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SEMIGROUPS AS GRAPHS

Semigroup as Graphs

W. B. Vasantha Kandasamy
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ZIP PUBLISHING
Ohio
2012

This book can be ordered from:

Zip Publishing
1313 Chesapeake Ave.
Columbus, Ohio 43212, USA
Toll Free: (614) 485-0721
E-mail: info@zippublishing.com
Website: www.zippublishing.com

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ISBN-13: 978-1-59973-191-9
EAN: 9781599731919

Printed in the United States of America

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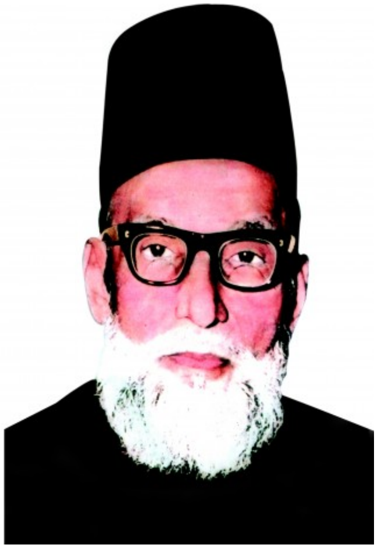
DEDICATION



QUAID-E-MILLETH MUHAMMAD ISMAIL
(1896-1972)

We dedicate this book to the Indian politician Quaid-e-Milleth Muhammad Ismail who was the backbone of the political evolution of Muslims in South India. Hailing from Tirunelveli in Tamil Nadu, he made his mark on the national scene following the split of the Muslim League. Even as he recognized and celebrated the need for sovereignty for the Muslims, he was also keen to earn them their rightful place in the Indian state. The Muslims who decided to stay back in India and not migrate to Pakistan formed the Indian Union Muslim League. As its first President Quaid-e-Milleth Sahib worked courageously and helped to consolidate the future of Muslims in India. He was immensely popular among the people of Tamil Nadu and Kerala, and we pay homage to his efforts for peace and mutual understanding, through this simple dedication.

PREFACE

In this book the authors study the zero divisor graph and unit graph of a semigroup.

The zero divisor graphs of semigroups Z_n under multiplication is studied and characterized. The zero divisor graphs of the semigroups $(Z_p, \times)$, p a prime are only trees with p vertices. However the zero divisor graph of $(Z_n, \times)$, n a non prime is not a tree. We also define the special zero divisor graph of a semigroup and the S-zero divisor of a semigroup.

We introduce a new notion called the tree covering pseudo lattice. We see the zero divisor graph of Z_{2p}, p a prime is a tree covering pseudo lattice where as the zero divisor graph of Z_{2n}, n a composite number is not a tree covering pseudo lattice.

This study is carried out in chapter two of this book. Chapter one is introductory in nature to make this book a self contained one.

In chapter three we study the unit graphs of semigroups. The unit graphs of the semigroups Z_n and S(n) are described. Further unit graphs of the finite complex modulo integers $C(Z_n)$ are analysed.

The S-unit graph of Z_p, p a prime is a complete graph with (p – 1) vertices. In the fourth chapter we suggest over 50 problems.

We thank Dr. K.Kandasamy for proof reading and being extremely supportive.

W.B.VASANTHA KANDASAMY
FLORENTIN SMARANDACHE

Chapter One

BASIC CONCEPTS

In this chapter we introduce some basic concepts about semigroups and graphs. These concepts are mainly given here to make the book a self contained one. This chapter has two sections. Section one gives the types of finite semigroups used in this book. In section two some basic notion about graphs are recalled.

1.1 Semigroups

In this section we introduce the notion of finite semigroups and describe their properties.

A non empty set S is said to be a semigroup if on S there is defined a closed binary operation '*' which is also associative. That is for a, b $\in$ S we have a * b $\in$ S and a * (b*c) = (a*b) * c for all a, b, c $\in$ S. If in S we have an element e such that a * e = e * a = a for all a $\in$ S then we say S is a semigroup with unit or a monoid.

If in S, '*' is defined in such a way that a * b = b * a for all a, b $\in$ S then we call (S, *) to be a commutative semigroup.

We will give examples of them.

Example 1.1.1: Let $Z_6 = \{0, 1, 2, 3, \ldots, 5\}$; Z_6 under modulo multiplication is a semigroup.

Example 1.1.2: Let $Z_{17} = \{0, 1, 2, 3, \ldots, 16\}$ be a semigroup under multiplication modulo 17.

Example 1.1.3: Let $P = Z_4 \times Z_4 \times Z_4 = \{(a, b, c) \mid a, b, c \in Z_4\}$ be a semigroup under product. That is if (3, 1, 0) and (2, 3, 1) are in P, then $(3, 1, 0) \times (2, 3, 1) = (2, 3, 0)$. The $\times$ is a modulo multiplication four.

Example 1.1.4: Let

$$M = \{Z_7 \times Z_7 \times Z_7\} = \{(a, b, c) \mid a, b, c \in Z_7\};$$

M is a semigroup under product.

We would be considering semigroups only under multiplication as our study pertains to graphs and graphs are of two types zero divisor graphs of the semigroup and the unit graphs of the semigroup.

Suppose $(S, *)$ is a semigroup with 0. If in S we have $a, b \in S \setminus \{0\}$ such that $a * b = 0$ then we say $(S, *)$ has zero divisors. If the semigroup S is non commutative we may have in S right zero divisors which are not left zero divisors and vice versa.

Let $(S, *)$ be a monoid. We say an element $a \in S \setminus \{1, 0\}$ is a unit if there exists a $b \in S \setminus \{1, 0\}$ such that $a * b = 1$.

We will illustrate these situations by some examples.

Example 1.1.5: Let $S = Z_{14} = \{0, 1, 2, \ldots, 13\}$ be the semigroup under multiplication modulo 14.

$$2 \times 7 \equiv 0 \pmod{14}$$
$$4 \times 7 \equiv 0 \pmod{14}$$
$$6 \times 7 \equiv 0 \pmod{14}$$

$$8 \times 7 \equiv 0 \pmod{14} \text{ and}$$
$$10 \times 7 \equiv 0 \pmod{14}.$$

We see this semigroup also has units; for

$$3.5 \equiv 1 \pmod{14} \text{ and}$$
$$13.13 \equiv 1 \pmod{14} \text{ are units in S.}$$

Example 1.1.6: Let S = Z_{18} = {0, 1, 2, 3, …, 17} be a semigroup under modulo multiplication.

Consider
$$\begin{aligned} 2.9 &\equiv 0 \pmod{18} \\ 3.6 &\equiv 0 \pmod{18} \\ 3.12 &\equiv 0 \pmod{18} \\ 4.9 &\equiv 0 \pmod{18} \\ 6.6 &\equiv 0 \pmod{18} \\ 6.12 &\equiv 0 \pmod{18} \\ 8.9 &\equiv 0 \pmod{18} \\ 9.6 &\equiv 0 \pmod{18} \\ 9.12 &\equiv 0 \pmod{18} \\ 10.9 &\equiv 0 \pmod{18} \end{aligned}$$

$12 \times 9 \equiv 0 \pmod{18}$ and so on are zero divisors in S = Z_{18}.

Now S = Z_{18} has also units given in the following.

$$17 \times 17 \equiv 1 \pmod{18}$$
$$11 \times 5 \equiv 1 \pmod{18} \quad \text{and}$$
$$13 \times 7 \equiv 1 \pmod{18}.$$

Thus we have semigroups which has units and zero divisors.

Now we can have units and zero divisors in case of complex modulo integer semigroups $C(Z_n)$ under multiplication [16].

Recall $C(Z_n) = \{a + bi_F \mid a, b \in Z_n;\ i_F^2 = n-1\}$.

Example 1.1.7: Let $C(Z_4) = \{a + bi_F \mid a, b \in Z_4;\ i_F^2 = 3\}$ be a semigroup under product.

$$\begin{aligned} \text{Consider}\quad & 2 \times 2 \equiv 0 \pmod 4 \\ & 2 \times (2 + 2i_F) \equiv 0 \pmod 4 \\ & 2i_F \times (2+2i_F) \equiv 0 \pmod 4 \\ & (2 + 2i_F) \times (2+2i_F) \equiv 0 \pmod 4 \\ & 3i_F \times i_F = 1 \pmod 4 \\ & \text{and } 3 \times 3 \equiv 1 \pmod 4. \end{aligned}$$

Thus this finite complex modulo integer semigroup has both units and zero divisors.

Example 1.1.8: Let $C(Z_6) = \{a + bi_F \mid i_F^2 = 5, a, b \in Z_6\}$ be a complex modulo integer semigroup. Clearly $C(Z_6)$ has both units and zero divisors.

Example 1.1.9: Let $C(Z_7) = \{a + bi_F \mid a, b \in Z_7,\ i_F^2 = 6\}$

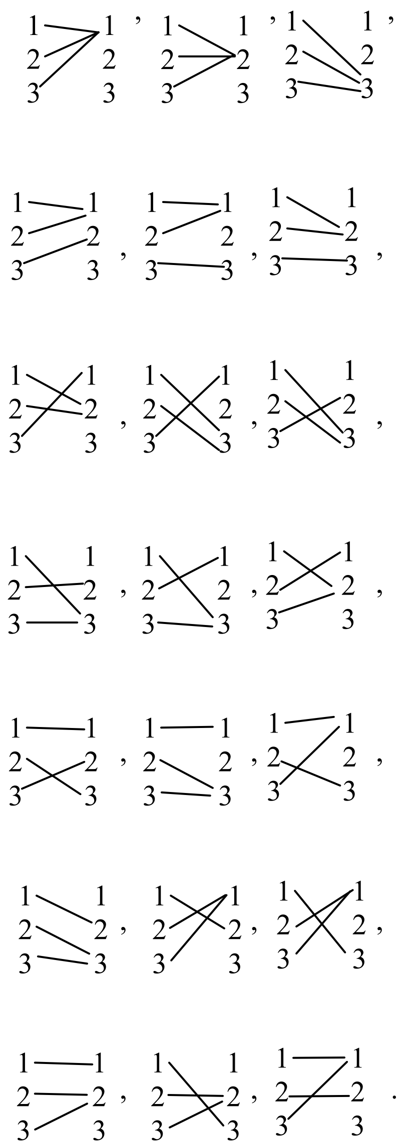
$$\begin{aligned} 6 \times 6 &\equiv 1 \pmod 7 \\ 5 \times 3 &\equiv 1 \pmod 7 \\ 2 \times 4 &\equiv 1 \pmod 7 \\ 6i_F \times i_F &\equiv 1 \pmod 7). \end{aligned}$$

However $C(Z_7)$ has no zero divisors.

We now proceed onto recall the notion of symmetric semigroup S(n); that is set of all maps from the set (1, 2, 3, …, n) to itself. Clearly S(n) is not one to one maps alone [15].

For instance

$$S(3)= \left\{ \begin{matrix} 1 \to 1 \\ 2 \to 2, \\ 3 \to 3 \end{matrix} \begin{matrix} 1 \to 2 \\ 2 \to 1, \\ 3 \to 3 \end{matrix} \begin{matrix} 1 \to 3 \\ 2 \to 1, \\ 3 \to 2 \end{matrix} \begin{matrix} 1 \to 3 \\ 2 \to 1, \\ 3 \to 2 \end{matrix} \begin{matrix} 1 \to 3 \\ 2 \to 2, \\ 3 \to 1 \end{matrix} \begin{matrix} 1 \to 1 \\ 2 \to 2 \\ 3 \to 3 \end{matrix} \right.$$



Clearly this has no zero divisors but has only units.

Example 1.1.10: Let S(5) be a symmetric semigroup. Clearly S(5) has no zero divisors and only 120 elements have inverse including the identity.

Now we proceed onto give examples of semigroups got under the natural product $\times_n$ and usual matrix product $\times$.

Example 1.1.11: Let M = $\left\{\begin{bmatrix} a \\ b \end{bmatrix} \middle| a, b \in Z_3\right\}$ be a semigroup under natural product. M has both units and zero divisors.

$$\text{Consider } \begin{bmatrix} 2 \\ 2 \end{bmatrix} \times_n \begin{bmatrix} 2 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \text{ is a unit}$$

$\begin{bmatrix} 2 \\ 1 \end{bmatrix} \times_n \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 2 \end{bmatrix} \times_n \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ are units in M.

Take x = $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and y = $\begin{bmatrix} 0 \\ 2 \end{bmatrix}$ in M; clearly $x \times_n y = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$.

Likewise x = $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ and y = $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ in M contribute to zero divisors.

Example 1.1.12: Let T = {(a, b, c) | a, b, c $\in Z_6$} be a semigroup under the matrix product. It is important to note that in case of row matrices natural product coincides with the usual matrix product.

Take x = (2, 0, 4) and y = (3, 5, 3) in T.

Clearly $x \times y = x \times_n y$ = (0 0 0).

Take x = (1, 5, 1) and y = (1, 5, 1) in T.

Clearly x × y = (1, 1, 1) is a unit in T.

Also if a = (5, 5, 5) are in T we see a × b = (1, 1, 1)

Consider x = (1 0 0) and y = (0 0 3) in T, we see x × y = (0, 0, 0). Thus these semigroups have plenty of zero divisors and units. Further o (T) = 6 × 6 × 6 = 216.

Example 1.1.13: Let

$$M = \left\{ \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{bmatrix} \text{ where } a_i \in Z_2;\ 1 \leq i \leq 4 \right\}$$

be a semigroup under natural product $\times_n$.

M has zero divisors and no units for $\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$ is the unit of M.

Take $x = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}$ and $y = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}$ in M, $x \times_n y = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$.

Likewise if $x = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$ we see $y_1 = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix}$, $y_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}$, $y_3 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}$,

$$y_4 = \begin{bmatrix}0\\1\\1\\0\end{bmatrix},\ y_5 = \begin{bmatrix}0\\1\\0\\0\end{bmatrix},\ y_6 = \begin{bmatrix}0\\0\\1\\0\end{bmatrix} \text{ and } y_7 = \begin{bmatrix}0\\0\\0\\1\end{bmatrix} \text{ in M are such that}$$

$$x \times_n y_i = \begin{bmatrix}0\\0\\0\\0\end{bmatrix} \text{ for every } i = 1, 2, \ldots, 7.$$

Example 1.1.14: Let $S = \{(x_1, x_2, x_3, x_4, x_5) \mid x_i \in Z_2; 1 \leq i \leq 5\}$ be a semigroup under product

$x = (1, 0, 0, 1, 0)$ in S has $y_1 = (0, 1, 1, 0, 1)$,

$y_2 = (0, 0, 1, 0, 1)$, $y_3 = (0, 0, 0, 0, 1)$, $y_4 = (0, 1, 0, 0, 0)$,

$y_5 = (0, 0, 1, 0, 0)$, $y_6 = (0, 1, 1, 0, 0)$ and $y_7 = (0, 1, 0, 0, 1)$

in S are such that

$x \times y_i = (0, 0, 0, 0, 0)$, $1 \leq i \leq 7$.

