}{m_3(B)} = \frac{m_1(A)m_2(B)m_3(B)}{m_1(A) + m_2(B) + m_3(B)}$$

$$\frac{x_A^{PCR6}}{0.6} = \frac{x_{B,2}^{PCR6}}{0.3} = \frac{x_{B,3}^{PCR6}}{0.1} = \frac{0.018}{0.6 + 0.3 + 0.1} = 0.018$$

whence

$$\begin{cases} x_A^{PCR6} = 0.6 \cdot 0.018 = 0.0108 \\ x_{B,2}^{PCR6} = 0.3 \cdot 0.018 = 0.0054 \\ x_{B,3}^{PCR6} = 0.1 \cdot 0.018 = 0.0018 \end{cases}$$

Therefore, one gets

$$\begin{cases} x_A^{PCR6} = 0.0108 \\ x_B^{PCR6} = x_{B,2}^{PCR6} + x_{B,3}^{PCR6} = 0.0054 + 0.0018 = 0.0072 \end{cases}$$

Note: PCR6 is more simple to implement than PCR5 (see MatLab Code)

Zadeh's Example (1979)

$\Theta = \{A, B, C\}$,

Shafer's model

Inputs

	A	B	C
$m_1(.)$	0.9	0	0.1
$m_2(.)$	0	0.9	0.1
$m_{12}(.)$	0	0	0.01

Partial conflicts: $m_{12}(A \cap B) = 0.81, m_{12}(A \cap C) = m_{12}(B \cap C) = 0.09$

Total conflict: $k_{12} = m_1(A)m_2(B) + m_1(A)m_2(C) + m_2(B)m_1(C) = 0.81 + 0.09 + 0.09 = 0.99$

Comparison of Fusion results

(DS)	(DSmH=DP)	(PCR5)	(Yager)
$m_{DS}(C) = 1$	$m_{DSmH}(A \cup B) = 0.81$	$m_{PCR5}(A) = 0.486$	$m_Y(A \cup B \cup C) = 0.99$
	$m_{DSmH}(A \cup C) = 0.09$	$m_{PCR5}(B) = 0.486$	$m_Y(C) = 0.01$
	$m_{DSmH}(B \cup C) = 0.09$	$m_{PCR5}(C) = 0.028$	
	$m_{DSmH}(C) = 0.01$		

What is the most reasonable/trustable result ?

No definitive answer since ~ 30 years !!! but simulations can be done based on groundtruth to compare performances of different rules.

Smarandache's example (non Bayesian case)

$\Theta = \{A, B, C, D\}$ Shafer's model

Inputs

	A	B	$C \cup D$
$m_1(.)$	0.99	0	0.01
$m_2(.)$	0	0.99	0.01
$m_{12}(.)$	0	0	0.0001

Partial conflicts:

$$m_{12}(A \cap B) = m_1(A)m_2(B) = 0.9801$$

$$m_{12}(A \cap (C \cup D)) = m_1(A)m_2(C \cup D) = 0.0099$$

$$m_{12}(B \cap (C \cup D)) = m_1(C \cup D)m_2(B) = 0.0099$$

Total conflict:

$$k_{12} = m_1(A)m_2(B) + m_1(A)m_2(C \cup D) + m_1(C \cup D)m_2(B) = 0.9801 + 0.0099 + 0.0099 = 0.9999$$

With (DS) rule, one will get $m_{DS}(C \cup D) = 1$

With (DSmH) rule, one will get

$$m_{DSmH}(A \cup B) = 0.9801$$

$$m_{DSmH}(A \cup C \cup D) = 0.0099$$

$$m_{DSmH}(B \cup C \cup D) = 0.0099$$

$$m_{DSmH}(C \cup D) = 0.0001$$

With (PCR5) rule, one will get

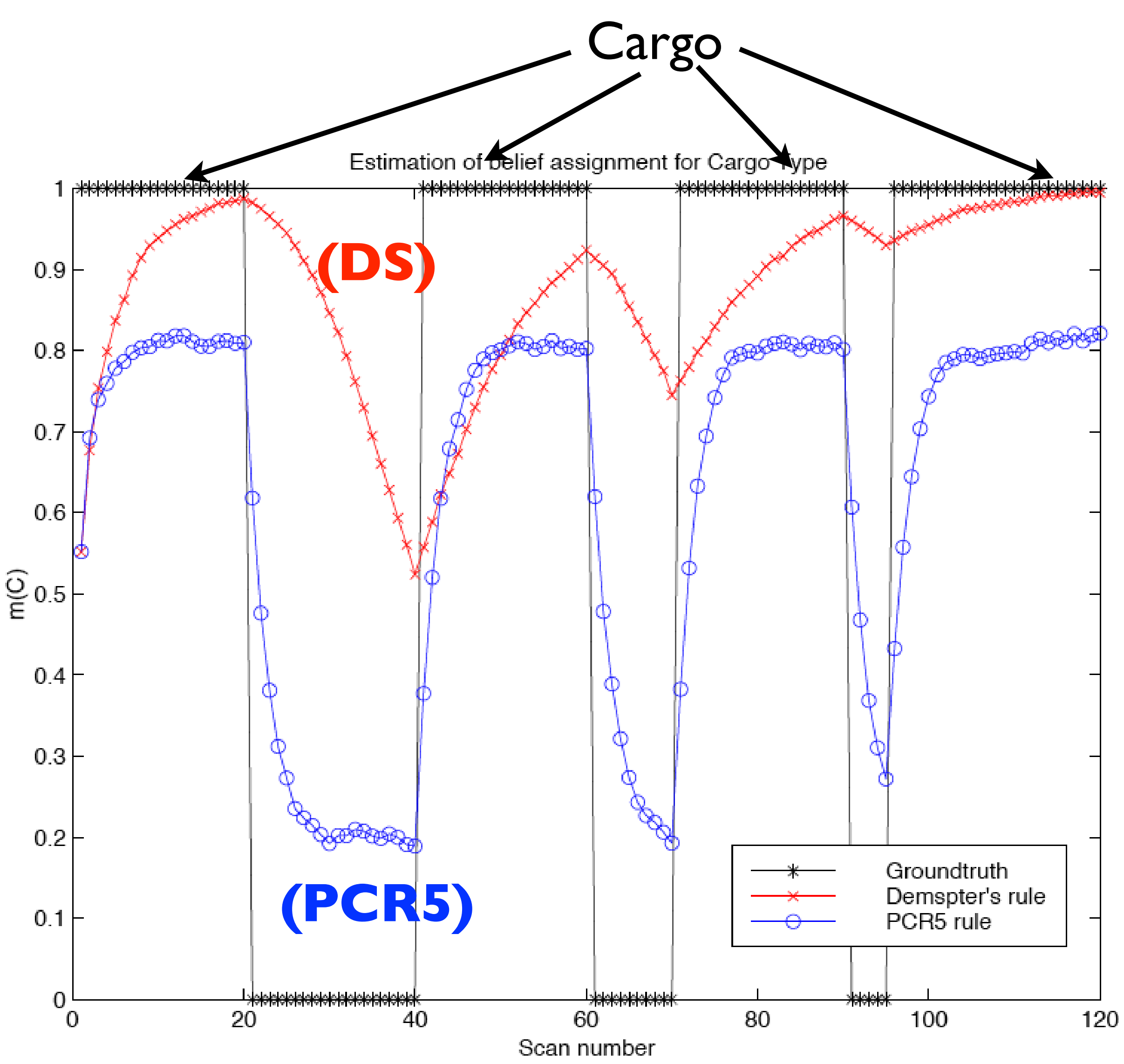
$$m_{PCR5}(A) = m_{PCR5}(B) = 0.499851$$

$$m_{PCR5}(C \cup D) = 0.000298$$

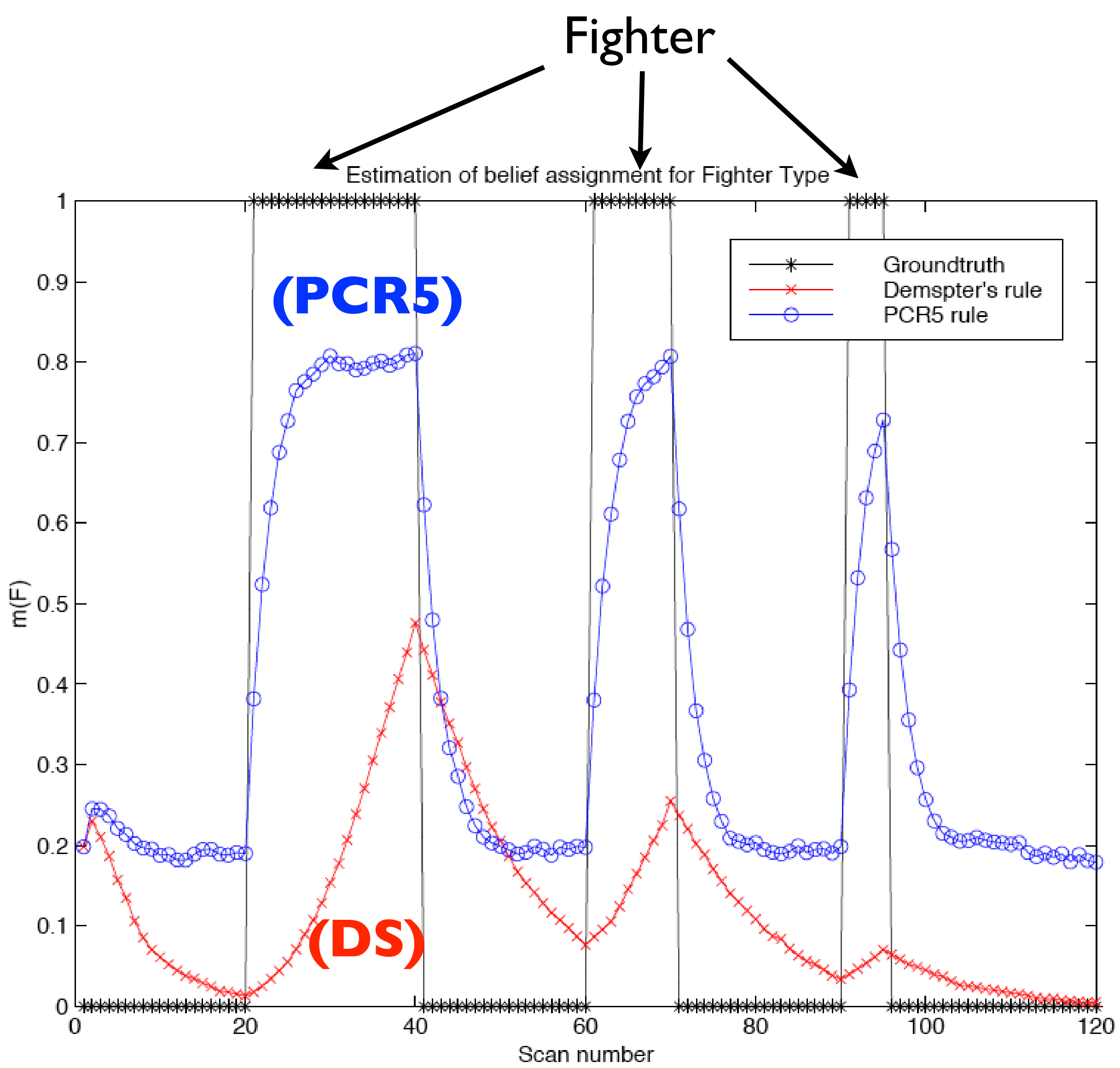
With TBM and Smets' rule, one gets $m_S(\emptyset) = 0.9999$ $m_S(C \cup D) = 0.0001$

Target type tracking with (DS) and (PCR5)

2 targets sequentially observed and classified with $\mathbf{C}_2 = \begin{bmatrix} 0.75 & 0.25 \\ 0.25 & 0.75 \end{bmatrix}$



Cargo Type Tracking

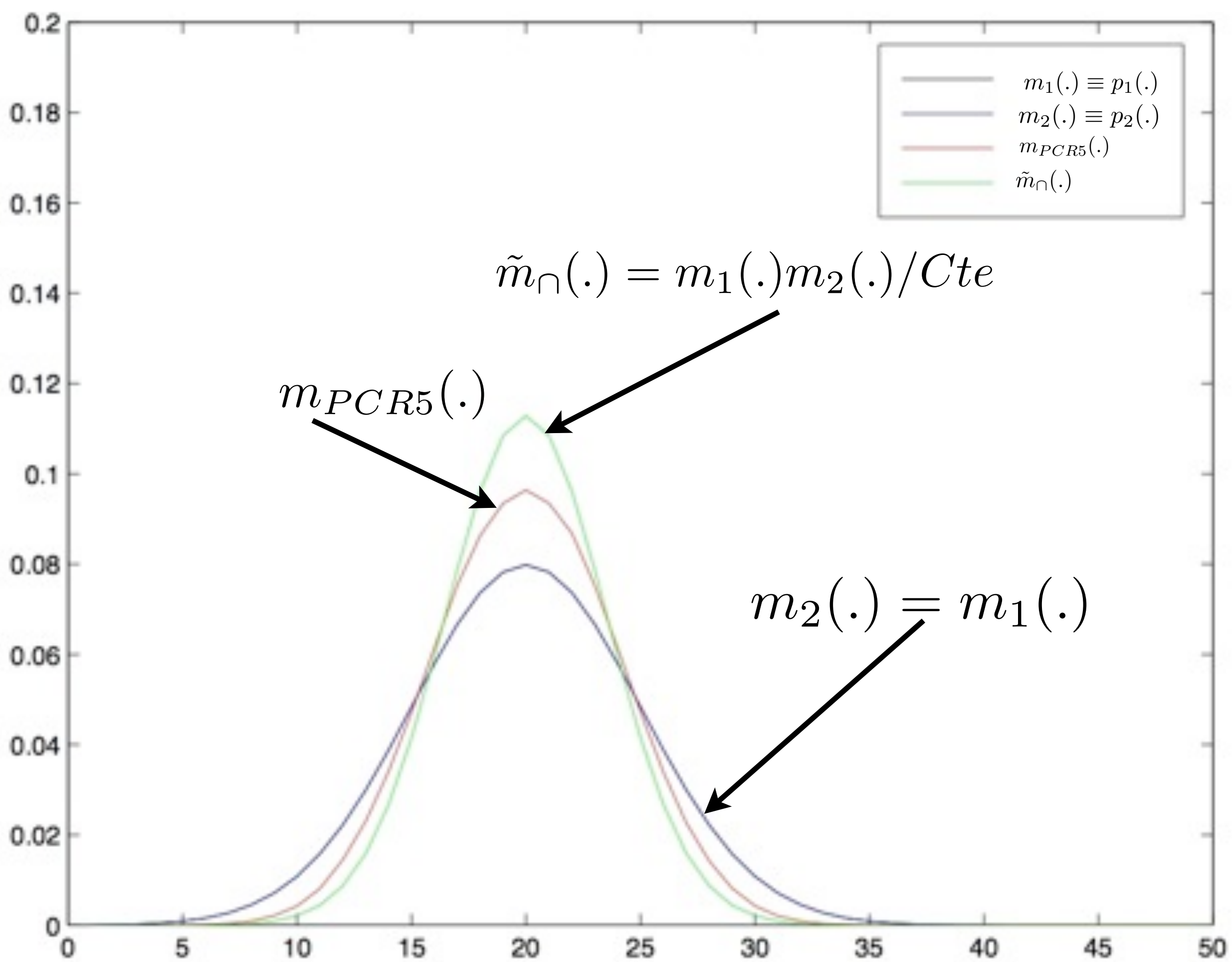


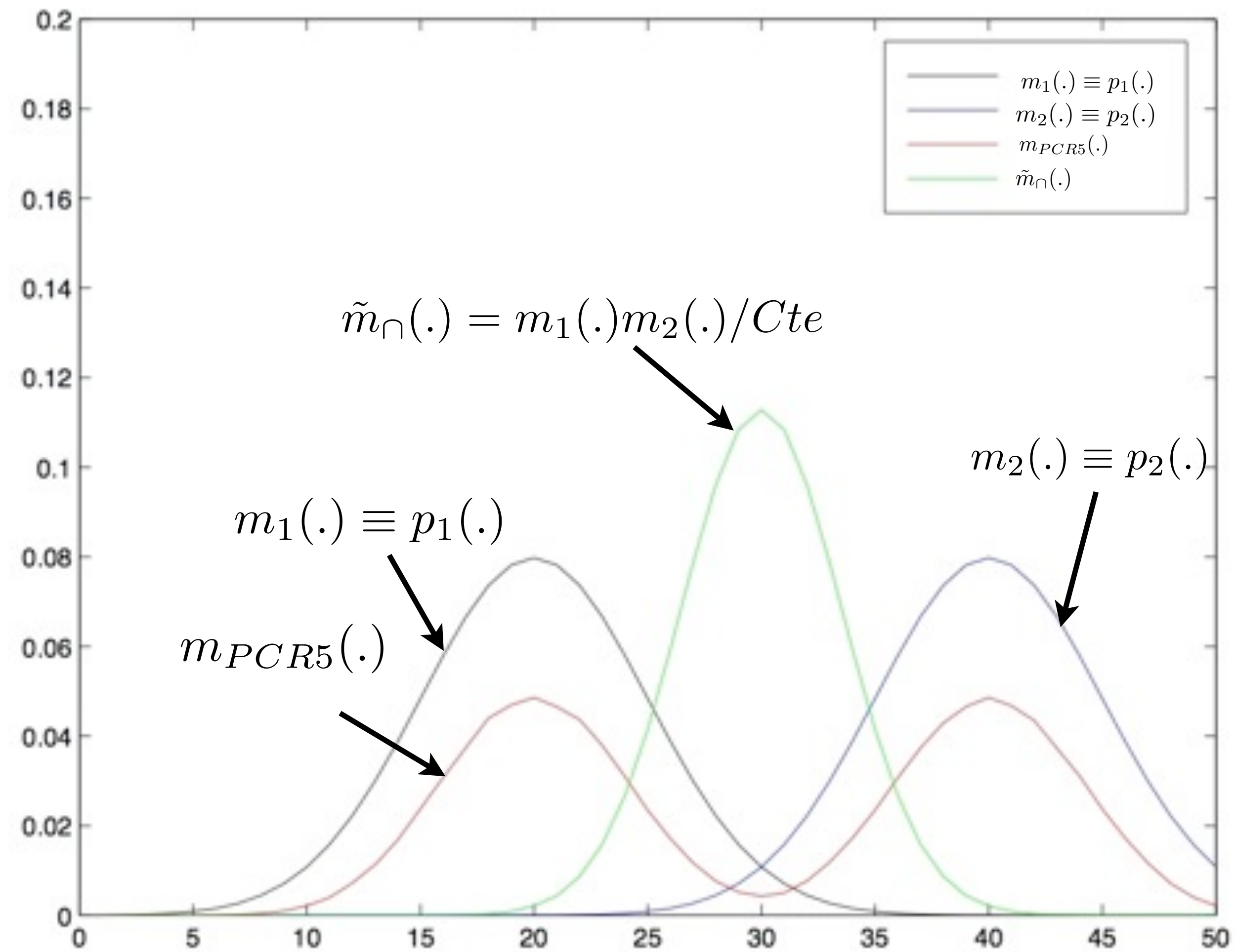
Fighter Type Tracking

[Dezert, Tchamova, Konstantinova, Smarandache 2006]