Example 1.1.15: Let

$$X = \left\{\begin{bmatrix}a_1 & a_2\\a_3 & a_4\\a_5 & a_6\end{bmatrix} \middle|\ a_i \in Z_5, 1 \leq i \leq 6\right\}$$

be a semigroup under natural product $\times_n$.

We see X has both zero divisors and units.

For consider x = $\begin{bmatrix} 1 & 3 \\ 4 & 2 \\ 1 & 4 \end{bmatrix}$ and y = $\begin{bmatrix} 1 & 2 \\ 4 & 3 \\ 1 & 4 \end{bmatrix}$ in X.

It is easy to verify x $\times_n$ y = $\begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 1 & 1 \end{bmatrix}$.

Consider x = $\begin{bmatrix} 0 & 1 \\ 2 & 3 \\ 0 & 4 \end{bmatrix}$ and y = $\begin{bmatrix} 3 & 0 \\ 0 & 0 \\ 4 & 0 \end{bmatrix}$ in X, we see

$$x \times_n y = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

We can have several elements y in X, which are such that

$$x \times_n y = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

Example 1.1.16: Let

$$Y = \left\{\begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \end{bmatrix} \text{ where } a_i \in Z_6, 1 \leq i \leq 6\right\}$$

be a semigroup under natural product Y has both units and zero divisors.

Example 1.1.17: Let

$$M = \left\{ \begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ \vdots & \vdots & \vdots \\ a_{28} & a_{29} & a_{30} \end{bmatrix} \text{ where } a_i \in Z_{12}, 1 \leq i \leq 30 \right\}$$

be a semigroup under natural product. M has both zero divisors and units.

Example 1.1.18: Let

$$V = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \middle| a, b, c, d \in Z_{10} \right\};$$

V is a semigroup under usual matrix product and V is a non commutative semigroup with $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ as its unit.

Consider the same V with natural $\times_n$; clearly V is a commutative semigroup with $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ as its unit.

We see for $x = \begin{bmatrix} 9 & 1 \\ 7 & 3 \end{bmatrix}$ we have $y = \begin{bmatrix} 9 & 1 \\ 3 & 7 \end{bmatrix}$ in V is

such that

$$x \times_n y = \begin{bmatrix} 9 & 1 \\ 7 & 3 \end{bmatrix} \times_n \begin{bmatrix} 9 & 1 \\ 3 & 7 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}; \text{ where as}$$

$$x \times y = \begin{bmatrix} 9 & 1 \\ 7 & 3 \end{bmatrix} \times \begin{bmatrix} 9 & 1 \\ 3 & 7 \end{bmatrix} = \begin{bmatrix} 4 & 6 \\ 4 & 8 \end{bmatrix} \in V.$$

$$\text{Now } y \times x = \begin{bmatrix} 9 & 1 \\ 3 & 7 \end{bmatrix} \times \begin{bmatrix} 9 & 1 \\ 7 & 3 \end{bmatrix} = \begin{bmatrix} 8 & 2 \\ 6 & 4 \end{bmatrix} \in V,$$

however $x \times y \neq y \times x$.

Thus under natural product x and y are inverses of each other.

Example 1.1.19: Let

$$M = \left\{ \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \text{ where } a, b, c, d, e, f, g, h, i \in Z_3 \right\}$$

be a semigroup under usual product. Clearly M is non commutative and $I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ acts as the multiplicative identity.

Suppose we consider M under the natural product $\times_n$, then M is commutative and $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$ is the multiplicative identity under the natural product $\times_n$.

However number of elements in M is the same. M has both units and zero divisors both under usual product $\times$ and natural product $\times_n$.

Example 1.1.20: Let

$$P = \left\{ \begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \\ a_{10} & a_{11} & a_{12} \\ a_{13} & a_{14} & a_{15} \end{bmatrix} \text{ where } a_i \in Z_2, 1 \leq i \leq 15 \right\}$$

be a semigroup under natural product $\times_n$. P is a commutative group of finite order.

However P has multiplicative identity $I = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$.

We see P is not a semigroup under usual product as it is not compatible under usual product. We see P has several zero divisors. Further this semigroup is not a Smarandache semigroup.

THEOREM 1.1.1: *Let P = {collection of all $m \times n$ matrices with entries from Z_2} ($m \neq n$ or $m < n$ or $m > n$, m and n fixed positive integers). P under natural product $\times_n$ is a commutative semigroup with unit. P is not a Smarandache semigroup.*

Proof is left as an exercise to the reader. The only simple reasoning is that every element in P is a $m \times n$ matrix whose entries are either 0 or 1. We see there is one and only one matrix in P with all one's. This matrix serves as the unit. However every other matrix in P has atleast one zero as its entry so it is not invertible.

THEOREM 1.1.2: *Let P = {collection of all $m \times n$ matrices with entries from Z_t; $t \neq 2$; $m = n$ or $m > n$ or $m < n$ m and n* *positive integers} be a semigroup under natural product $\times_n$. P is a Smarandache semigroup.*

The proof is direct for P has more than one m × n matrix X

such that we have a Y with $X \times_n Y = \begin{bmatrix} 1 & 1 & 1 & \ldots & 1 \\ 1 & 1 & 1 & \ldots & 1 \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & 1 & 1 & \ldots & 1 \end{bmatrix}_{m \times n}$.

For example take P = [5 × 4 matrices with entries from Z_8]. P under natural product $\times_n$ is a semigroup. Take

$$X = \begin{bmatrix} 1 & 7 & 3 & 5 \\ 3 & 5 & 7 & 1 \\ 1 & 3 & 3 & 3 \\ 7 & 7 & 5 & 5 \\ 5 & 3 & 5 & 7 \end{bmatrix} \in P; \text{ we have}$$

$$X \times_n X = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}. \text{ If } X = \begin{bmatrix} 7 & 7 & 7 & 7 \\ 7 & 7 & 7 & 7 \\ 7 & 7 & 7 & 7 \\ 7 & 7 & 7 & 7 \\ 7 & 7 & 7 & 7 \end{bmatrix} \text{ then also}$$

$$X \times_n X = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}.$$

Further
M = {all 5 × 4 matrices with entries from {1, 3, 5, 7} ⊆ Z_8} is such that every element in M is invertible. Infact M is a group so P is a Smarandache semigroup under $\times_n$. P also has zero divisors. [17]

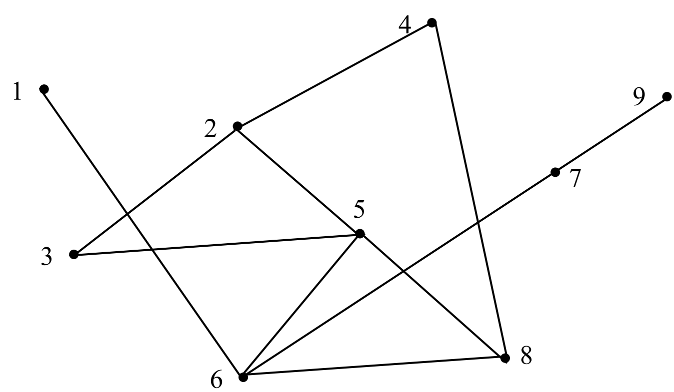
Now we proceed onto recall some basic notions about graphs.

1.2 Basic notions on Graphs

In this section we just recall some of the basic properties associated with graphs which are essential in this study.

A graph is a pair G = (V, E) of sets satisfying $E \subseteq [V]^2$, thus the elements of E are 2-element subsets of V. To avoid notations ambiguities we shall always assume tacitly that V ∩ E = ϕ. The elements of V are the vertices (or nodes or points) of the graph G, the elements of E are its edges (or lines) [5].

We will denote a simple graph with vertices {1, 2, 3, …, 9}.

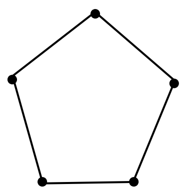


The graph on V = {1, 2, …, 9} with edge set and

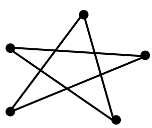
E = {{1, 6}, {2, 3}, {2, 4}, {3, 5}, {2, 5}, {5, 6}, {6, 7}, {7, 9}, {4, 8}, {5, 8}, {6, 8}} as its edge set.

We denote the edge set by E (G) and vertex set by V(G).

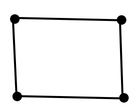
Let G be a graph



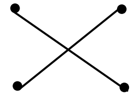
the complement of G is G′;which is as follows:



Let H be a graph



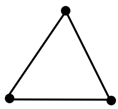
H′ the complement of H is

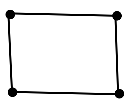


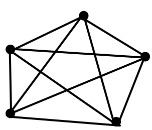
Now we also make use of the notion of complete graphs in this book. We see the set of units of a semigroup $(Z_n, \times)$ is a complete graph.

Recall a graph G in which all the vertices in G are pairwise adjacent is called complete.

We see this is a complete graph with two vertices.

 is a complete graph with these vertices.

 is a complete graph with four vertices and



is a complete graph with 5 vertices.

We also make use of trees and for basic notion about trees refer [5].

Chapter Two

ZERO DIVISOR GRAPHS OF SEMIGROUPS

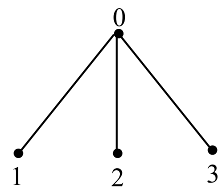
In this chapter we define, derive and give the properties of zero divisor graphs associated with finite semigroups. Further we give a simple model of the zero divisor graphs in case of certain infinite semigroups.

Throughout this book we assume the semigroup is only under the binary operation product. If the semigroup has no zero divisors or zero element under product we see the zero divisor graph is trivial.

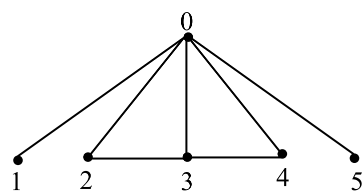
DEFINITION 2.1: *A semigroup S is considered as a simple graph G whose vertices are elements of G such that two different elements x and y in G are adjacent if and only if x is a zero divisor of y.*

We will call this graph G as the zero divisor graph of the semigroup S.

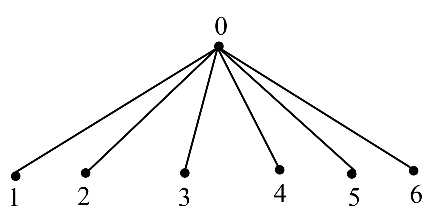
Example 2.1: Let $(Z_4, \times) = S$ be a semigroup. The zero divisor graph of S is as follows:



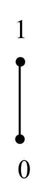
Example 2.2: Let $P = (Z_6, \times)$ be a semigroup. The zero divisor graph of P is as



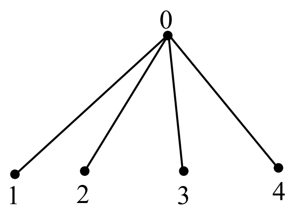
Example 2.3: Let $S = (Z_7, \times)$ be semigroup under product. The zero divisor graph of S is as follows:



Example 2.4: Let $S = \{Z_2, \times\}$ be the semigroup under product. The zero divisor graph of S is a tree with two vertices given by

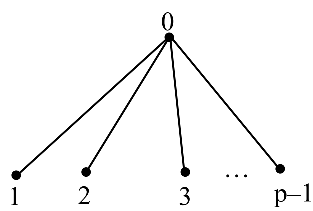


Example 2.5: Let S = {Z_5, ×} be the semigroup under product. The zero divisor graph of S is a tree with five vertices given by the following figure.



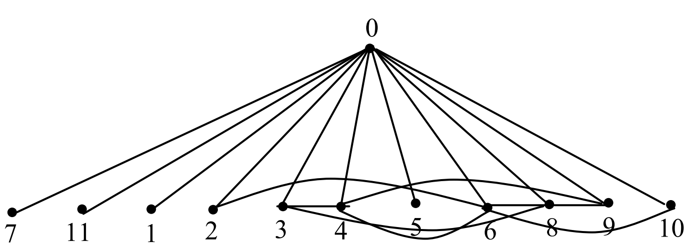
The following theorem is direct.

THEOREM 2.1: *Let S = {Z_p, ×}; (p a prime) be a semigroup under product. The zero divisor graph of S is a tree with p vertices given by the following diagram.*

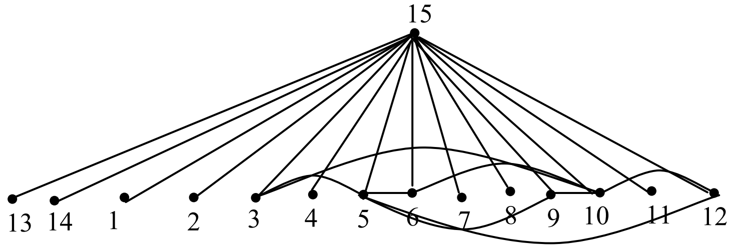


Proof is direct and hence left as an exercise to the reader.

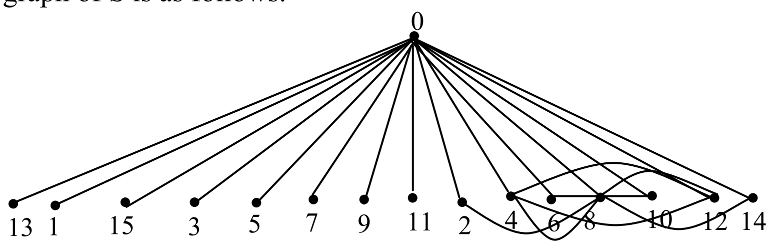
Example 2.6: Let S = (Z_{12}, ×) be a semigroup. The zero divisor graph of S is as follows:



Example 2.7: Let S = (Z_{15}, ×) be a semigroup. The zero divisor graph of S is as follows.

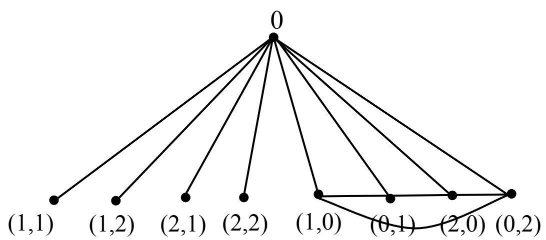


Example 2.8: Let S = (Z_{16}, ×) be a semigroup. The zero divisor graph of S is as follows.