Example : (PCR5) for Gaussian Bayesian belief distributions

Here we restrict masses to be Bayesian and we extend PCR5 to work on a continuous frame





where Cte is a normalization constant

Case 1 : $m_2(.) = m_1(.)$

Case 2 : $m_2(.) \neq m_1(.)$

Application: Particle Filtering for target tracking [Fusion 2007]

Fusion of beliefs based on sampling

[Frédéric Dambreville, Chap.6, DSmT Book 3,2009]

Dempster's rule obtained from sampling approach

The estimate $\widehat{m}_{DS}(.)$ of $m_{DS}(.)$ is obtained by the following sampling process:

1. Repeat from $n = 1$ to $n = N$:
   (a) Generate Y_1 and Y_2 by means of $m_1(.)$ and $m_2(.)$ respectively,
   (b) If $Y_1 \cap Y_2 = \emptyset$, then set $X_n = \text{rejected}$,
   (c) Otherwise, keep $X_n = Y_1 \cap Y_2$,
2. Compute the rejection rate $\widehat{z} = \frac{1}{N}\sum_{n=1}^{N} I[X_n = \text{rejected}]$,
3. For any $X \in G^{\Theta}$, compute $\widehat{m}_{DS}(X)$ by:

$$\widehat{m}_{DS}(X) = \frac{1}{N(1-\widehat{z})}\sum_{n=1}^{N} I[X_n = X] .$$

Fusion of beliefs based on sampling

PCR5 rule obtained from sampling approach

The estimate $\widehat{m}_{PCR5}(.)$ of $m_{PCR5}(.)$ is obtained by the sampling process:

1. Repeat from $n = 1$ to $n = N$:
   (a) Generate Y_1 and Y_2 by means of $m_1(.)$ and $m_2(.)$ respectively,
   (b) If $Y_1 \cap Y_2 \neq \emptyset$, then take $X_n = Y_1 \cap Y_2$,
   (c) Otherwise, do:
      i. Compute $\theta = \frac{m_1(Y_1)}{m_1(Y_1)+m_2(Y_2)}$,
      ii. Generate a random number u uniformly distributed on $[0, 1]$,
      iii. If $u < \theta$, set $X_n = Y_1$; otherwise, set $X_n = Y_2$,
2. For any $X \in G^\Theta$, compute $\widehat{m}_{PCR5}(X)$ by:

$$\widehat{m}_{PCR5}(X) = \frac{1}{N} \sum_{n=1}^{N} I[X_n = X] .$$

A general theoretical framework for the fusion based on sampling techniques has been developed by Dambreville [DSmT book 3]

Simple MatLab Code for PCR5 and PCR6

(For Shafer's model only)

File : PCR5fusion.m

```
function [mPCR5,TotalConflict]=PCR5fusion(BBA)
% Author and copyrights: Jean Dezert
% Input: BBA matrix
% Output: mPCR5 = resulting bba after fusion with PCR5
% TotalConflict = level of total conflict between sources
NbrSources=size(BBA,2);
CardTheta=log2(size(BBA,1)+1);
if(NbrSources==1)
mPCR5=BBA(:,1);TotalConflict=0;return
end
Card2PowerTheta=2^(CardTheta)-1;
% All possible combinations
vec=[1:Card2PowerTheta];
Combinations=vec;
for s=1:NbrSources-1
Combinations=combvec(Combinations,vec);
end
Combinations=Combinations';
mPCR5=zeros(Card2PowerTheta,1);
TotalConflict=0;
NbrComb=size(Combinations,1);
for c=1:NbrComb
PC=Combinations(c,:);
mConj=zeros(1,NbrSources);
for s=1:NbrSources
mConj(s)=BBA(PC(s),s);
end
massConj=prod(mConj,2);
if(massConj>0)
% Check if this is a real partial conflict or not
Intersections=PC(1);
for s=2:NbrSources
X=PC(s);
Intersections=bitand(Intersections,X);
end
if(Intersections~=0) % the intersection is not empty
mPCR5(Intersections)=mPCR5(Intersections)+massConj;
else % the intersection is empty
TotalConflict=TotalConflict+massConj;
% Let's apply PCR5 rule principle
UQ=unique(PC);
Proportions=0*UQ;
DenPCR5=0;
for u=1:size(UQ,2)
SamePropositions=find(PC==UQ(u));
MassProd=prod(mConj(SamePropositions));
Proportions(u)= MassProd*massConj;
DenPCR5=DenPCR5+MassProd;
end
Proportions=Proportions/DenPCR5;
% PCR5 redistribution
for u=1:size(UQ,2)
mPCR5(UQ(u))=mPCR5(UQ(u))+Proportions(u);
end, end, end, end, return
```

File : PCR6fusion.m

```
function [mPCR6,TotalConflict]=PCR6fusion(BBA)
% Author and copyrights: Jean Dezert
% Input: BBA matrix
% Output: mPCR6 = resulting bba after fusion with PCR6
% TotalConflict = level of total conflict between sources
NbrSources=size(BBA,2);
CardTheta=log2(size(BBA,1)+1);
if(NbrSources==1)
mPCR6=BBA(:,1);
TotalConflict=0;
return
end
Card2PowerTheta=2^(CardTheta)-1;
% All possible combinations
vec=[1:Card2PowerTheta];
Combinations=vec;
for s=1:NbrSources-1
Combinations=combvec(Combinations,vec);
end
Combinations=Combinations';
mPCR6=zeros(Card2PowerTheta,1);
TotalConflict=0;
NbrComb=size(Combinations,1);
for c=1:NbrComb
PC=Combinations(c,:);  % particular combination
mConj=zeros(1,NbrSources);
for s=1:NbrSources
mConj(s)=BBA(PC(s),s);
end
massConj=prod(mConj,2);
if(massConj>0)
Intersections=PC(1);
for s=2:NbrSources
X=PC(s);
Intersections=bitand(Intersections,X);
end
if(Intersections~=0) % intersection not empty
mPCR6(Intersections)=mPCR6(Intersections)+massConj;
else % empty intersection
TotalConflict=TotalConflict+massConj;
% PCR6 rule principle
for s=1:NbrSources
Proportion= mConj(s)*(massConj/(sum(mConj,2)));
% Redistribution back to element PC(s)
mPCR6(PC(s))=mPCR6(PC(s))+Proportion;
end, end, end, end, return
```

Sophisticated toolboxes for DSmT are available for research purpose:

By A. Martin - See DSmT Book 3 and upon request to this author

By F. Dambreville - http://refereefunction.fredericdambreville.com

On the associativity of DSm rules

General case : Hybrid DSm model

DSmH and PCR5 rules are **commutative** and **quasi-associative**, i.e. in order to preserve the associativity we keep the result of the conjunctive rule and, when new evidence comes in, this result is combined with the new evidence and then one applies the redistribution of the conficiting mass using (DSmH).

$$\underbrace{[m_1 \oplus m_2 \oplus m_3](.)}_{\text{Optimal Fusion}} \neq \underbrace{[(m_1 \oplus m_2) \oplus m_3](.)}_{\text{Suboptimal fusion}} \neq \underbrace{[m_1 \oplus (m_2 \oplus m_3)](.)}_{\text{Suboptimal fusion}} \neq \underbrace{[m_2 \oplus (m_1 \oplus m_3)](.)}_{\text{Suboptimal fusion}}$$

To preserve optimality and coherence of the fusion result, all the sources have to be combined altogether at same fusion level (centralized fusion), not sequentially.

Sequential/decentralized fusion is only suboptimal since part of information is lost during intermediate fusion steps.

Special case : Free DSm model (no constraint)

DSmH reduces to DSmC (i.e. the conjunctive consensus over hyper-power set).

DSmC is commutative and associative on free DSm models whatever values bba's take.

DS rule is commutative and associative but provides counter-intuitive results when the conflict between sources becomes high.

On the refinement of the frame

How to refine ?
Why ?

$$\Theta = \{\theta_1 = \text{Small}, \theta_2 = \text{Tall}\}$$

$$m_1(\theta_1) = 0.4 \qquad m_1(\theta_2) = 0.5 \qquad m_1(\theta_1 \cup \theta_2) = 0.1$$

$$m_2(\theta_1) = 0.6 \qquad m_2(\theta_2) = 0.2 \qquad m_2(\theta_1 \cup \theta_2) = 0.2$$

$$k_{12} = m_1(\theta_1)m_2(\theta_2) + m_2(\theta_1)m_1(\theta_2) = 0.38$$

Case 1: Assume Shafer's model holds

(DS) $m(\emptyset) = 0 \qquad m(\theta_1) = \frac{0.38}{1-0.38} = 0.613 \qquad m(\theta_2) = \frac{0.22}{1-0.38} = 0.355 \qquad m(\theta_1 \cup \theta_2) = \frac{0.02}{1-0.38} = 0.032$

(DSmH) $m(\emptyset) = 0 \qquad m(\theta_1) = 0.38 \qquad m(\theta_2) = 0.22 \qquad m(\theta_1 \cup \theta_2) = 0.02 + 0.38 = 0.40$

DSmH is not equivalent to Dempster's rule (DS)

For this simple 2D static fusion problem, DSmH coincides with Yager's and Dubois & Prade's rules.

Case 2: Assume Shafer's model doesn't hold

because of the continuity and vagueness of elements and their relative interpretation

Possible appraoches: 1) use DSmC with free model, or 2) use DS on a refined frame

On the refinement of the frame (cont'd)

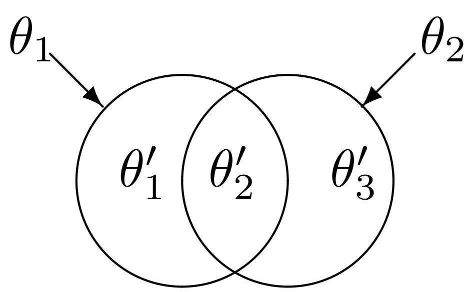
Case 2: Assume Shafer's model doesn't hold

Approach 1: work directly on DSm free model with DSmC

(DSmC)

$m(\emptyset) = 0 \quad m(\theta_1 \cap \theta_2) = 0.38 \quad m(\theta_1) = 0.38 \quad m(\theta_2) = 0.22 \quad m(\theta_1 \cup \theta_2) = 0.02$

Approach 2: refine the frame and see what DS provides



$\theta_1 \cap \theta_2 = \theta'_2$

$\theta_2 = \theta'_2 \cup \theta'_3$

$\theta_1 = \theta'_1 \cup \theta'_2$

$\Theta_{ref} \triangleq \{\theta'_1 = \text{Small'}, \theta'_2 \triangleq \text{Medium}, \theta'_3 = \text{Tall'}\}$

$m'_1(\theta'_1 \cup \theta'_2) = 0.4 \quad m'_1(\theta'_2 \cup \theta'_3) = 0.5 \quad m'_1(\theta'_1 \cup \theta'_2 \cup \theta'_3) = 0.1$

$m'_2(\theta'_1 \cup \theta'_2) = 0.6 \quad m'_2(\theta'_2 \cup \theta'_3) = 0.2 \quad m'_2(\theta'_1 \cup \theta'_2 \cup \theta'_3) = 0.2$

Applying DS rule (there is NO conflict now)

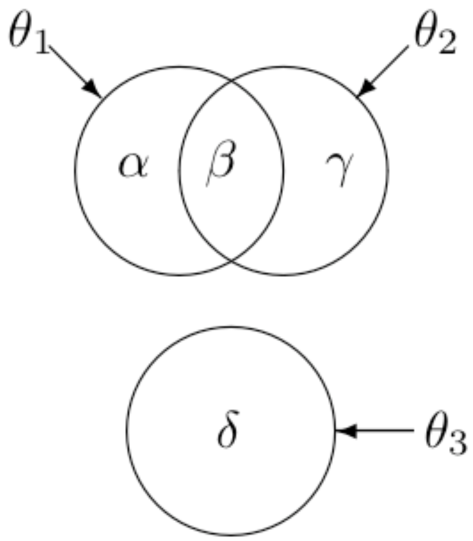
(DS)

$$
\begin{aligned}
m(\emptyset) &= 0 \\
m(\theta'_2) &= m'_1(\theta'_1 \cup \theta'_2)m'_2(\theta'_2 \cup \theta'_3) + m'_2(\theta'_1 \cup \theta'_2)m'_1(\theta'_2 \cup \theta'_3) = 0.38 \\
m(\theta'_1 \cup \theta'_2) &= m'_1(\theta'_1 \cup \theta'_2)m'_2(\theta'_1 \cup \theta'_2) + m'_1(\theta'_1 \cup \theta'_2 \cup \theta'_3)m'_2(\theta'_1 \cup \theta'_2) + m'_2(\theta'_1 \cup \theta'_2 \cup \theta'_3)m'_1(\theta'_1 \cup \theta'_2) = 0.38 \\
m(\theta'_2 \cup \theta'_3) &= m'_1(\theta'_2 \cup \theta'_3)m'_2(\theta'_2 \cup \theta'_3) + m'_1(\theta'_1 \cup \theta'_2 \cup \theta'_3)m'_2(\theta'_2 \cup \theta'_3) + m'_2(\theta'_1 \cup \theta'_2 \cup \theta'_3)m'_1(\theta'_2 \cup \theta'_3) = 0.22 \\
m(\theta'_1 \cup \theta'_2 \cup \theta'_3) &= m'_1(\theta'_1 \cup \theta'_2 \cup \theta'_3)m'_2(\theta'_1 \cup \theta'_2 \cup \theta'_3) = 0.02
\end{aligned}
$$

Thus (DS) reduces to (DSmC) with the necessity and justification (?) of the existence of a possible refinement. It introduces useless complexity w.r.t the direct DSmT formalism.
Just work directly on hyper power set !!!

Example of refinement with hybrid model

$$\Theta = \{\theta_1, \theta_2, \theta_3\}$$



$$\Theta_{\text{ref}} = \{\alpha, \beta, \gamma, \delta\}$$

$m_1(\theta_1) = 0.6$ $\quad m_1(\theta_2) = 0.3$ $\quad m_1(\theta_3) = 0.1$

$m_2(\theta_1) = 0.4$ $\quad m_2(\theta_2) = 0.4$ $\quad m_2(\theta_3) = 0.2$

$$k_{12} = 0.04 + 0.04 + 0.12 + 0.06 = 0.26$$

Conjunctive consensus	$m_1(\theta_1 = \alpha \cup \beta) = 0.6$	$m_1(\theta_2 = \beta \cup \gamma) = 0.3$	$m_1(\theta_3 = \delta) = 0.1$
$m_2(\theta_1 = \alpha \cup \beta) = 0.4$	$0.4 \times 0.6 \to \alpha \cup \beta$	$0.4 \times 0.3 \to \beta$	$0.4 \times 0.1 \to \emptyset$
$m_2(\theta_2 = \beta \cup \gamma) = 0.4$	$0.4 \times 0.6 \to \beta$	$0.4 \times 0.3 \to \beta \cup \gamma$	$0.4 \times 0.1 \to \emptyset$
$m_2(\theta_3 = \delta) = 0.2$	$0.2 \times 0.6 \to \emptyset$	$0.2 \times 0.3 \to \emptyset$	$0.2 \times 0.1 \to \delta$

(DS)

$$m_{DS}(\alpha \cup \beta = \theta_1) = 0.24/(1 - k_{12}) = 0.324324$$
$$m_{DS}(\beta = \theta_1 \cap \theta_2) = 0.36/(1 - k_{12}) = 0.486486$$
$$m_{DS}(\beta \cup \gamma = \theta_2) = 0.12/(1 - k_{12}) = 0.162162$$
$$m_{DS}(\delta = \theta_3) = 0.02/(1 - k_{12}) = 0.027028$$

(DSmH)

$m_{DSmH}(\alpha \cup \beta = \theta_1) = 0.24$ $\quad m_{DSmH}(\delta = \theta_3) = 0.02$

$m_{DSmH}(\beta = \theta_1 \cap \theta_2) = 0.36$ $\quad m_{DSmH}(\theta_1 \cup \theta_3) = 0.16$

$m_{DSmH}(\beta \cup \gamma = \theta_2) = 0.12$ $\quad m_{DSmH}(\theta_2 \cup \theta_3) = 0.10$

(PCR5)

$$m_{PCR5}(\alpha \cup \beta = \theta_1) = 0.362$$
$$m_{PCR5}(\beta = \theta_1 \cap \theta_2) = 0.360$$
$$m_{PCR5}(\beta \cup \gamma = \theta_2) = 0.188$$
$$m_{PCR5}(\delta = \theta_3) = 0.090$$

Conclusion: when working on hybrid models, Dempster's rule applied on refined frame is **different** from DSmT rules (DSmH and PCR5).

Problem with Smets rule (TBM framework)

$\Theta = \{A, B, C\}$

Shafer's model

Sequential/Temporal Fusion

Sequential Fusion of 2 sources

	A	B	C	$\emptyset$	$A \cup B$	$A \cup C$	$B \cup C$
$m_1(.)$	0.4	0	0.6				
$m_2(.)$	0.7	0.3	0				
TBM model $m^{12}_{TBM}(.)$	0.28	0	0	0.72			
Shafer's model $m^{12}_{DS}(.)$	1						
Shafer's model $m^{12}_{DSmH}(.)$	0.28	0	0	0	0.12	0.42	0.18
Shafer's model $m^{12}_{PCR5}(.)$	0.574725	0.111429	0.313846				

In the dynamic fusion suppose that a new source $m_3(.)$ provides the information below. Then one sequentially combines the results obtained by $m^{12}_{TBM}(.)$, $m^{12}_{DS}(.)$, $m^{12}_{DSmH}(.)$ and $m^{12}_{PCR5}(.)$ with $m_3(.)$ and one gets:

	A	B	C	$\emptyset$	$A \cup B$	$A \cup C$	$B \cup C$	$A \cup B \cup C$
$m_3(.)$	0	0.8	0.2					
TBM model $m^{123}_{TBM}(.)$	0	0	0	1	The specificity is lost forever			
Shafer's model $m^{123}_{DS}(.)$	(DS) not working (division by 0)							
Shafer's model $m^{123}_{DSmH}(.)$	0	0.240	0.120	0	0.224	0.056	0	0.360
Shafer's model $m^{123}_{PCR5}(.)$	0.277490	0.545010	0.177500					

If again a fourth, fifth, etc. source provide information and we need to sequentially combine each such source with the previous result one gets for TBM:

$$m_{TBM}(\emptyset) = m^{1234}_{TBM}(\emptyset) = 1 \qquad m_{TBM}(\emptyset) = m^{12345}_{TBM}(\emptyset) = 1 \quad \ldots \quad m_{TBM}(\emptyset) = m^{12\ldots n}_{TBM}(\emptyset) = 1$$

TBM approach does not respond to new information while DSm rules (DSmH and/or PCR5) respond to new information to combine. (DS) is not working at all.