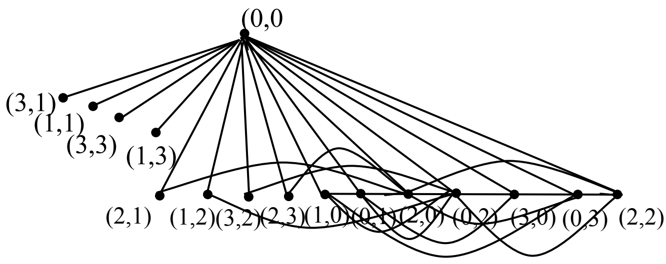
Now we will provide examples of matrix semigroups some commutative and others non commutative.

Example 2.9: Let V = {(a, b) | a, b ∈ Z_3} be a semigroup under product. The zero divisor graph of V is as follows.



Example 2.10: Let S = {(a, b) | a, b ∈ Z_4} be the semigroup.

The zero divisor graph associated with S is as follows:



Likewise we can get the zero divisor semigroup for any

S = $\{(Z_{p_1} \times Z_{p_2} \times \ldots \times Z_{p_n}) \mid p_i$ are positive integers$\}$.

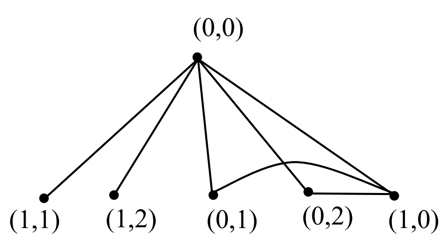
THEOREM 2.2: *Let*

$$S = \{(Z_{n_1} \times Z_{n_2} \times \ldots \times Z_{n_t}) = (a_1, a_2, \ldots, a_t) \mid a_i \in Z_{n_i};$$
$$1 \leq i \leq t,\ 1 < n_i < \infty\}$$

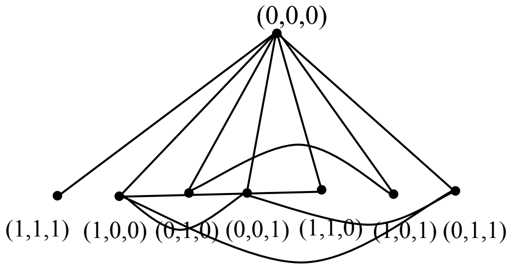
be a semigroup under product. The zero divisor graph of S is not a tree.

Proof is direct and hence is left as an exercise to the reader.

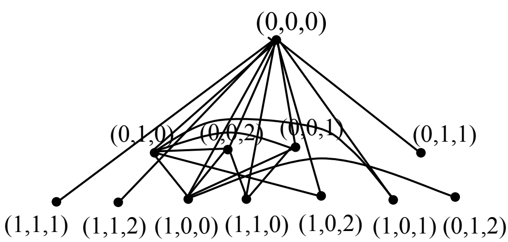
Example 2.11: Let M = $\{Z_2 \times Z_3 = (a, b) \mid a \in Z_2$ and $b \in Z_3\}$ be a semigroup under product. The zero divisor graph of M is as follows:



Example 2.12: Let M = $\{(a, b, c) \mid a, b, c \in Z_2\}$ be a semigroup under product. The zero divisor graph associated with M is as follows:



Example 2.13: Let P = {(a, b, c) | a, b $\in Z_2$ and c $\in Z_3$} be a semigroup. The zero divisor graph associated with P is as follows:



Now we proceed onto show we have a class of semigroups for which we do not have any zero divisor graph associated with them.

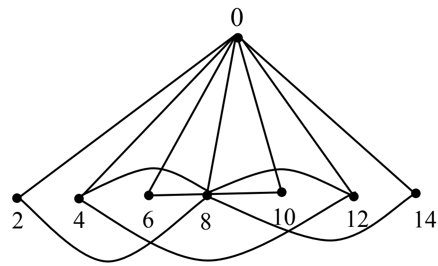
Example 2.14: Let S(3) be the symmetric semigroup of three symbols. |S(3)| = 27 = 3^3. Clearly S(3) has no zero divisor graph associated with it.

Example 2.15: Let S(5) be the symmetric semigroup of degree 5. S(5) has no zero divisor. So S(5) has no zero divisor graph associated with it.

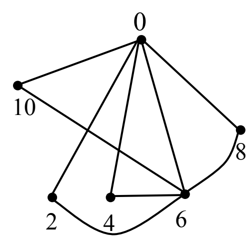
THEOREM 2.3: *Let S(n) be the symmetric semigroup. S(n) has no zero divisor graph associated with it.*

The proof follows from the simple fact S(n) has no zero.

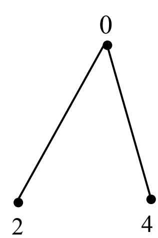
Example 2.16: Let M = {0, 2, 4, 6, 8. 10, 12, 14} $\subseteq Z_{16}$ be the semigroup under multiplication modulo 16. The zero divisor graph associated with M is as follows:



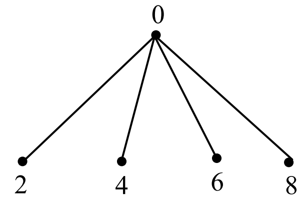
Example 2.17: Let S = {0, 2, 4, 6, 8, 10} be a semigroup under multiplication modulo 12. The zero divisor graph associated with S is as follows:



Example 2.18: Let S = {0, 2, 4} be a semigroup under multiplication modulo 6. The zero divisor graph of S is as follows:

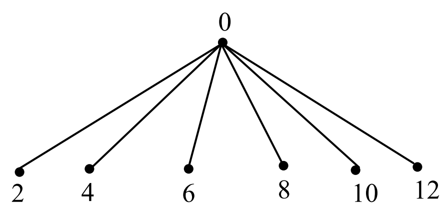


Example 2.19: Let S = {0, 2, 4, 6, 8} be a semigroup under multiplication modulo 10. The zero divisor graph associated with S is as follows:



Example 2.20: Let S = {0, 2, 4, 6, 8, 10, 12} be a semigroup under multiplication modulo 14.

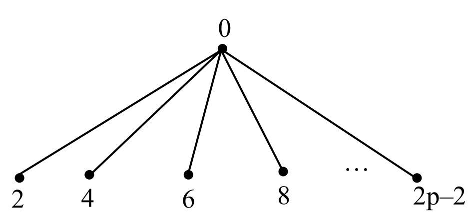
The zero divisor graph associated with S is as follows:



In view of all these examples we have the following theorem.

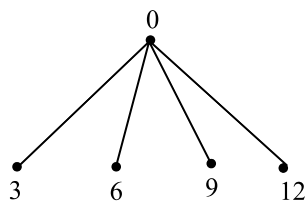
THEOREM 2.4: *Let S = {0, 2, 4, 6, ..., 2p–2} be a semigroup under multiplication modulo 2p p a prime. The zero divisor graph of S is a tree with p vertices.*

Proof: The zero divisor graph associated with S is as follows:

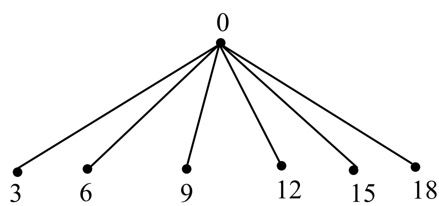


Consider the following examples.

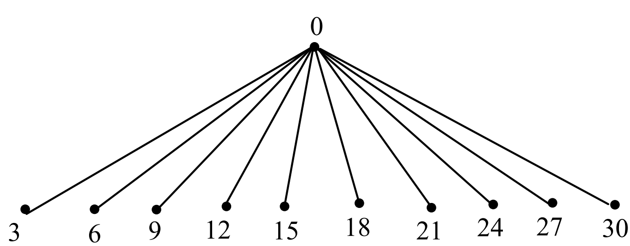
Example 2.21: Let S = {0, 3, 6, 9, 12} $\subseteq Z_{15}$ be the semigroup under product modulo 15. The zero divisor graph associated with S is as follows:



Example 2.22: Let S = {0, 3, 6, 9, 12, 15, 18} be a semigroup under multiplication modulo 21. The zero divisor graph associated with S is a tree with 7 vertices.



Example 2.23: Let S = {0, 3, 6, 9, 12, 15, …, 30} $\subseteq Z_{33}$ be a semigroup under multiplication modulo 33. The zero divisor graph associated with S is a tree with 11 vertices which is as follows:



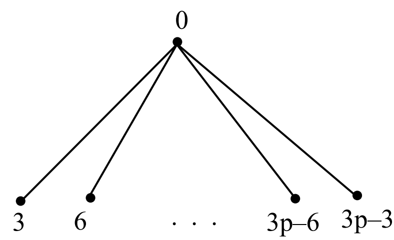
In view of all these we have the following theorem.

THEOREM 2.5: *Let $S = \{0, 3, 6, 9, 12, \ldots, 3p-3 \mid p$ is a prime$\}$ be a semigroup under multiplication modulo 3p.*

The zero divisor graph of S is a tree with p vertices.

Proof: We see S has no zero divisors other than the trivial ones that is $0s = 0$ for all $s \in S$.

The zero divisor graph of S is as follows:



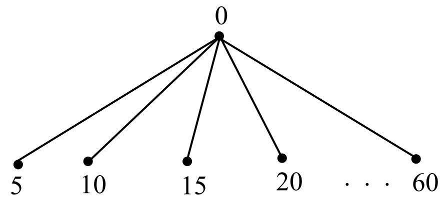
Clearly this graph is a tree with p vertices.

Consider Z_n where $n = pq$ where p and q are primes $p \neq q$. Take $S_1 = \{0, p, 2p, \ldots, qp - p\}$ and $S_2 = \{0, q, 2q, 3q, \ldots, pq-q\}$ two proper subsets of Z_n. Both S_1 and S_2 are semigroups under multiplication modulo pq. Further the zero divisor graph associated with S_i are trees; $1 \leq i \leq 2$. The graph of S_1 has q vertices and that of S_2 has p vertices.

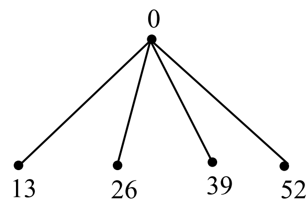
We illustrate this by examples.

Example 2.24: Let $S = Z_{65}$ be a semigroup under multiplication modulo 65. Consider $P_1 = \{0, 5, 10, \ldots, 60\} \subseteq S$ and $P_2 = \{0, 13, 26, \ldots, 52\} \subseteq S$ subsemigroups of S. Both P_1 and P_2 have zero divisor graphs to be trees with 13 and 5 vertices respectively.

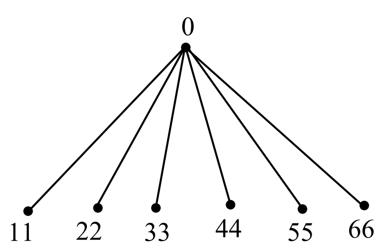
The zero divisor graph of P_1 is as follows:

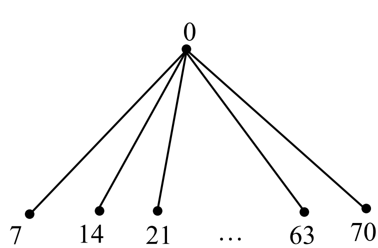


The zero divisor graph associated with P_2 is a tree with 5 vertices.



Example 2.25: Let S = Z_{77} be a semigroup under multiplication modulo 77. Consider $P_1 = \{0, 11, 22, 33, 44, 55, 66\} \subseteq S$ and $P_2 = \{0, 7, 4, \ldots, 70\} \subseteq S$; both P_1 and P_2 are subsemigroups and the associated zero divisor graphs of P_1 and P_2 are trees with 7 and 11 vertices respectively given by the following graphs.



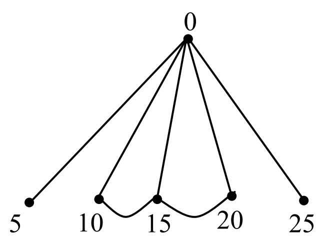


In view of this we have the following theorem.

THEOREM 2.6: *Let $S = Z_{pq}$, p and q two distinct primes be a semigroup under multiplication modulo pq. $P_1 = \{0, p, 2p, \ldots, pq-p\}$ and $P_2 = \{0, q, 2q, \ldots, pq-q\}$ be two subsemigroups of S. Clearly the zero divisor graphs associated with P_1 and P_2 are trees with q and p vertices respectively.*

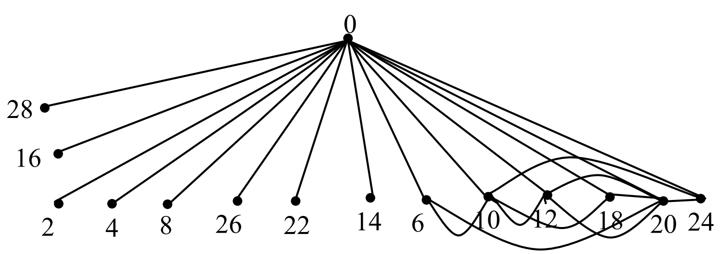
The proof is direct hence left as an exercise to the reader.

Example 2.26: Let $S = Z_{30}$ be a semigroup under multiplication modulo 30. Take $P_1 = \{0, 2, 4, 6, \ldots, 28\}$, $P_2 = \{0, 3, 6, 9, \ldots, 27\}$ and $P_3 = \{0, 5, 10, 15, 20, 25\}$ be subsemigroups of S. Clearly the zero divisor graphs of P_1, P_2 and P_3 are not trees. For

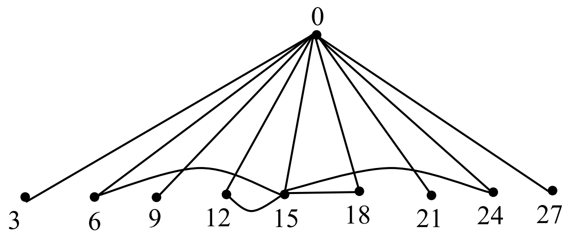


is the zero divisor graph of P_3.

The zero divisor graph of P_1 is



Clearly this is not a tree. The zero divisor graph of P_2 is



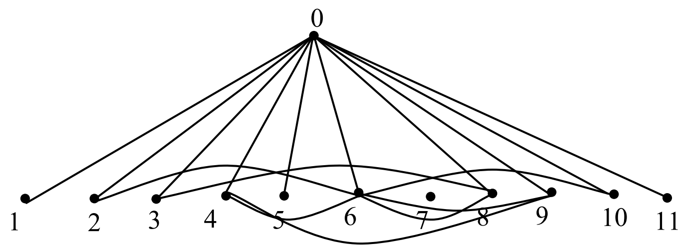
which is not a tree.

In view of this example we have the following theorem.