The only ad-hoc solution to overcome this behavior is to introduce some temporal discounting factors and/or avoid to fall into such pathological cases ....

Dynamic versus static fusion of three sources

The masses m1(.),m2(.), m3(.) are those used in the previous example

Dynamic/temporal Fusion

The three sources are combined sequentially

		A	B	C	$\emptyset$	$A \cup B$	$A \cup C$	$B \cup C$	$A \cup B \cup C$
	$m_3(.)$	0	0.8	0.2					
TBM model	$m_{TBM}^{123}(.)$	0	0	0	1	TBM not responding and the specificity is lost			
Shafer's model	$m_{DS}^{123}(.)$	(DS) not working (division by 0)							
Shafer's model	$m_{DSmH}^{123}(.)$	0	0.240	0.120	0	0.224	0.056	0	0.360
Shafer's model	$m_{PCR5}^{123}(.)$	0.277490	0.545010	0.177500					

Dynamic Fusion $\rightarrow [(m_1 \oplus m_2) \oplus m_3](.)$

Static Fusion

The three sources are combined alltogether

		A	B	C	$\emptyset$	$A \cup B$	$A \cup C$	$B \cup C$	$A \cup B \cup C$
	$m_1(.)$	0.4	0	0.6					
	$m_2(.)$	0.7	0.3	0					
	$m_3(.)$	0	0.8	0.2					
TBM model	$m_{TBM}^{123}(.)$	0	0	0	1	TBM not responding and the specificity is lost			
Shafer's model	$m_{DS}^{123}(.)$	(DS) not working (division by 0)							
Shafer's model	$m_{DSmH}^{123}(.)$	0	0	0	0	0.32	0.14	0.18	0.36
Shafer's model	$m_{PCR5}^{123}(.)$	0.345115	0.404783	0.250102					

Static Fusion $\rightarrow [m_1 \oplus m_2 \oplus m_3](.)$

Belief conditioning and Non-Bayesian Reasoning

Approach 1: Following Shafer's idea based on fusion

1) Shafer's "conditioning" rule (SCR)

subjective certainty committed to A by source # 2

$$\mathrm{m}_1(.|A) = [m_1 \oplus m_2](.) \quad \text{with} \quad \begin{cases} m_2(A) = 1 \\ \oplus = \text{Dempster's rule} \end{cases}$$

SCR = Bayesian reasoning with plausibilities

2) PCR5 conditioning rule (PCR5CR) [Smarandache Dezert, Brest 2010]

We replace Dempster rule by PCR5 fusion rule

PCR5CR = Non Bayesian reasoning (NBR)

Approach 2: Direct Belief Conditioning Rules (BCR)

Approach 1 (based on fusion)

1) Extension of Bayesian Reasoning (Shafer's cond.)

$$m(X|Y) = m_{DS}(X) = [m_1 \oplus m_2](X) \quad \text{with} \quad m_2(Y) = 1$$

$$Bel(X|Y) = \sum_{\substack{Z \in 2^\Theta \\ Z \subseteq X}} m_{DS}(Z|Y) = \frac{Bel_1(X \cup \bar{Y}) - Bel_1(\bar{Y})}{1 - Bel_1(\bar{Y})}$$

$$Bel(X|Y) \leq P(X|Y) \leq Pl(X|Y)$$

$$Pl(X|Y) = \sum_{\substack{Z \in 2^\Theta \\ Z \cap X \neq \emptyset}} m_{DS}(Z|Y) = \frac{Pl_1(X \cap Y)}{Pl_1(Y)}$$

Consistency with Bayes formula using Bayesian bba's:

$$P(X|Y) = \frac{P(X \cap Y)}{P(Y)}$$

Bayesian principle:

When $Y = X$ and as soon as $Bel(\bar{X}) < 1$, one gets $Bel(X|X) = 1$ because $Bel_1(X \cup \bar{Y}) = Bel_1(X \cup \bar{X}) = Bel_1(\Theta) = 1$. For Bayesian belief, this implies $P(X|X) = 1$ for any X such that $P_1(X) > 0$.

Property of DS rule: $\text{Bayesian} \oplus \text{Non-Bayesian} = \text{Bayesian}$

Approach 1 (based on fusion)

2) Non Bayesian Reasoning (NBR or PCR5CR)

Principle: $\mathrm{m}_1(.|A) = [m_1 \oplus m_2](.)$ with $\begin{cases} m_2(A) = 1 \\ \oplus = \text{PCR5 Fusion rule} \end{cases}$

Result:

$$m(X \parallel Y) = \sum_{\substack{X_1 \in 2^\Theta \\ X_1 \cap Y = X}} m_1(X_1) + \delta(X \cap Y = \emptyset) \cdot \frac{m_1(X)^2}{1 + m_1(X)} + \delta(X = Y) \cdot \sum_{\substack{X_2 \in 2^\Theta \\ X_2 \cap Y = \emptyset}} \frac{m_1(X_2)}{1 + m_1(X_2)}$$

$$Bel(X \parallel Y) = \sum_{\substack{Z \in 2^\Theta \\ Z \subseteq X}} m(Z \parallel Y)$$

$$Bel(X \parallel Y) \leq P(X||Y) \leq Pl(X||Y)$$

$$Pl(X \parallel Y) = \sum_{\substack{Z \in 2^\Theta \\ Z \cap X \neq \emptyset}} m(Z \parallel Y)$$

This conditioning is truly Non-Bayesian since Bel(Y||Y) ≤ 1

Property of PCR5 rule: $\text{Bayesian} \oplus \text{Non-Bayesian} = \text{Non-Bayesian (in general)}$

Deconditioning: It is the inverse (dual) problem of conditioning. It consists to retrieve the prior belief function from a given posterior/conditional belief function. Useful for revising/reconditioning knowledge w.r.t. other conditional hypothesis. More simply stated, we want to see if for any given conditional bba m(.||Y), we can compute m_1(.) such that m(.||Y)=PCR5(m_1(.),m_2(.) with m_2(Y)=1.

Example of NBR with Bayesian prior

Example 1: **with Bayesian prior** $\Theta = \{A, B, C\}$ $Y = A \cup B$

Prior Bayesian bba's

Focal Elem.	m_1	$m'_1(.)$
A	0.49	0.01
B	0.49	0.01
C	0.02	0.98

Shafer's conditioning

Focal Elem.	$m(.\|Y)$	$m'(.\|Y)$
A	0.5	0.5
B	0.5	0.5
C	0	0
$A \cup B$	0	0

$Bel(Y|Y) = 1$ $Bel'(Y|Y) = 1$

Non Bayesian conditioning

Focal Elem.	$m(.\|\|Y)$	$m'(.\|\|Y)$
A	0.4900	0.0100
B	0.4900	0.0100
C	0.00039215	0.48505051
$A \cup B$	0.01960785	0.49494949

$Bel(Y||Y) = 0.99960785 < 1$

$Bel'(Y||Y) = 0,51494949 < 1$

Unique deconditioning of PCR5 conditioning is possible (see paper) contrariwise to SCR

2^Θ	$\Delta(.\|Y) = \Delta'(.\|Y)$	$\Delta(.\|\|Y)$	$\Delta'(.\|\|Y)$
$\emptyset$	[0,0]	[0,0]	[0,0]
A	[0.5,0.5]	[0.4900, 0.5096]	[0.0100, 0.5050]
B	[0.5,0.5]	[0.4900, 0.5096]	[0.0100, 0.5050]
C	[0,0]	[0.0004, 0.0004]	[0.4850,0.4850]
$Y = A \cup B$	[1,1]	[0.9996,0.9996]	[0.5150,0.5150]
$A \cup C$	[0.5,0.5]	[0.4904, 0.5100]	[0.4950, 0.9900]
$B \cup C$	[0.5,0.5]	[0.4904, 0.5100]	[0.4950, 0.9900]
$A \cup B \cup C$	[1,1]	[1,1]	[1,1]

$$\Delta(.|Y) = [Bel(.|Y), Pl(.|Y)]$$

see Smarandache-Dezert, Brest 2010 paper for details

Example of NBR with NON-Bayesian prior

Example 2: **with Non-Bayesian prior** $\Theta = \{A, B, C\}$ $Y = A \cup B$

Prior bba's

Focal Elem.	m_1	$m'_1(.)$
A	0.20	0.20
B	0.30	0.30
C	0.10	0.10
$A \cup B$	0.25	0.15
$A \cup B \cup C$	0.15	0.25

Shafer conditioning

Focal Elem.	$m(.\|Y)$	$m'(.\|Y)$
A	0.222	0.222
B	0.333	0.333
C	0	0
$A \cup B$	0.445	0.445

$Bel(Y|Y) = 1$ $Bel'(Y|Y) = 1$

Non Bayesian conditioning

Focal Elem.	$m(. \parallel Y)$	$m'(. \parallel Y)$
A	0.20	0.20
B	0.30	0.30
C	0.01	0.01
$A \cup B$	0.49	0.49

$Bel(Y||Y) = Bel'(Y||Y) = 0.99 < 1$

Unique deconditioning of PCR5 conditioning is not possible in general with non Bayesian prior bba, unless additional constraints are introduced.

2^Θ	$\Delta(.\|Y) = \Delta'(.\|Y)$	$\Delta(.\|\|Y) = \Delta'(.\|\|Y)$
$\emptyset$	[0,0]	[0,0]
A	[0.2220,0.6670]	[0.2000,0.6900]
B	[0.3330,0.7780]	[0.3000,0.7900]
C	[0,0]	[0.0100,0.0100]
$Y = A \cup B$	[1,1]	[0.9900,0.9900]
$A \cup C$	[0.2220,0.6670]	[0.2100,0.7000]
$B \cup C$	[0.3330,0.7780]	[0.3100,0.8000]
$A \cup B \cup C$	[1,1]	[1,1]

$$\Delta(.|Y) = [Bel(.|Y), Pl(.|Y)]$$

see Smarandache-Dezert, Brest 2010 paper for details

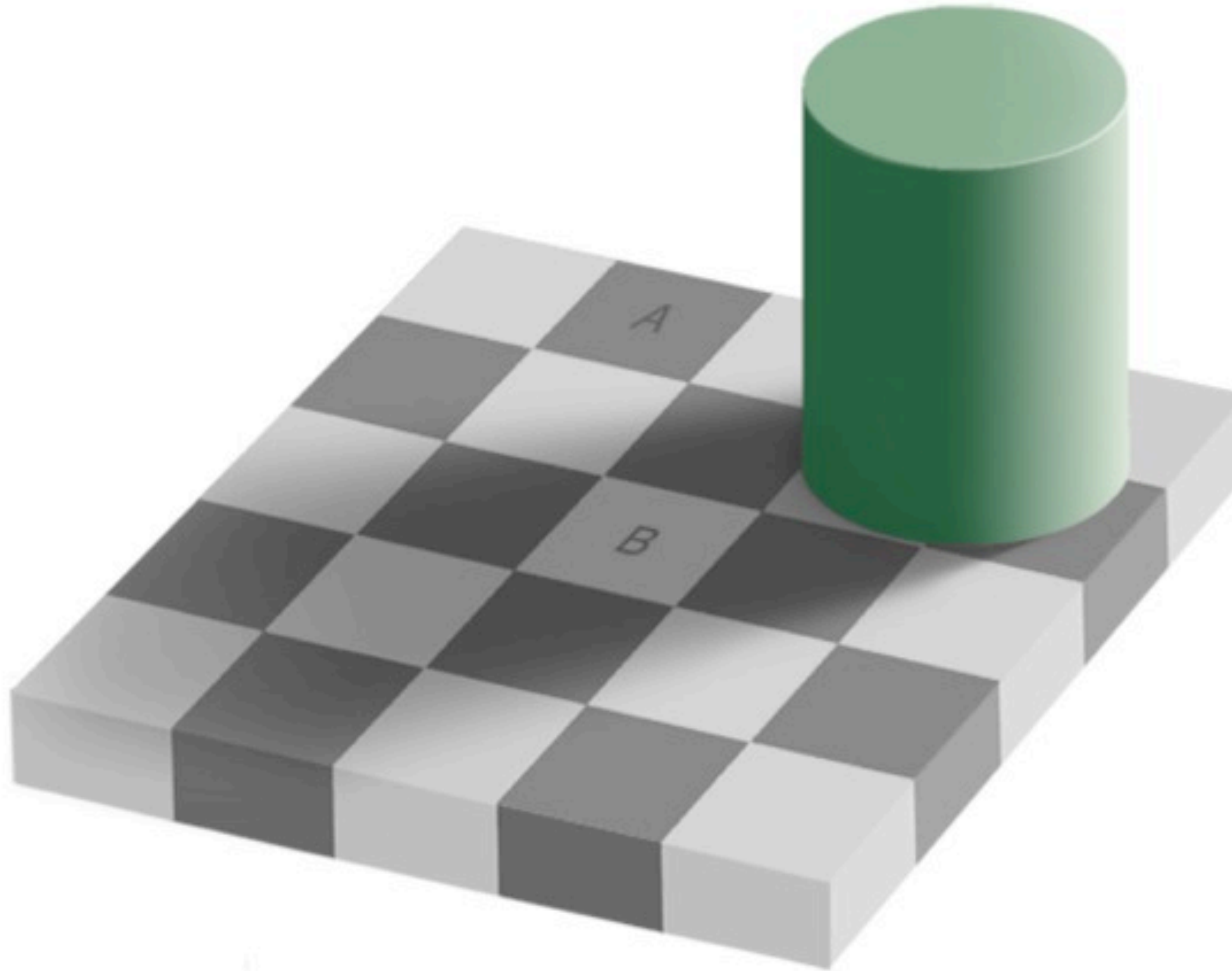
Approach 2 : Direct Belief Conditioning Rules (BCR)

Justification : One makes a clear and fundamental distinction between fusion of a prior bba $m_1(.)$ with a source focused on a given set A (Shafer's approach) and belief revision conditioned by the fact that absolute truth is in A (BCRs approach).

To compute $m_1(.|A)$, and because the conditioning event A contains the absolute truth, one proposes to revise the prior bba $m_1(.)$ based on NEW mass transfer, but NOT based on the fusion of $m_1(.)$ with specialized bba $m_2(A)=1$. Many BCRs (BCR1-31) have been recently developed.

BCR12 and BCR17 seems to be the most appealing so far (see justification in next slides).

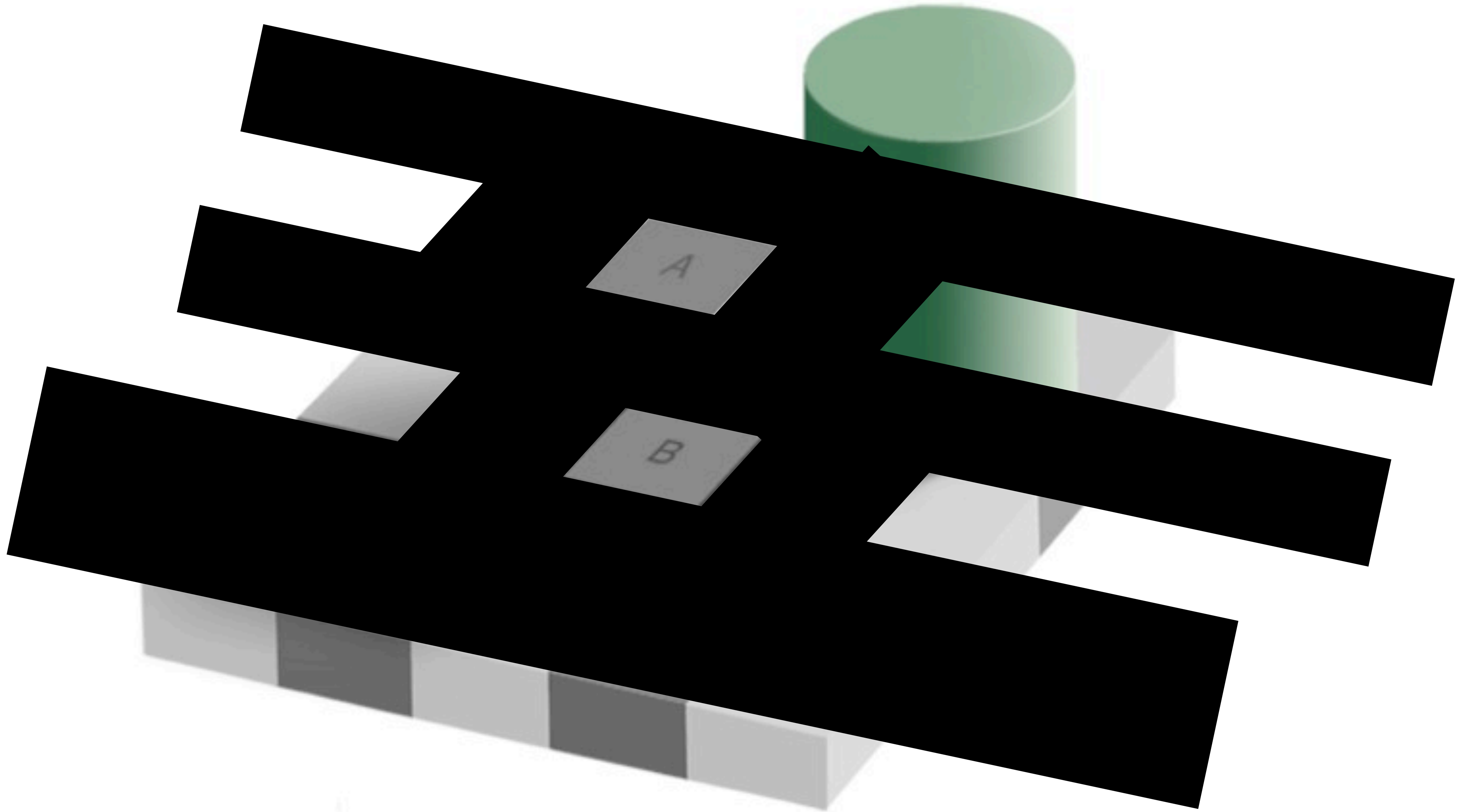
Example: visual perception and subjective certainty



Question: Is the color of squares A and B the same or different ?