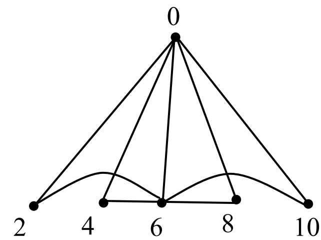
THEOREM 2.7: *Let $S = Z_n$ where $n = p\,q\,r$ where p, q and r are three distinct primes. Suppose $P_1 = \{0, p, 2p, \ldots, pqr–p\}$ and $P_2 = \{0, q, 2q, \ldots, pqr – q\}$ and $P_3 = \{0, r, 23r, \ldots, pqr – r\}$ be three subsemigroups of S. The zero divisor graphs of P_1, P_2 and P_3 are not trees.*

Proof follows from the simple fact p_i's have non trivial zero divisors $1 \leq i \leq 3$.

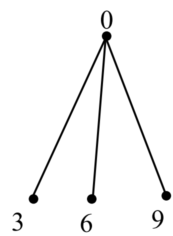
Example 2.27: Let $S = Z_{12}$ be a semigroup under product modulo 12. The zero divisor graph of S is as follows:



Consider $P_1 = \{0, 2, 4, 6, 8, 10\}$ the subsemigroup of S. The zero divisor graph of P_1 is as follows:



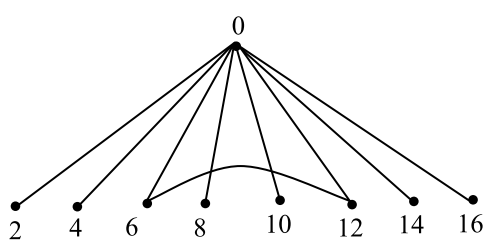
Consider the subsemigroup $P_2 = \{0, 3, 6, 9\}$ of S. The zero divisor graph of P_2 is as follows:



P_2 is a tree with four vertices.

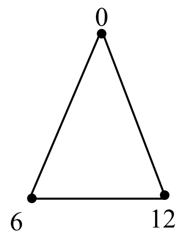
Example 2.28: Let $S = Z_{18}$ be a semigroup under product. $P_1 = \{0, 2, 4, \ldots, 16\}$, $P_2 = \{0, 3, 6, 9, 12, 15\}$, $P_3 = \{0, 9\}$ and $P_4 = \{0, 6, 12\}$ be subsemigroups of S. The zero divisor graphs associated with them are as follows:

The zero divisor graph of P is

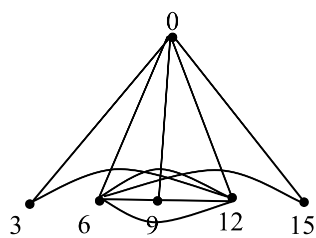


Clearly it is not a tree. The zero divisor graph of P_3 is a tree with two vertices.

The zero divisor graph of P_4 is as follows:

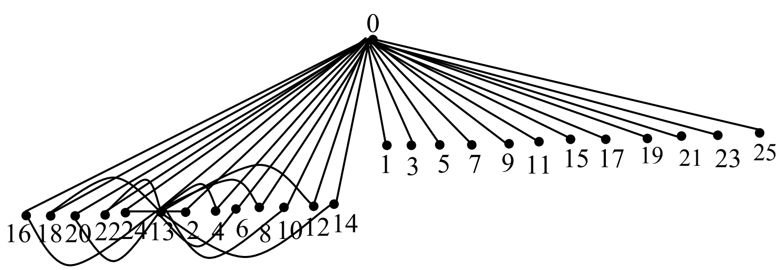


Clearly this is not a tree. The zero divisor graph of P_2 is as follows:



Clearly this is also not a tree.

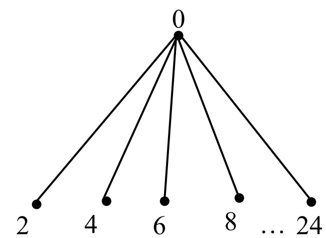
Example 2.29: Let S = Z_{26} be a semigroup under product.



S has the following subsemigroups.
$P_1 = \{0, 2, 4, \ldots, 24\}$ and $P_2 = \{0, 13\}$.

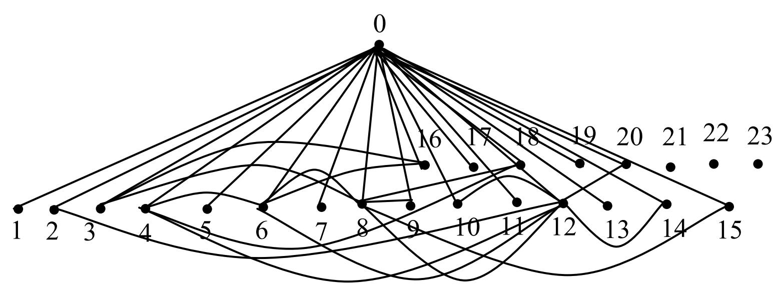
The zero divisor graph of P_2 is a tree.

The zero divisor graph of P_1 is as follows:

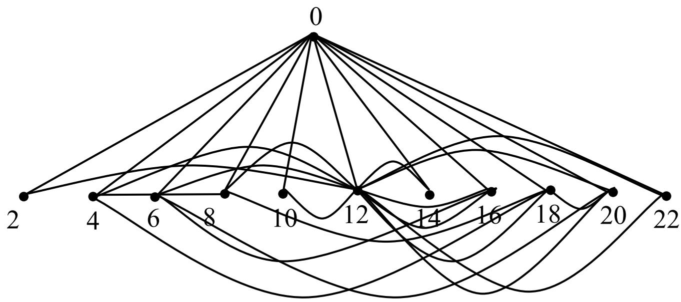


This is a tree with 13 vertices we call these graphs as special subgraphs.

Example 2.30: Let $S = Z_{24}$ be a semigroup. Consider the zero divisor graph of S.

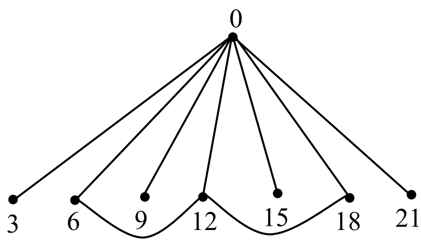


The subsemigroups of Z_{24} are $P_1 = \{0, 2, \ldots, 22\}$, $P_2 = \{0, 3, 6, \ldots, 21\}$, $P_3 = \{0, 4, 8, 12, 16, \ldots, 20\}$, $P_4 = \{0, 6, 12, 18\}$, $P_5 = \{0, 12\}$ and $P_6 = \{0, 8, 16\}$ are subsemigroups. The zero divisor graphs of them are as follows:

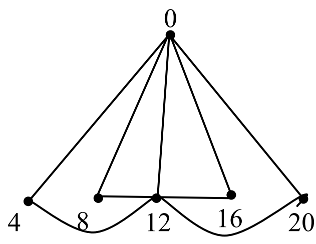


Clearly the zero divisor graph not a tree.

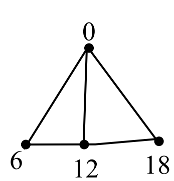
Consider the zero divisor graph of P_2.



This is not a tree. Consider the zero divisor graph of P_3.

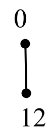


Consider the zero divisor graph of P_4.

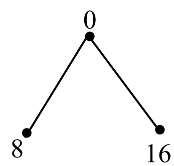


This is not a tree.

The zero divisor graph of P_5 is a tree.



The zero divisor graph of P_6 is



is a tree.

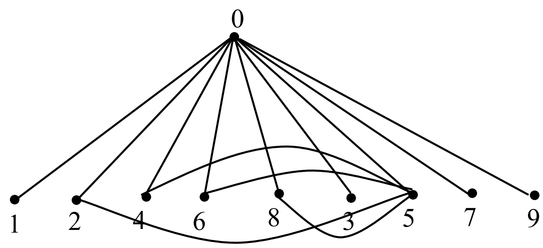
Now we define special subgraphs in view of these examples.

DEFINITION 2.2: *Let S be a semigroup. The zero divisor graph of S be G. Suppose $P_1, P_2, \ldots, P_t$ are subsemigroups of S. Suppose $G_1, G_2, \ldots, G_t$ be the zero divisor graphs of $P_1, P_2, \ldots, P_t$ respectively. Then we define $G_1, G_2, \ldots, G_t$ as special subgraphs of G.*

THEOREM 2.8: *Let S be a semigroup. G be the zero divisor graph of S. Suppose $P_1, P_2, \ldots, P_t$ are subsemigroups and $G_1, G_2, \ldots, G_t$ are the associated special subgraphs of G respectively. Then every special subgraph of G is a subgraph but a subgraph of G in general is not a special subgraph of G.*

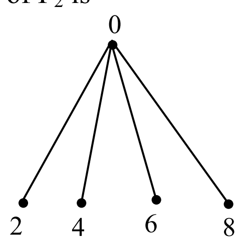
Proof: The first part is clear as every special subgraph of G is also a subgraph of G. To prove the converse we illustrate this situation by an example.

Consider $S = Z_{10}$ the semigroup under multiplication modulo 10. The zero divisor graph associated with S is as follows:

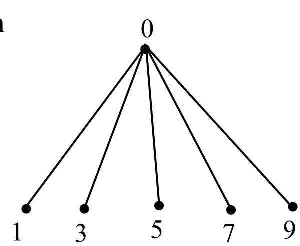


The subsemigroups of S are $P_1 = \{0, 5\}$ and $P_2 = \{0, 2, 4, 6, 8\}$. Now the zero graph of P_1 is

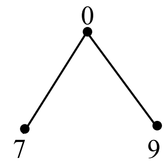
The zero divisor graph of P_2 is



Both of them are trees with two and five vertices respectively. Consider the subgraph

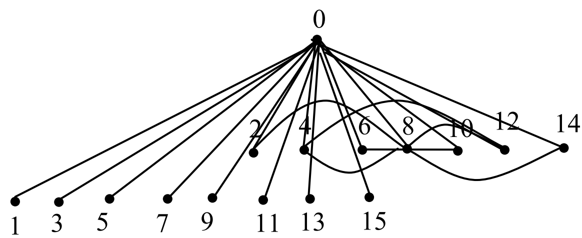


Clearly $S_1 = \{0, 1, 3, 5, 7, 9\}$ is a subsemigroup of S. So this is also a special subgraph.



This is a subgraph of G but is not a subsemigroup as P = {0, 7, 9} is only a subset of S and 7.9 ≡ 63 (mod 10) = 3 ∉ P. Hence a subgraph in general is not a special subgraph of S.

Example 2.31: Let S = Z_{16} be a semigroup under multiplication modulo 16. The zero divisor graph of Z_{16} is as follows:

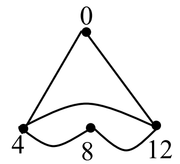


We see the subsemigroups of Z_{16} are as follows:

P_1 = {0, 2, 4, 6, …, 14},
P_2 = {0, 4, 8, 12} and P_3 = {0, 8}.
Finally P_4 = {0, 1, 3, 9, 11}.

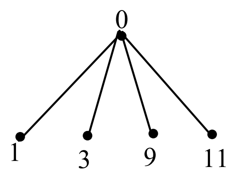
The zero divisor graphs of P_1, P_2, P_3 and P_4 are as follows:
The zero divisor graph of P_3 is

The zero divisor graph of P_2 is



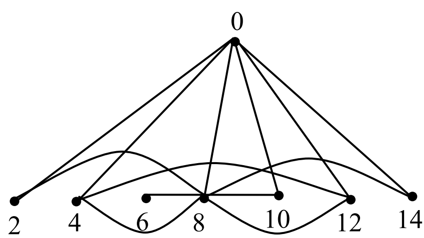
Clearly the graph is not a tree.

The zero divisor graph of P_4 is as follows:



Clearly P_4 is associated with a tree.

The zero divisor graph of P_1 is as follows:



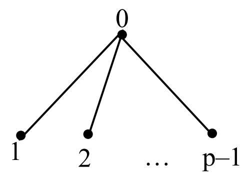
However we have subgraph which are not special subgraph.

Example 2.32: Let S = Z_{28} be a semigroup under product. The zero divisor graph of S has both special subgraphs as well as subgraphs some of which are trees.

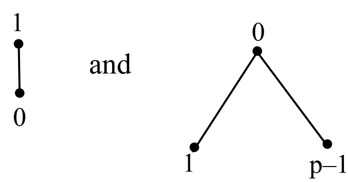
Example 2.33: Let S = Z_{31} be a semigroup under product modulo 31. The zero divisor graph of S is a tree and it has only one special subgraph which is also tree with three vertices.

THEOREM 2.9: *Let $S = Z_p$ be a semigroup under multiplication modulo p. The zero divisor graph of Z_p has only two special subgraphs which is a tree with 3 vertices and 2 vertices.*

Proof: Obvious from the fact that the zero divisor graph associated with the semigroup $S = Z_p$ is a tree with p vertices.



Now the two subsemigroups of S are $P_1 = \{0, 1\}$ and $P_2 = \{0, 1, p–1\}$. The special subgraphs associated with P_1 and P_2 are respectively.



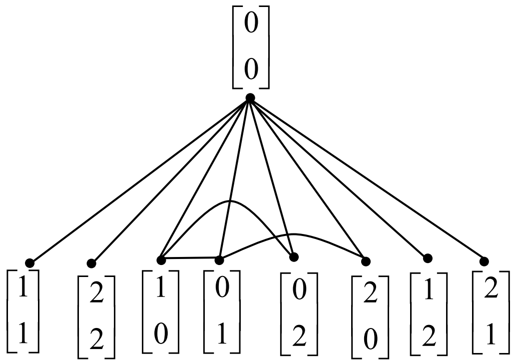
Hence the theorem.

However the graph has several subgraphs which are not special subgraphs.

Now we proceed onto give the zero divisor graphs of semigroups which are matrices.

Example 2.34: Let $M = \left\{\begin{bmatrix} a \\ b \end{bmatrix} \middle| a, b \in Z_3\right\}$ be the semigroup under natural product $\times_n$.

The zero divisor graph of M is as follows:



The subsemigroups of M are

$$P_1 = \left\{\begin{bmatrix}0\\0\end{bmatrix},\begin{bmatrix}1\\0\end{bmatrix},\begin{bmatrix}2\\0\end{bmatrix}\right\},$$

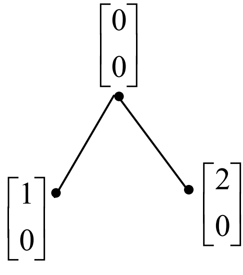
$$P_2 = \left\{\begin{bmatrix}0\\0\end{bmatrix},\begin{bmatrix}1\\1\end{bmatrix},\begin{bmatrix}2\\2\end{bmatrix}\right\},\ P_3 = \left\{\begin{bmatrix}0\\0\end{bmatrix},\begin{bmatrix}0\\1\end{bmatrix},\begin{bmatrix}0\\2\end{bmatrix}\right\}$$

$$P_4 = \left\{\begin{bmatrix}0\\0\end{bmatrix},\begin{bmatrix}1\\1\end{bmatrix},\begin{bmatrix}1\\2\end{bmatrix},\begin{bmatrix}2\\1\end{bmatrix},\begin{bmatrix}2\\2\end{bmatrix}\right\} \text{ and } P_5 = \left\{\begin{bmatrix}0\\0\end{bmatrix},\begin{bmatrix}0\\1\end{bmatrix},\begin{bmatrix}1\\0\end{bmatrix},\begin{bmatrix}0\\2\end{bmatrix},\begin{bmatrix}2\\0\end{bmatrix}\right\}$$

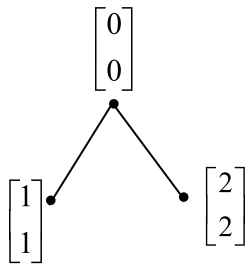
are subsemigroups of M.

The zero divisor graph associated with them are given.

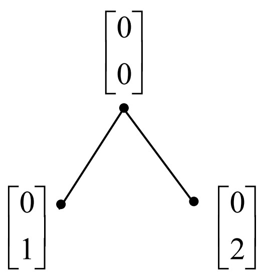
The zero divisor graph of P_1 is



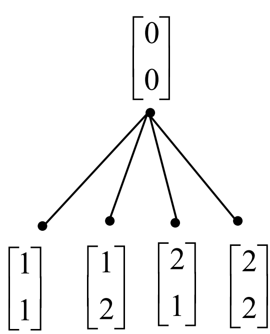
The zero divisor graph of P_2 is



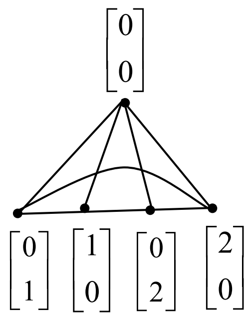
The zero divisor graph of P_3 is



The zero divisor graph of P_4 is



The zero divisor graph of P_5 is as follows:

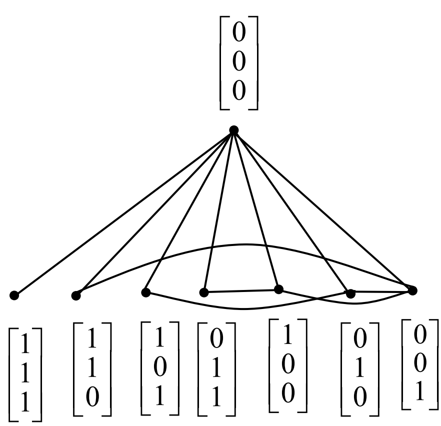


All the special subgraphs of M are not trees.

Now suppose we have

$$M = \left\{ \begin{bmatrix} a \\ b \\ c \end{bmatrix} \middle| \, a, b, c \in Z_2 \right\}$$

be a semigroup. To find the zero divisor graph of M.



The subsemigroup of M are as follows:

$$P_1 = \left\{\begin{bmatrix}1\\1\\1\end{bmatrix}, \begin{bmatrix}0\\0\\0\end{bmatrix}\right\}, P_2 = \left\{\begin{bmatrix}0\\0\\1\end{bmatrix}, \begin{bmatrix}0\\0\\0\end{bmatrix}\right\}, P_3 = \left\{\begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\0\\0\end{bmatrix}\right\},$$

$$P_4 = \left\{\begin{bmatrix}0\\1\\0\end{bmatrix}, \begin{bmatrix}0\\0\\0\end{bmatrix}\right\}, P_5 = \left\{\begin{bmatrix}0\\0\\0\end{bmatrix}, \begin{bmatrix}1\\1\\0\end{bmatrix}\right\}, P_6 = \left\{\begin{bmatrix}0\\0\\0\end{bmatrix}, \begin{bmatrix}1\\0\\1\end{bmatrix}\right\},$$

$$P_7 = \left\{\begin{bmatrix}0\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\1\end{bmatrix}\right\}, P_8 = \left\{\begin{bmatrix}0\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\1\end{bmatrix}, \begin{bmatrix}0\\0\\1\end{bmatrix}\right\} \text{ and so on.}$$

Some of them are trees with two vertices. We have seven special subgraphs with two vertices.

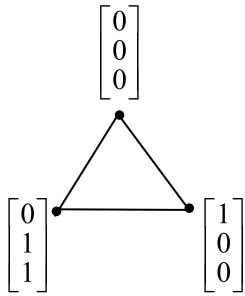
We have 21 special subgraphs with 3 vertices, of which some of them are trees and others are not trees.

$$\text{Take } R_1 = \left\{\begin{bmatrix}0\\0\\0\end{bmatrix}, \begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\1\end{bmatrix}\right\}, R_2 = \left\{\begin{bmatrix}0\\0\\0\end{bmatrix}, \begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}\right\},$$

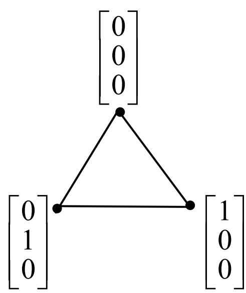
$$R_3 = \left\{\begin{bmatrix}0\\0\\0\end{bmatrix}, \begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\end{bmatrix}\right\}, R_4 = \left\{\begin{bmatrix}0\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\1\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}\right\} \text{ and so on.}$$

The special subgraphs associated with them are as follows:

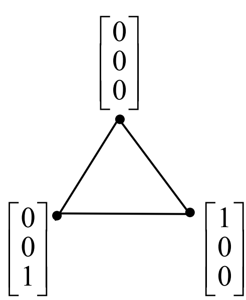
The zero divisor graph of R_1 is



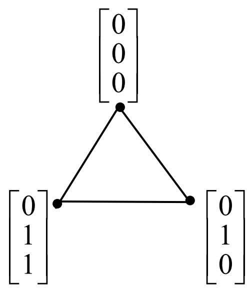
The zero divisor graph of R_2 is



The zero divisor graph associated with R_3 is



The zero divisor graph associated with R_4 is



Thus some special subgraphs are trees and others are not.

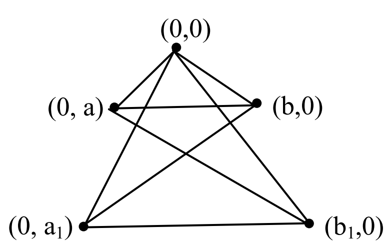
Now suppose we have an infinite semigroup but the zero divisors can be modeled then we say.

The zero divisor graph pattern of that semigroup.

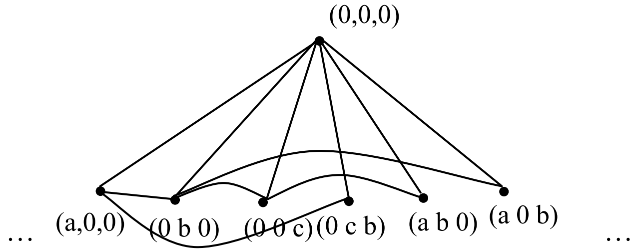
We will illustrate such situation. However all infinite semigroups cannot have pattern of the zero divisor graph.

Example 2.35: Let S = $\{(Z \times Z) = (a, b) \mid a, b \in S\}$ be a semigroup. The zero divisors in S are of the form $\{(a, 0)$ and $(0, b) \mid a, b \in Z\}$.

The pattern of the zero divisor graph of S will contain subgraphs of the form which is not a tree.



Example 2.36: Suppose S = {(a, b, c) | a, b, c ∈ R} be an infinite semigroup under product. The pattern of the zero divisor subgraph of S is as follows:



Likewise if we have the semigroup to be a row vector X (or column vector semigroup under natural product $\times_n$) we will get the same pattern of the zero divisor graph.

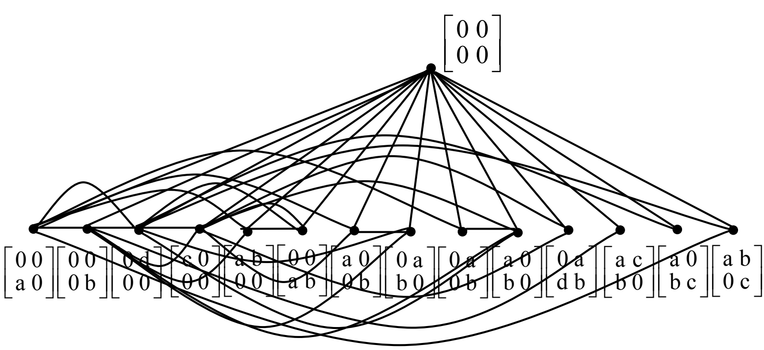
We have different types of zero divisor graphs if we use matrices, however matrices, under natural product have different patterns.

Example 2.37: Let

$$S = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \middle| \text{a, b, c, d} \in \text{Q (or Z or R)} \right\}$$

be a semigroup under natural product $\times_n$.

The pattern of the zero divisor subgraph of S is as follows:



This will be the universal pattern of the zero divisor subgraph graph when S is an infinite semigroup or a finite semigroup.

This is the zero divisor graph pattern associated with 2×2 matrices with entries from Z or Q or R or Z_n.

Likewise one can get the zero divisor graph patterns of any $m \times n$ matrix $m \neq n$.

The following result is interesting.

THEOREM 2.10: *Let*

$$S = \{m \times n \text{ matrices with entries from } Z \text{ or } Q \text{ or } R \text{ or } Z_n\}$$

be a semigroup under natural product. The zero divisor graph pattern of S is not a tree. But there exists subsemigroups of S which has the pattern of special subgraphs to be trees.

Proof is direct hence left as an exercise to the reader.

We give an example to substantiate the latter part of the theorem.

Example 2.38: Let

$$S = \left\{\begin{bmatrix} a & b \\ c & d \end{bmatrix} \middle| \; a, b, c, d \in Z \text{ or } Q \text{ or } R \text{ or } Z_n\right\}$$

be the semigroup under natural product $\times_n$.

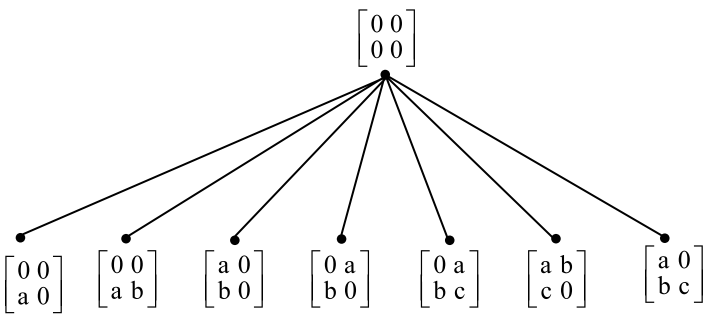
Suppose

$$P = \left\{\begin{bmatrix} 0 & 0 \\ a & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} a & 0 \\ b & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ a & b \end{bmatrix},\right.$$

$$\begin{bmatrix} 0 & b \\ a & 0 \end{bmatrix}, \begin{bmatrix} 0 & a \\ c & b \end{bmatrix}, \begin{bmatrix} a & 0 \\ b & c \end{bmatrix}, \begin{bmatrix} a & b \\ c & 0 \end{bmatrix}$$

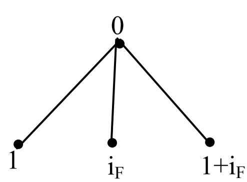
where a, b, c, d $\in$ Z or Q or R or Z_n}

be a subsemigroup of S. The pattern of the zero divisor graph of P is a tree with 8 vertices.



Now we proceed onto give complex finite modulo integer semigroups and their zero divisor graphs.

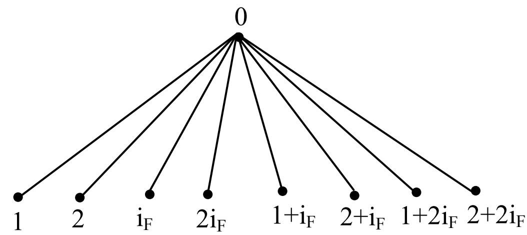
Example 3.39: Let S = $\{a + bi_F \mid a, b \in Z_2, i_F^2 = 1\}$ be a complex semigroup under product the zero divisor graph of S is as follows:



is a tree with four vertices.

Example 3.40: Let S = $\{a + bi_F \mid a, b \in Z_3, i_F^2 = 2\}$ be a complex modulo integer semigroup.

The zero divisor graph of S is as follows:

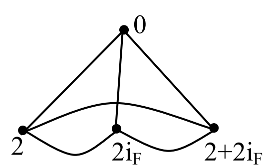


It is a tree with 9 vertices.

Example 3.41: Let S = $\{a + bi_F \mid a, b \in Z_4,\ i_F^2 = 3\}$ be a semigroup under multiplication modulo four.

The zero divisor graph associated with S is not a tree.

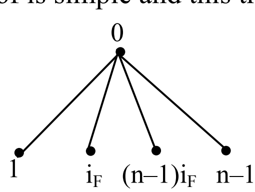
However this has subgraphs which are trees.



This is a special graph with four vertices which is not a tree. Inview of all these we have the following theorem.

THEOREM 2.11: *Let V = $\{a + bi_F \mid a, b \in Z_n,\ i_F^2 = n-1\}$ be a semigroup under product modulo n. This has a zero divisor graph with n^2 vertices. Further V has subsemigroups which have special subgraphs some of which are trees.*

Proof: The proof is simple and this tree



with 5 vertices is associated with a subsemigroup $H = \{0, 1, i_F, (n-1)i_F, n-1\}$.

If n is not a prime certainly this has special subgraphs which are not trees.

Now we just recall the notion of Smarandache zero divisors and the related Smarandache zero divisor graphs.

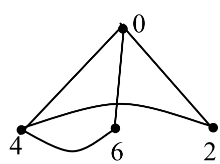
Let $(S, \times)$ be a semigroup with 0. A element $x \in S$ is said to be a Smarandache zero divisor of S if $x \neq 0$ and there exists $y \in S$ with $x \times y = 0$, further there exists $a, b \in S \setminus \{0, x, y\}$ with $xa = 0$ or $ax = 0$ and $yb = 0$ and $by = 0$ but $ab \neq 0$ or $ba \neq 0$.

We will illustrate this situation by some examples.

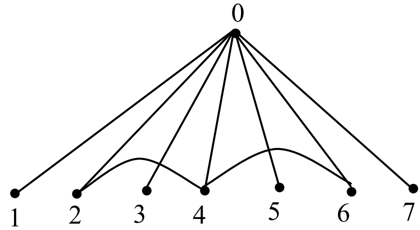
Example 2.42: Let $S = \{Z_8, \times\}$ be a semigroup with zero.