Credit: Example borrowed from Edward H. Adelson

Let’s check



Conclusion:

Subjective certainty ≠ Objective (i.e. absolute) certainty

Hyper-power set decomposition (HPSD)

BCRs are based on a particular hyper-power set decomposition imposed by the conditioning event, say A.

$$\boxed{D^{\Theta} \setminus \{\emptyset\} = D_1 \cup D_2 \cup D_3}$$

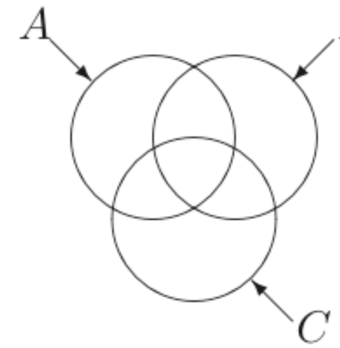
- $D_1 \triangleq \mathcal{P}_{\mathcal{D}}(A) = 2^A \cap D^{\Theta} \setminus \{\emptyset\} =$ all non-empty parts of D^{Θ} which are included in A;
- $D_2 \triangleq \{(\Theta \setminus s(A)), \cup, \cap\} \setminus \{\emptyset\} =$ the sub-hyper-power set generated by $\Theta \setminus s(A)$ under $\cup$ and $\cap$, without the empty set.
- $D_3 \triangleq (D^{\Theta} \setminus \{\emptyset\}) \setminus (D_1 \cup D_2)$.

where $s(A) = \{\theta_{i_1}, \theta_{i_2}, \ldots, \theta_{i_p}\}$, $1 \leq p \leq n$, be the singletons/atoms that compose A.

Example: if $A = \theta_1 \cup (\theta_3 \cap \theta_4)$ then $s(A) = \{\theta_1, \theta_3, \theta_4\}$.

The masses of D_2 and D_3 elements are redistributed to D_1 non-empty elements according to many ways (i.e.BCR1-BCR31)

Examples of HPSD



Let's consider $\Theta = \{A, B, C\}$ and the free DSm model.

Example 1 : If the truth is in A

$D_1 = \{A, A\cap B, A\cap C, A\cap B\cap C\} \equiv \mathcal{P}(A)\cap(D^\Theta \setminus \emptyset)$

$D_2 = (\{B, C\}, \cup, \cap) = D^{\{B,C\}} = \{B, C, B\cup C, B\cap C\}$

$D_3 = \{A\cup B, A\cup C, A\cup B\cup C, A\cup(B\cap C)\}$

Example 2 : If the truth is in $A\cap B$

$D_2 = \{C\}$

$D1 = \{A\cap B, A\cap B\cap C\}$

$D_3 = \{A, B, A\cup B, A\cap C, B\cap C, \ldots\} = (D^\Theta \setminus \{\emptyset\}) \setminus (D_1 \cup D_2)$

Example 3 : If the truth is in $A\cup B$

$D_1 = \{A, B, A\cap B, A\cup B, \ldots\}$

all other sets included in these four ones, i.e.
$A\cap C$, $B\cap C$, $A\cap B\cap C$, $A\cup(B\cap C)$, $B\cup(A\cap C)$, etc.

$D_2 = \{C\}$

$D_3 = \{A\cup C, B\cup C, A\cup B\cup C, C\cup(A\cap B)\}$

Example 4 : If the truth is in $A\cup B\cup C$ $D_1 = D^\Theta \setminus \{\emptyset\}$ D_2 and D_3 do not exist.

BCR #17

BCR17 does the most refined/precise redistribution among all possible BCR, i.e.

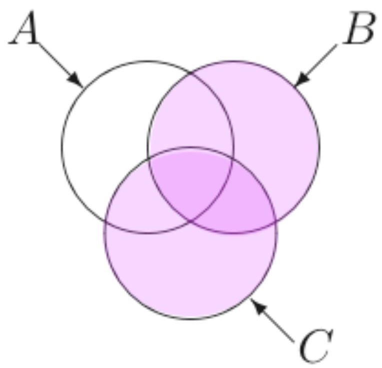
- the mass m(W) of each element W in $D_2 \cup D_3$ is transferred to the elements X in D_1 which are included in W (if any) proportionally with respect to their non-empty masses;

- if no such X exists, the mass m(W) is transferred in a pessimistic/prudent way to the k-largest elements from D_1 which are included in W (in equal parts) if any;

- if neither this way is possible, then m(W) is indiscriminately distributed to all X in D_1 proportionally with respect to their nonzero masses.

$$m_{BCR17}(X|A) = m(X) \cdot \left[\Big[\sum_{\substack{Z \in D_1, \\ \text{or } Z \in D_2 \,\mid\, \nexists Y \in D_1 \text{ with } Y \subset Z}} m(Z) \Big] / \sum_{Y \in D_1} m(Y) + \sum_{\substack{W \in D_2 \cup D_3 \\ X \subset W \\ S(W) \neq 0}} \frac{m(W)}{S(W)} \right] + \sum_{\substack{W \in D_2 \cup D_3 \\ X \subset W, \text{ X is } k\text{-largest} \\ S(W) = 0}} m(W)/k$$

Example #1 for BCR17



$\Theta = \{A, B, C\}$ **free DSm model** with **non-Bayesian** bba

$m(A) = 0.2$ $m(B) = 0.1$ $m(C) = 0.2$

$m(A \cap B) = 0.1$ $m(A \cup B) = 0.1$ $m(B \cup C) = 0.1$

$m(A \cup (B \cap C)) = 0.1$ $m(A \cup B \cup C) = 0.1$

Let's assume that the truth is in $B \cup C$, i.e. the conditioning term is $B \cup C$

HPSD:

$D_1 = \{A \cap B \cap C, B \cap C\ A \cap B, A \cap C, (A \cap B) \cup (B \cap C), (B \cap C) \cup (A \cap C), (A \cap B) \cup (A \cap C)$
$(A \cap B) \cup (A \cap C) \cup (B \cap C), B, C, (A \cap C) \cup B, (A \cap B) \cup C, B \cup C\}$

$D_2 = \{A\}$ $D_3 = \{A \cup (B \cap C), A \cup B, A \cup C, A \cup B \cup C\}$

$m(A \cup C) = 0$

BCR17 conditioning:

For D_2: $m(A) = 0.2$, where $A \in D_2$, is transferred to $B \cap A$ since $B \cap A \subset A$ and $m(B \cap A) > 0$.

For D_3: $m(A \cup B) = 0.1$ is transferred to B and $B \cap A$ since these are the only D_1 elements included in $A \cup B$ whose masses are non-zero, proportionally to their corresponding masses, i.e.

$\frac{x_B}{0.1} = \frac{w_{B \cap A}}{0.1} = \frac{0.1}{0.2} = 0.5$ whence $x_B = 0.05$ and $w_{B \cap A} = 0.05$.

$m(A \cup B \cup C) = 0.1$ is transferred to B, C, $B \cap A$, $B \cup C$, i.e. $\frac{x_B}{0.1} = \frac{y_C}{0.2} = \frac{z_{B \cup C}}{0.1} = \frac{w_{B \cap A}}{0.1} = \frac{0.1}{0.5} = 0.2$

whence $x_B = 0.02$, $y_C = 0.04$, $z_{B \cup C} = 0.02$ and $w_{B \cap A} = 0.02$.

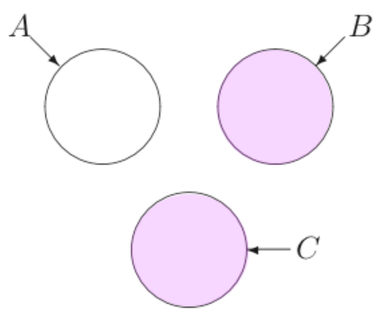
$m(A \cup (B \cap C)) = 0.1$ is transferred to $B \cap A$ only since no other D_1 element with non-zero mass is included in $A \cup (B \cap C)$.

BCR17 result:

$m_{BCR17}(B|B \cup C) = 0.10 + 0.05 + 0.02 = 0.17$ $m_{BCR17}(B \cup C|B \cup C) = 0.10 + 0.02 = 0.12$

$m_{BCR17}(C|B \cup C) = 0.20 + 0.04 = 0.24$ $m_{BCR17}(B \cap A|B \cup C) = 0.1 + 0.2 + 0.05 + 0.02 + 0.1 = 0.47$

Example #2 for BCR17



$\Theta = \{A, B, C\}$ **Shafer's model** with **non-Bayesian** bba

$m(A) = 0.2 \quad m(B) = 0.1 \quad m(C) = 0.2$

$m(A \cup B) = 0.1 \quad m(B \cup C) = 0.1 \quad m(A \cup B \cup C) = 0.3$

Let's assume as conditioning constraint that the truth is in $B \cup C$.

HPSD: $D_1 = \{B, C, B \cup C\}$ $D_2 = \{A\}$ $D_3 = \{A \cup (B \cap C), A \cup B, A \cup C, A \cup B \cup C\}$

BCR17 conditioning:

For D_2, $m(A) = 0.2$ is transferred proportionally to all elements of D_1, i.e. $\frac{x_B}{0.1} = \frac{y_C}{0.2} = \frac{z_{B\cup C}}{0.1} = \frac{0.2}{0.4} = 0.5$ whence $x_B = 0.05$, $y_C = 0.10$, and $z_{B\cup C} = 0.05$.

For D_3, $m(A \cup B) = 0.1$ is transferred to B (no case of k-elements herein);

$m(A \cup B \cup C) = 0.3$ is transferred to B, C, $B \cup C$ proportionally to their corresponding masses:

$\frac{x_B}{0.1} = \frac{y_C}{0.2} = \frac{z_{B\cup C}}{0.1} = \frac{0.3}{0.4} = 0.75$ whence $x_B = 0.075$, $y_C = 0.15$, and $z_{B\cup C} = 0.075$.