The zero divisors of S are
$\{0, 2 \times 4 \equiv 0 \pmod 8), 4 \times 6 \equiv 0 \pmod 8)$ and $4^2 \equiv 0 \pmod 8)\}$.

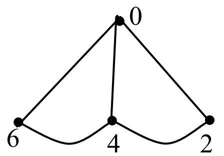
The special zero divisor graph of S is as follows:



The zero divisor graph of S is

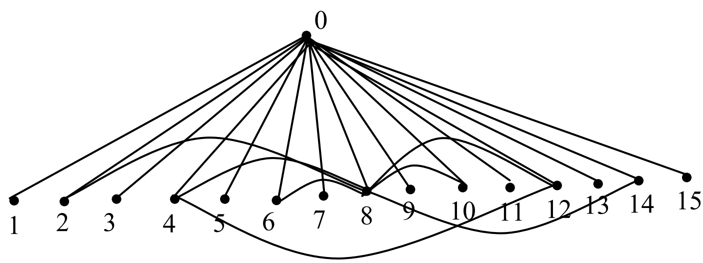


The special zero divisor graph of is a subgraph of the zero divisor graph.



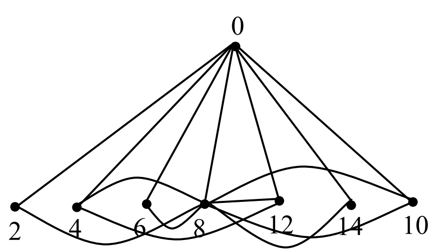
However $S = \{Z_8, \times\}$ has Smarandache zero divisors.

Example 2.43: Let $S = \{Z_{16}, \times\}$ be a semigroup. The zero divisor graph of S is as follows:



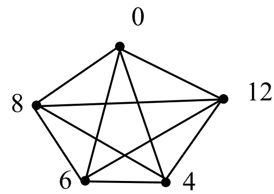
The special zero divisor graph of S is as follows: The set of zero divisors of S are $T = \{0, 2, 4, 6, 8, 12, 14, 10\} \subseteq S$.

The special zero divisor graph associated with T is



Consider $8 \in S$, $8 \times 12 \equiv 0 \pmod{16}$ 6 and 4 in S are such that $6 \times 8 \equiv 0 \pmod{16}$ $12 \times 4 \equiv 0 \pmod{16}$ but $6 \times 4 \equiv 0$ (mod 16).

Thus 12 and 8 are a S-unit of S. The S-unit graph of 12 and 8 is as follows:



Clearly the S-unit set {0, 4, 6, 8, 12} forms a subsemigroup of S, seen from the following table.

$\times$	0	6	4	8	12
0	0	0	0	0	0
6	0	4	8	0	8
4	0	8	0	0	0
8	0	0	0	0	0
12	0	8	0	0	0

Thus the S-zero divisor graph of {0, 6, 4, 8, 12} is a complete graph with 5 vertices.

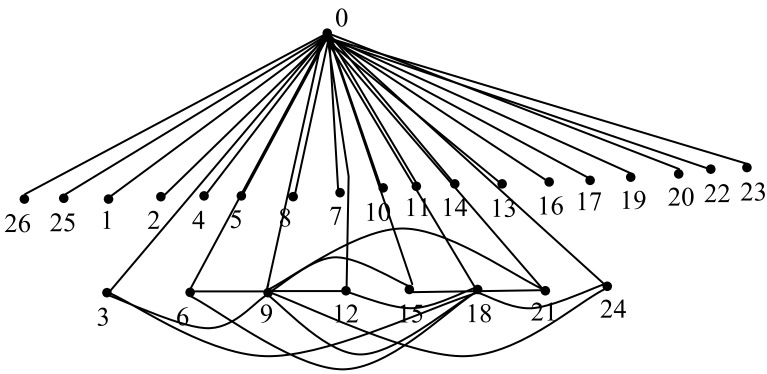
Clearly we see 4 is also a S-zero divisor of S for $8 \times 4 \equiv 0$ (mod 16). We have (6, 12) is S such that $8.6 \equiv 0$ (mod 16) $12 \times 4 \equiv 0$ (mod 16) and $6 \times 12 \not\equiv 0$ (mod 16). The S-zero divisor graph of 4 is the same as that of 12. Both are identical.

Example 2.44: Let S = $\{Z_{27}, \times\}$ be a semigroup under $\times$.
The zero divisors of S are $\{0, 3, 6, 9, 12, 15, 18, 21, 24\} \subseteq$ S.

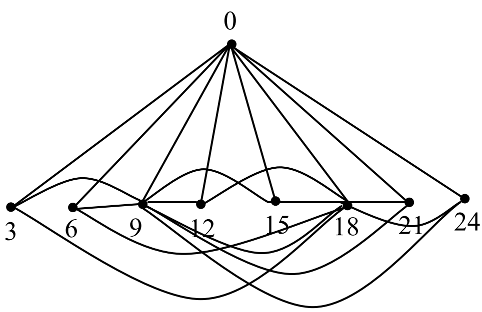
$$\begin{aligned} \text{We see } 3 \times 9 &\equiv 0 \pmod{27} \\ 6 \times 9 &\equiv 0 \pmod{27} \\ 12 \times 9 &\equiv 0 \pmod{27} \\ 15 \times 9 &\equiv 0 \pmod{27} \\ 18 \times 9 &\equiv 0 \pmod{27} \\ 18 \times 3 &\equiv 0 \pmod{27} \end{aligned}$$

$$15 \times 18 \equiv 0 \pmod{27}$$
$$24 \times 9 \equiv 0 \pmod{27}$$
$$24 \times 18 \equiv 0 \pmod{27}$$
$$21 \times 9 \equiv 0 \pmod{27} \text{ and}$$
$$21 \times 18 \equiv 0 \pmod{27}.$$

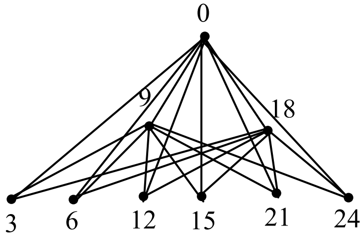
The zero divisor graph of S is



The special zero divisor graph of S is



or the same graph can be redrawn as



Now we proceed onto give the S-zero divisor graphs of S. We see 9 (and 18) alone is the S-zero divisor. $9 \times 18 \equiv 0 \pmod{27}$ and we can have 15 pairs (x, y) of elements such that $y \times 18 \equiv 0 \pmod{27}$ and $x \times 9 \equiv 0 \pmod{27}$ with $x \times y \neq 0 \pmod{27}$.

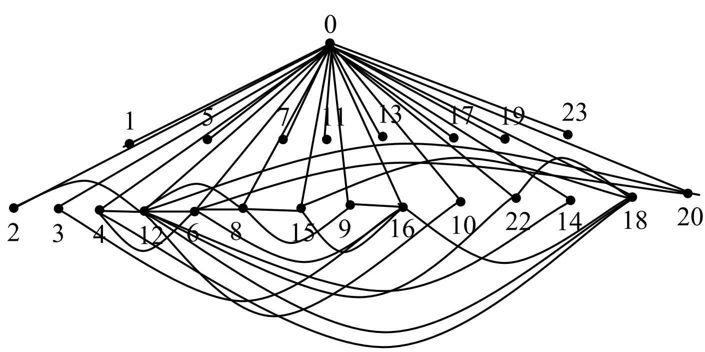
The 15 sets are (3, 6), (3,12), (3, 15), (3, 21), (3, 24), (6, 12), (6, 15), (6, 21), (6, 24), (12, 15), (12, 21), (12, 24), (21, 24), (21, 15) and (24, 15).

Can the semigroup $S = \{Z_{p^3}, \times\}$ p a prime have more than (p–1), S-zero divisors? p a prime.

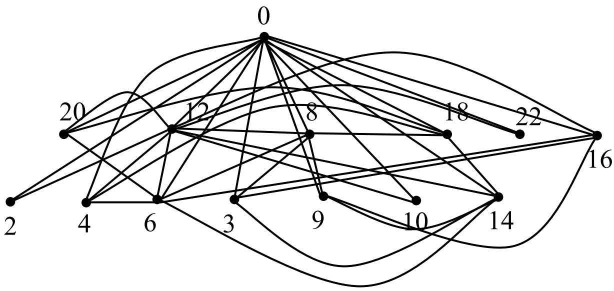
Example 2.45: Let $S = \{Z_{24}, \times\}$ be a semigroup. We find the S-zero divisors graph, special zero divisor graph and the zero divisor graph of S.

The zero divisors of S are 0, $2.12 \equiv 0 \pmod{24}$,
$8 \times 3 \equiv 0 \pmod{24}$, $4 \times 6 \equiv 0 \pmod{24}$, $4 \times 12 \equiv 0 \pmod{24}$,
$6 \times 8 \equiv 0 \pmod{24}$, $6 \times 12 \equiv 0 \pmod{24}$, $8 \times 15 \equiv 0 \pmod{20}$,
$10 \times 12 \equiv 0 \pmod{24}$, $12 \times 16 \equiv 0 \pmod{12}$,
$14 \times 12 \equiv 0 \pmod{24}$, $20 \times 12 \equiv 0 \pmod{22}$,
$21 \times 8 \equiv 0 \pmod{24}$, $9 \times 8 \equiv 0 \pmod{24}$.

The zero divisor graph of S is as follows:

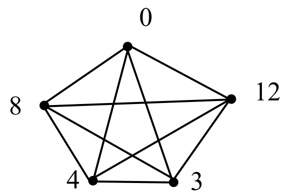


The special zero divisor graph of S is as follows:



The S-zero divisor graph is given in the following.

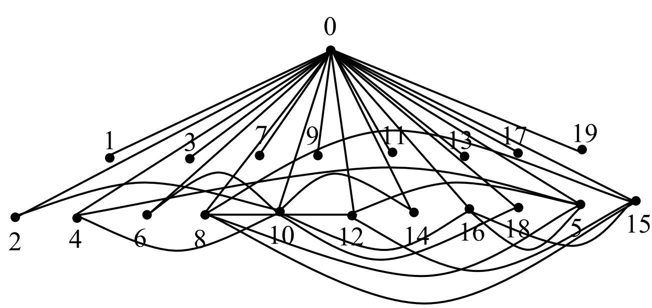
Clearly 8 (or 12) is a zero divisors. The S zero divisor graph of 8 (or 12) is as follows:



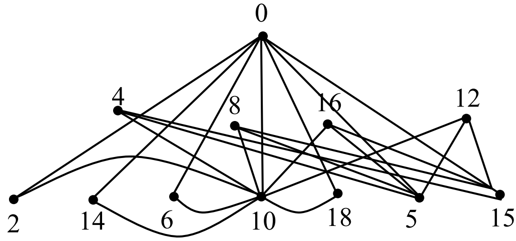
However (3, 4) is not a S-zero divisor in S.

Characterize those semigroups S = $\{Z_n, \times\}$ that has only a pair of S-zero divisors. By a pair (x, y) we mean $x \times y = 0$ and x is a S-zero divisor then y is also a S-zero divisor.

Example 2.46: Let S = $\{Z_{20}, \times\}$ be a semigroup. Consider the zero divisor of S; {0, 2, 10, 4, 5, 15, 6, 8, 12, 14, 16, 18} are the zero divisors of S. The zero divisor graph of S is as follows:



The special zero divisor graph of S is as follows:

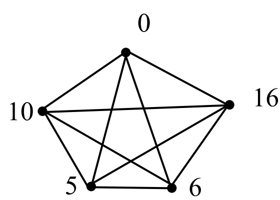


Now we find the S-zero divisor of S. Clearly 10 (or 16) is a S zero divisor of S for $10 \times 16 \equiv 0 \pmod{20}$ and $5 \times 6 \equiv 0 \pmod{20}$ but $16 \times 5 \equiv 0 \pmod{20}$ with $10 \times 6 \equiv 0 \pmod{20}$.

The table of semigroup for the 5 tuple {0, 10, 6, 5, 16} is as follows:

×	0	6	10	5	16
0	0	0	0	0	0
6	0	16	0	10	16
10	0	0	0	0	0
5	0	10	10	5	0
16	0	16	0	0	16

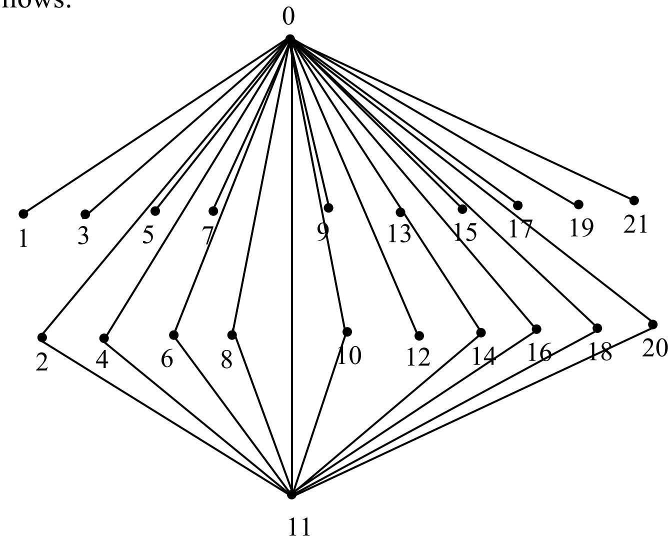
The S-zero divisor graph of 10 (or 16) is as follows:



Interested author may find any other S-zero divisor of $(Z_{20}, \times)$.

Finally we work for the semigroup $S = (Z_{2p}, \times)$.

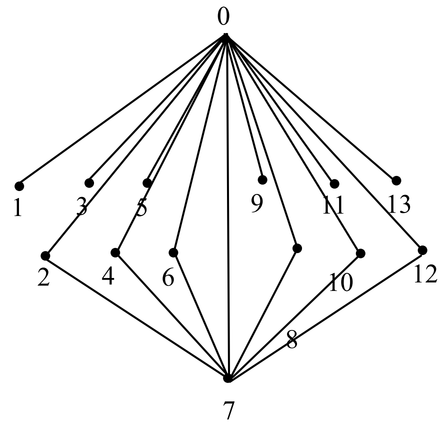
Example 2.47: Let S = (Z_{22}, ×) be a semigroup under product. The zero divisor of Z_{22} are
{0, 2, 11, 4, 6, 8, 10, 12, 14, 16, 18, 20}. The zero graph of S is as follows:



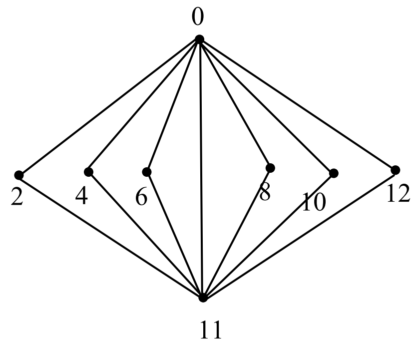
We see every element which is a zero divisor of the form xy ≡ 0 must necessarily have x = 11 or y = 11.

So S = (Z_{22}, ×) have S-zero divisors.

Example 2.48: let S = {Z_{14}, ×} be a semigroup. The zero divisors of S are {0, 2, 7, 4, 6, 8, 10, 12}. The zero divisor graph of S is as follows.

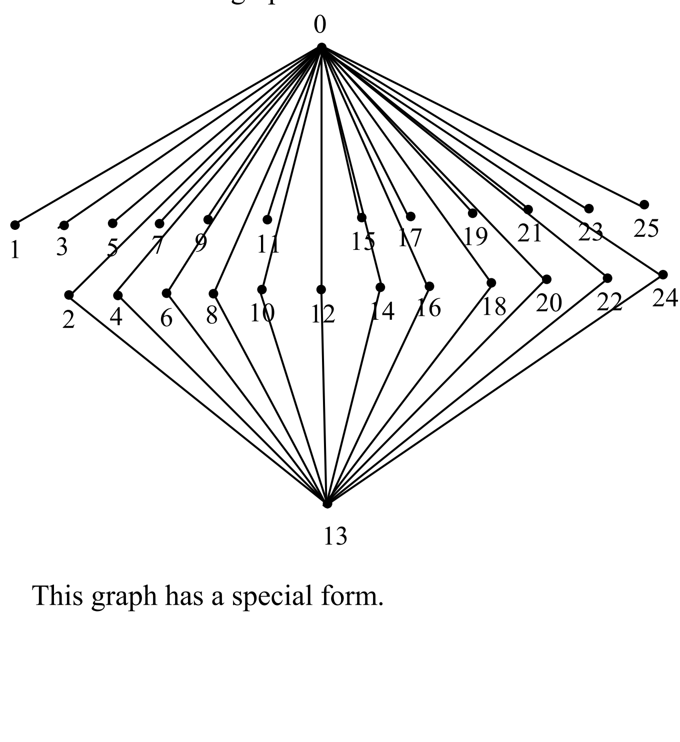


Clearly the special zero divisor graph is as follows:



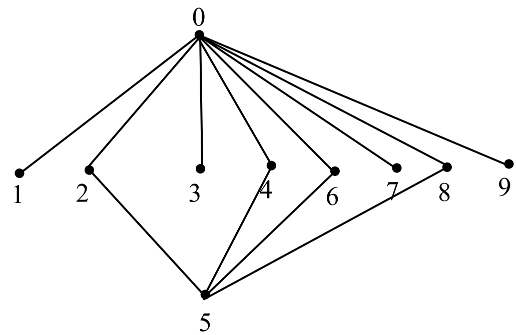
Example 2.49: Let S = $\{Z_{26}, \times\}$ be a semigroup. The zero divisors in S are {0, 2, 4, 6, 8, 10, 12, 14, 16, 18, 20, 22, 24, 13}.

The zero divisor graph of S is as follows:

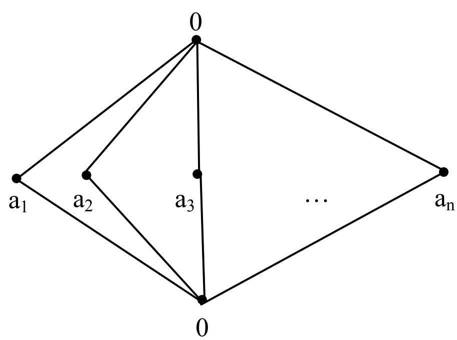


This graph has a special form.

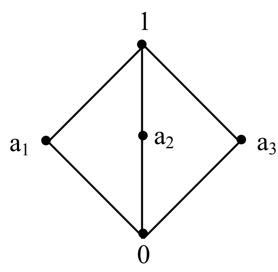
Example 2.50: Let S = {Z_{10}, ×} be a semigroup. The zero divisors of S are {0, 2, 4, 6, 8, 5}. The zero divisor graph of S is as follows:

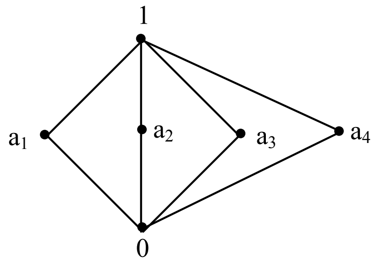


We denote by M_n a lattice of the form given below.



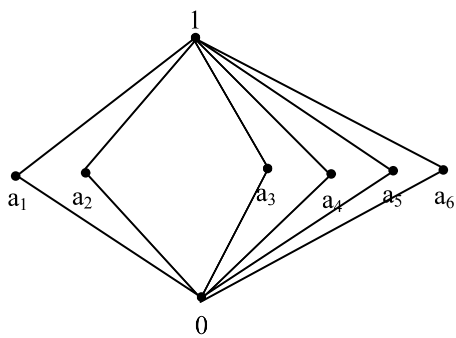
Thus M_3



M_4 = 

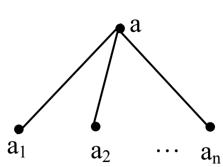
and

M_6 is

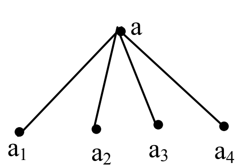


and so on.

We recall a tree T with (n+1) vertices is as follows:



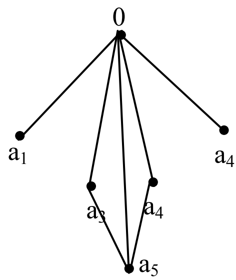
Tree with 5 vertices is as follows:



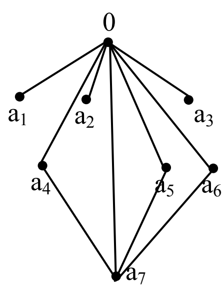
and so on.

We define a graph to be a tree covering lattice graph or pseudo lattice with a tree cover if the following conditions hold good.

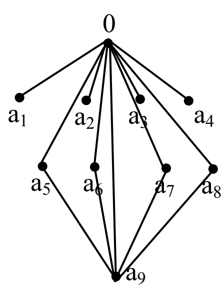
We only describe them.



This is the smallest pseudo lattice tree covering graph. We call a 'pseudo lattice' as the vertices are not compatible with ordering. The next one is

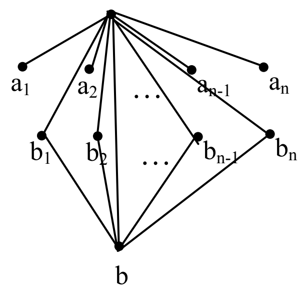


The next larger one is as follows:



and so on.
In general we get

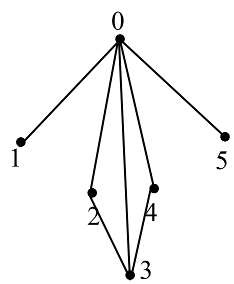
0



This is a tree covering pseudo lattice we have certain zero divisor graph of S = $\{Z_{2n}, \times\}$ happen to be a tree covering pseudo lattice.

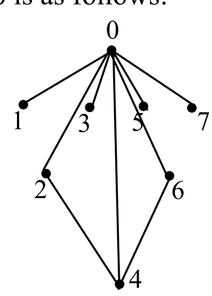
We will illustrate them by examples.

Example 2.51: Let S = $\{Z_6, \times\}$ be a semigroup. The zero divisor graph of S is as follows:



Clearly this is a tree covering pseudo lattice.

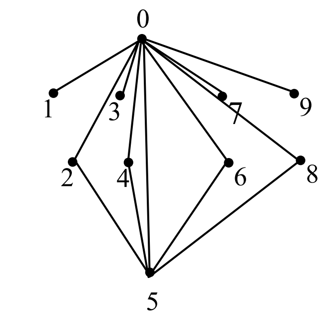
Example 2.52: Let S = $\{Z_8, \times\}$ be a semigroup. The zero divisor graph of S is as follows:



Clearly this graph is not a tree covering pseudo lattice as number of vertices of the tree is 5 that the lattice is two.

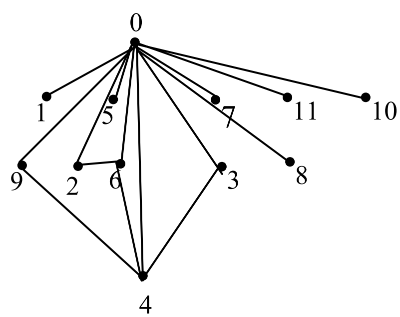
Example 2.53: Let S = {Z_{10}, ×} be a semigroup.

The zero divisor graph of S is as follows:



This is tree covering pseudo lattice.

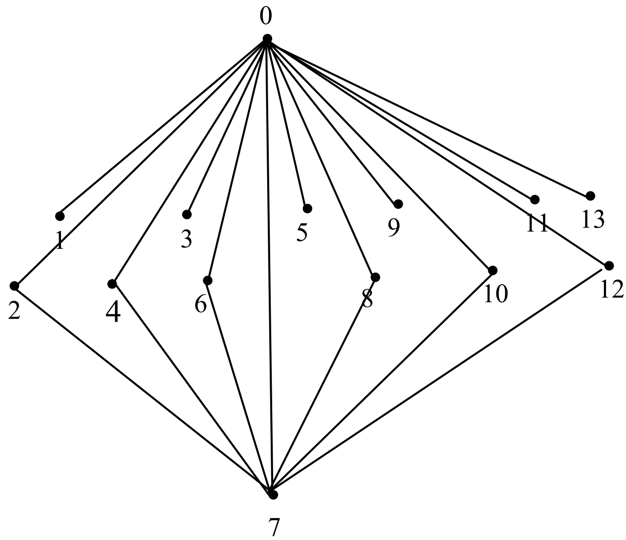
Example 2.54: Let S = {Z_{12}, ×} be a semigroup. The zero divisor graph of S is as follows:



Clearly this is not a tree covering pseudo lattice.

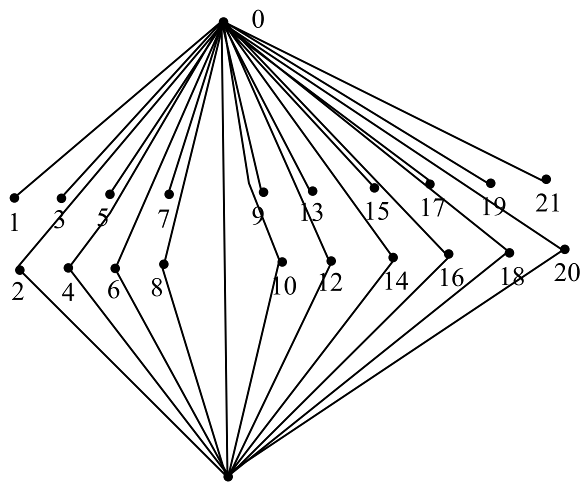
Example 2.55: Let S = {Z_{14}, ×} be a semigroup.

The zero divisor graph of S is as follows:



Clearly the graph is a tree covering pseudo lattice graph.

Example 2.56: Let S = {Z_{22}, ×} be a semigroup. The zero divisor graph of S is as follows:



In view of all these we have the following result.

THEOREM 2.12: *Let $S = \{Z_{2p}, \times \mid p$ is a prime$\}$ be a semigroup. The zero divisor graph of S is a tree covering pseudo lattice graph. p vertices for the tree and the pseudo lattice has (p+1) vertices.*

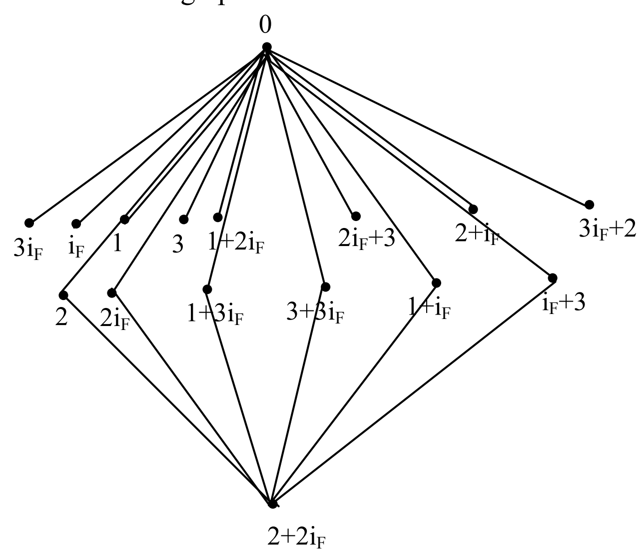
The proof is simple and hence is left as an exercise to the reader.

THEOREM 2.13: *Let $S = \{Z_{2n}, \times$, n is not a prime$\}$ be a semigroup. The zero divisor graph of S is not a tree covering pseudo lattice.*

This is evident from several examples given earlier. We can also call these graphs as top tree covering pseudo lattice graph. However we have to test for the certain semigroups.

Example 2.57: Let $S = \{C(Z_4), \times\}$ be a semigroup. To find S-zero divisors in S. The zero divisors of S are $\{0, 2, 2i_F, 2+2i_F, (2+2i_F), (3+i_F), (2+2i_F), (1+3i_F), (3+3i_F) (2+2i_F) (1+i_F) (2+2i_F)\}$

The zero divisors graph of S is as follows:

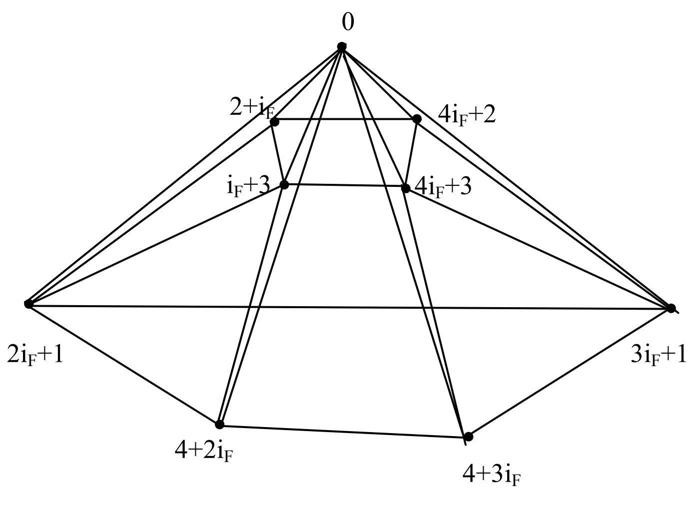


However the zero divisor graph of S is not a tree covering pseudo lattice.