Result with BCR17

$m_{BCR17}(B|B \cup C) = 0.10 + 0.05 + 0.10 + 0.075 = 0.325$

$m_{BCR17}(C|B \cup C) = 0.2 + 0.10 + 0.15 = 0.450$

$m_{BCR17}(B \cup C|B \cup C) = 0.10 + 0.05 + 0.075 = 0.225$

$\neq$

Result with SCR

$m_{SCR}(B|B \cup C) = 0.25$

$m_{SCR}(C|B \cup C) = 0.25$

$m_{SCR}(B \cup C|B \cup C) = 0.50$

Belief Conditioning Rule #12

$$m_{BCR12}(X|A) = \big[m(X) \cdot \sum_{\substack{Z\in D_1, \\ \text{or } Z\in D_2 \,|\, \nexists Y\in D_1 \text{ with } Y\subset Z}} m(Z)\big] / \sum_{Y\in D_1} m(Y)$$

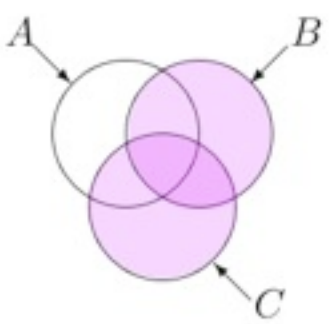
$$+ \sum_{\substack{W\in D_2\cup D_3 \\ X\subset W, \text{X is } k\text{-largest}}} m(W)/k$$

BCR12 does the most pessimistic/prudent redistribution among all possible BCR:

- the mass m(W) of each W in $D_2 U D_3$ is transferred in a pessimistic/prudent way to the k-largest elements X from D_1 which are included in W (in equal parts) if any;
- if this way is not possible, then m(W) is indiscriminately distributed to all X from D_1 proportionally with respect their nonzero masses.

BCR12 can be regarded as a generalization of SCR from the power set to the hyper-power set in the free DSm free model (all intersections non-empty). In this case the result of BCR12 is equal to that of $m_1(.)$ combined with $m_2(A)=1$, when the truth is in A, using (DSmC).

Example #1 for BCR12



$\Theta = \{A, B, C\}$ **free DSm model** with **non-Bayesian** bba

$m(A) = 0.2$ $m(B) = 0.1$ $m(C) = 0.2$

$m(A \cap B) = 0.1$ $m(A \cup B) = 0.1$ $m(B \cup C) = 0.1$

$m(A \cup (B \cap C)) = 0.1$ $m(A \cup B \cup C) = 0.1$

Let's assume that the truth is in $B \cup C$, i.e. the conditioning term is $B \cup C$

HPSD:

$D_1 = \{A \cap B \cap C, B \cap C\ A \cap B, A \cap C, (A \cap B) \cup (B \cap C), (B \cap C) \cup (A \cap C), (A \cap B) \cup (A \cap C),$
$(A \cap B) \cup (A \cap C) \cup (B \cap C), B, C, (A \cap C) \cup B, (A \cap B) \cup C, B \cup C\}$

$D_2 = \{A\}$ $D_3 = \{A \cup (B \cap C), A \cup B, A \cup C, A \cup B \cup C\}$

$m(A \cup C) = 0$

BCR12 conditioning:

$m(A) = 0.2$ is transferred to $(A \cap B) \cup (A \cap C)$ since it is the 1-largest element of D_1 included in A.

$m(A \cup (B \cap C)) = 0.1$ is transferred to $(A \cap B) \cup (A \cap C) \cup (B \cap C)$ since it is the 1-largest element of D_1 included in $A \cup (B \cap C)$.

$m(A \cup B) = 0.1$ is transferred to $(A \cap C) \cup B$ since it is the 1-largest element of D_1 included in $A \cup B$.

$m(A \cup B \cup C) = 0.1$ is transferred to $B \cup C$ since it is the 1-largest element of D_1 included in $A \cup B \cup C$.

BCR12 result:

$m_{BCR12}((A \cap B) \cup (A \cap C) \mid B \cup C) = 0.2$

$m_{BCR12}((A \cap B) \cup (A \cap C) \cup (B \cap C) \mid B \cup C) = 0.1$

$m_{BCR12}((A \cap C) \cup B \mid B \cup C) = 0.1$

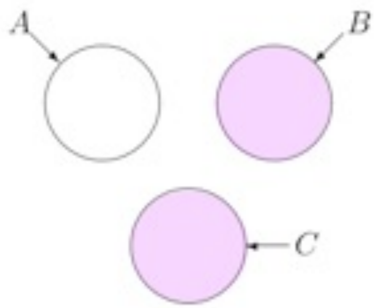
$m_{BCR12}(B \cup C \mid B \cup C) = 0.1 + 0.1 = 0.2$

$m_{BCR12}(B \mid B \cup C) = 0.1$

$m_{BCR12}(C \mid B \cup C) = 0.2$

$m_{BCR12}(A \cap B \mid B \cup C) = 0.1$

Example #2 for BCR12



$\Theta = \{A, B, C\}$ **Shafer's model** with **non-Bayesian** bba

$m(A) = 0.2 \quad m(B) = 0.1 \quad m(C) = 0.2$

$m(A \cup B) = 0.1 \quad m(B \cup C) = 0.1 \quad m(A \cup B \cup C) = 0.3$

Let's assume as conditioning constraint that the truth is in $B \cup C$.

HPSD: $D_1 = \{B, C, B \cup C\}$ $D_2 = \{A\}$ $D_3 = \{A \cup (B \cap C), A \cup B, A \cup C, A \cup B \cup C\}$

BCR12 conditioning:

$m(A) = 0.2$ is distributed to B, C and $B \cup C$ proportionally to their corresponding masses, i.e.

$$\frac{x_B}{0.1} = \frac{y_C}{0.2} = \frac{z_{B \cup C}}{0.1} = \frac{0.2}{0.1 + 0.2 + 0.1} = 0.5$$

(0.2 ← $m(A)$)

whence $x_B = 0.05$, $y_C = 0.10$ and $z_{B \cup C} = 0.05$.

$m(A \cup B) = 0.1$ is transferred to B, i.e. the 1-largest element of D_1 included in $A \cup B$.

$m(A \cup B \cup C) = 0.3$ is transferred to $B \cup C$, i.e. the 1-largest element of D_1 included in $A \cup B \cup C$.

Result with BCR12

$m_{BCR12}(B \mid B \cup C) = 0.1 + 0.1 + 0.05 = 0.25$

$m_{BCR12}(C \mid B \cup C) = 0.20 + 0.10 = 0.30$

$m_{BCR12}(B \cup C \mid B \cup C) = 0.1 + 0.05 + 0.3 = 0.45$

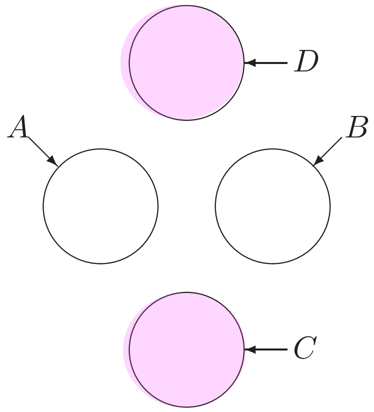
$\neq$

Result with SCR

$m_{SCR}(B \mid B \cup C) = 0.25$

$m_{SCR}(C \mid B \cup C) = 0.25$

$m_{SCR}(B \cup C \mid B \cup C) = 0.50$



Example #3 for BCR12

$\Theta = \{A, B, C, D\}$ **Shafer's model** with **Bayesian** bba

$m_1(A) = 0.4 \quad m_1(B) = 0.1 \quad m_1(C) = 0.2 \quad m_1(D) = 0.3$

Let's assume that one finds out that the truth is in $C \cup D$.

Actually we get same Result with all BCR

	A	B	C	D
$m_1(.)$	0.4	0.1	0.2	0.3
$m_{BCR1-31}(.\vert C \cup D)$	0	0	0.40	0.60

Result with SCR, based on Dempster's, DSmH and PCR5 fusion rules

	A	B	C	D	$C \cup D$	$A \cup C \cup D$	$B \cup C \cup D$
$m_{DS}(.\vert C \cup D)$	0	0	0.40	0.60	0	0	0
$m_{DSmH}(.\vert C \cup D)$	0	0	0.20	0.30	0	0.40	0.10
$m_{PCR5}(.\vert C \cup D)$	0.114286	0.009091	0.20	0.30	0.376623	0	0

Open questions

SCR and Dempster's combination rules commute because SCR is based on Dempster's rule and Dempster's rule is associative, but SCR is a special case of fusion, not a real conditioning dealing with absolute truth.

In general (but in Shafer's model with Bayesian bba's), BCRs do not commute with fusion operators, i.e.

$$m_{CF}(.|A) \neq m_{FC}(.|A)$$

$$m_{CF}(.|A) = \underbrace{Cond(m_1(.))}_{m_1(.|A)} \oplus \underbrace{Cond(m_2(.))}_{m_2(.|A)}$$

$$m_{FC}(.|A) = Cond(m_1(.) \oplus m_2(.))$$

Q1: How to compute $m(.|A)$ from $m_1(.)$ and $m_2(.)$?

Q2: How to justify if $m(.|A)=m_{FC}(.|A)$ or if $m(.|A)=m_{CF}(.|A)$?