Example 2.58: Let S = $\{C(Z_5), \times\}$ be a semigroup under product.

The zero divisors of S are $\{0, (4+2i_F)(2i_F + 1) = 0,$
$(3i_F+1), (4+3i_F) = 0, (2+i_F)(4i_F + 2) \equiv 0 \pmod 5,$
$(2+i_F)(3+i_F) = 0, (i_F + 3)(4i_F + 3) \equiv 0 \pmod 5,$
$(4i_F +2)(4i_F + 3) = 0, (2i_F+1)(i_F+2) = 0,$
$(2i_F + 1)(3i_F + 1) \equiv 0 \pmod 5,$
$(4i_F + 2)(3i_F + 1) = 0 \pmod 5$ and
$(4 + 3i_F)(4 + 2i_F) = 0 \pmod 5\}.$

The special zero divisor graph of S is as follows:



We see S = $\{C(Z_5), \times\}$ has S-zero divisors.

Infact this gives a layer of zero divisors which will be defined as the n-layered pseudo zero divisors.

DEFINITION 2.3: *Let S be a semigroup under product. Suppose $x_1 y_1 = 0$, $x_2 y_2 = 0$, …, $x_n y_n = 0$ with $x_i x_j = 0$ if $i \neq j$, $y_k y_t = 0$ if $k \neq t$ then we call the set of zero divisors to be a n-layered pseudo S-zero divisors.*

In example 2.58 the semigroup S = $\{C(Z_5), \times\}$ is a 4-layered pseudo zero divisor.

We see several interesting results in this direction can be derived we suggest some open problems.

Let S = $\{C\ (Z_{13}), \times\}$ be a semigroup. Will S have 12 layered pseudo S-zero divisor?

Chapter Three

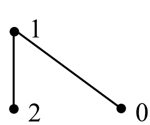
UNIT GRAPHS OF SEMIGROUPS

In this chapter we introduce the notion of special unit graphs of semigroups and study the properties associated with them.

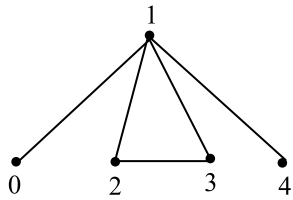
DEFINITION 3.1: *Let S be a semigroup with unit 1 under product. A semigroup S is considered as a simple graph G whose vertices are elements of G such that two different elements x and y in G are adjacent if and only if x is a inverse of y and vice versa all elements are adjacent with 1 that is an edge joins every element with 1. We call these graphs special unit graph.*

We will illustrate this situation by some examples.

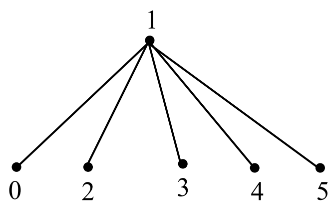
Example 3.1: Let S = Z_3 be a semigroup under multiplication the special unit graph of S is



Example 3.2: Let S = Z_5 be a semigroup under multiplication modulo five. The special unit graph of S is as follows:

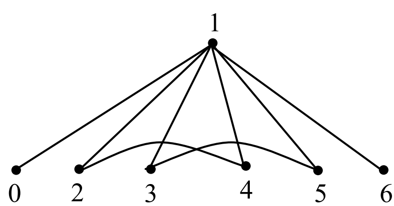


Example 3.3: Let S = Z_6 be a semigroup under product. The special unit graph of S is as follows:



This is a tree.

Example 3.4: Let S = Z_7 be a semigroup under product modulo 7. The special unit graph of S is as follows:



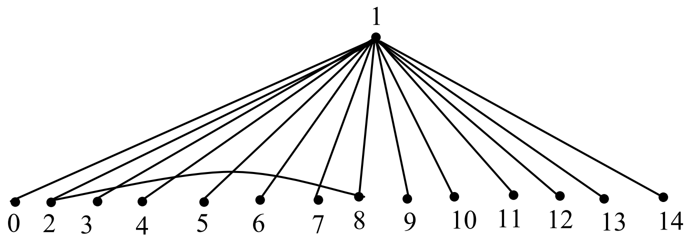
This graph is not a tree.

Now in view of this we have the following theorem.

THEOREM 3.1: *Let S = Z_p (p a prime) be a semigroup under product. The special unit graph of S is not a tree.*

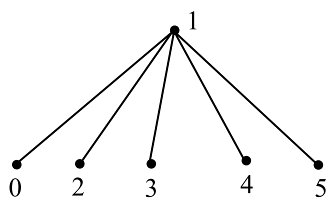
Proof : Follows from the simple observation every element in $Z_p \setminus \{0\}$ is a unit in S.

Example 3.5: Let S = Z_{15} be a semigroup under product. The special unit graph associated with S is as follows:



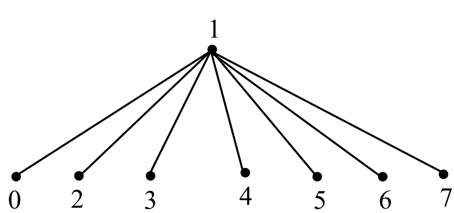
This graph is not a tree.

Example 3.6: Let S = Z_6 be a semigroup under product. The special unit graph associated with S is as follows:



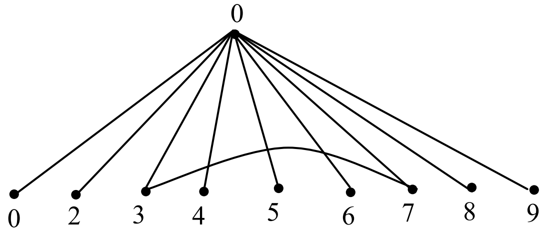
This is a tree with 6 vertices.

Example 3.7: Let S = Z_8 be a semigroup under product modulo 8. The special unit graph associated with S is as follows:



This is also a tree.

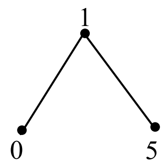
Example 3.8: Let Z_{10} = S be a semigroup under product modulo 10. The special unit graph of Z_{10} is as follows:



Clearly it is not a tree.

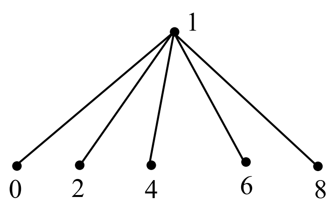
It has subsemigroups whose unit special subgraphs can only be trees.

The subsemigroups of Z_{10} are $P_1 = \{0, 1, 5\}$ and $P_2 = \{0, 2, 4, 6, 8, 1\}$. The unit special subgraphs of P_1 and P_2 are as follows:



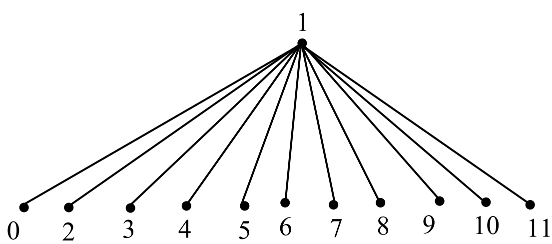
The special subgraph of P_1 is a tree.

Now we consider the subsemigroup P_2, the associated special subgraph of P_2 is as follows:



We see the special subgraph associated with P_2 is a tree with six vertices.

Example 3.9: Let $S = Z_{12}$ be a semigroup. The special unit graph associated with S is as follows:

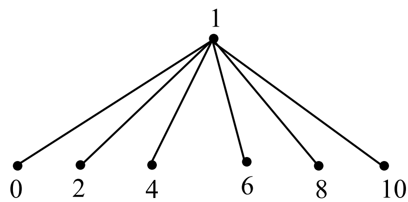


It is a tree with 12 vertices.

Consider the subsemigroups of S; $P_1 = \{1, 0, 2, 4, 6, 8, 10\}$ $P_2 = \{1, 0, 3, 6, 9\}$, $P_3 = \{0, 1, 6\}$ and $P_4 = \{0, 4, 1, 8\}$.

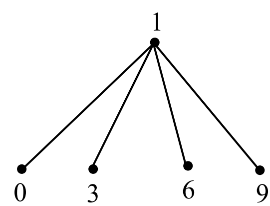
The special unit special subgraphs associated with P_1, P_2, P_3 and P_4 is as follows:

The special subgraph of P_1 is

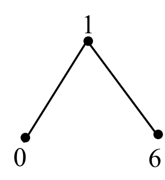


is a tree with 7 vertices.

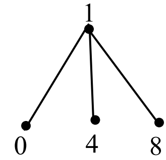
The special subgraph of P_2 is a tree with 5 vertices.



The special subgraph of P_3 is also a tree with three vertices.

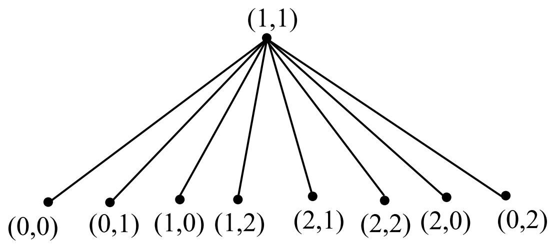


The special subgraph of P_4 is a tree with 4 vertices.



The special unit graph associated with row vector semigroup and column vector semigroup are as follows:

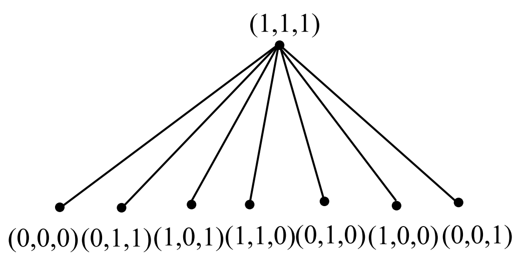
Example 3.10: Let S = {(a, b) | a, b $\in$ Z_3} be a semigroup under product. The special unit graph of S is as follows.



This is clearly a tree and has special subgraphs which are also trees.

All units are self units that is x^2 = (1,1) for x = (1, 2), (2,1) and (2,2).

Example 3.11: Let S = {(a, b, c) | a, b, c $\in$ Z_2} be a semigroup under product. The special unit graph of S is as followsL



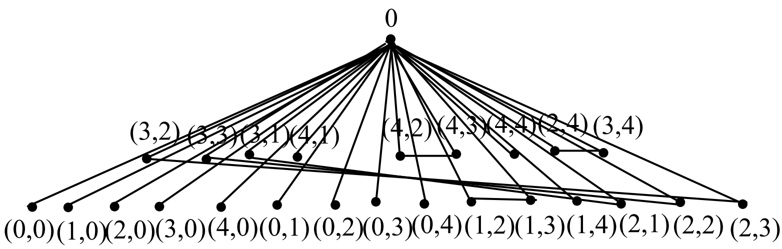
This graph is a tree with 8 vertices.
In view of this we have the following theorem.

THEOREM 3.2: *Let S = {(x_1, x_2, …, x_n) | $x_i \in Z_2$; $1 \leq i \leq n$} be a semigroup under product. The special unit graph of S is a tree with 2^n vertices.*

The proof is straight forward and hence is left as an exercise to the reader.

Example 3.12: Let S = {(a, b) | a, b $\in$ Z_5} be a semigroup under product.

The special unit graph of S is as follows:



Clearly the special unit graph of S is not a tree and the graph has 25 vertices.

Inview of this we have the following theorem.

THEOREM 3.3: *Let V = $Z_p \times Z_p$ = {(a, b) | a, b $\in Z_p$; p a prime} be a semigroup under product modulo p. V has a special unit graph with p^2 vertices which is not a tree.*

Proof: Follows from the simple fact

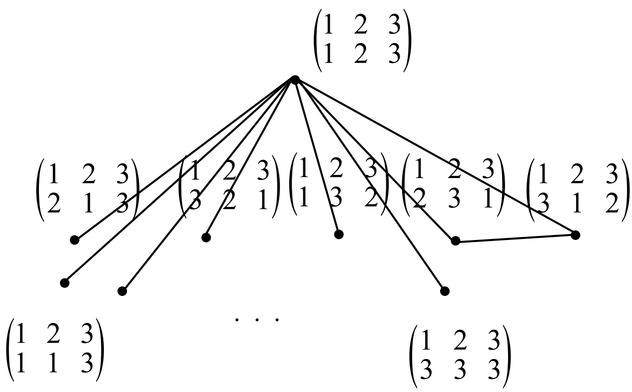
$$x = \left(\frac{(p+1)}{2}, \frac{p-1}{2}\right) \text{ and } \left(\frac{p-1}{2}, \frac{p+1}{2}\right) = y$$

in V are such that xy = (1,1).

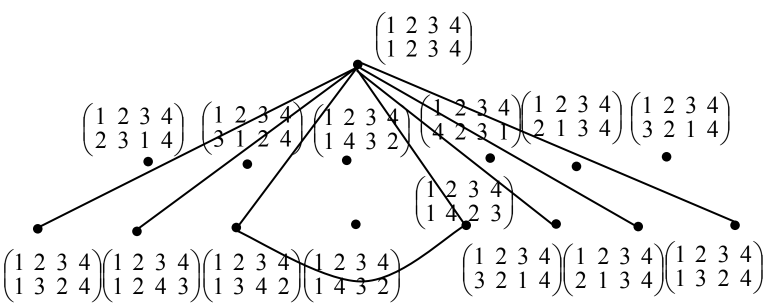
Hence is not a tree.

Now we proceed onto define symmetric semigroups and the special unit graphs associated with them.

Example 3.13: Let V = S(3) be the symmetric semigroup of degree 3. The special unit graph associated with V is as follows: Clearly $o(S(3)) = 3^3 = 27$.

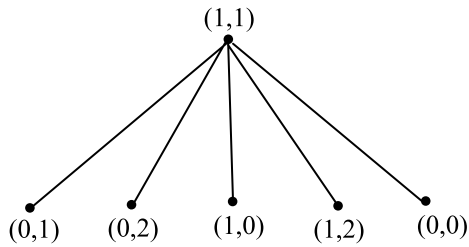


Example 3.14: Let S (4) be a symmetric semigroup of four elements (1, 2, 3, 4). The unit graph of S(4) is as follows:



The special unit graph is not a tree.

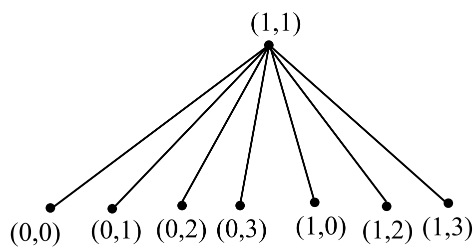
Example 3.15: Let $V = \{Z_2 \times Z_3 = (a, b) \mid a \in Z_2 \text{ and } b \in Z_3\}$ be a semigroup. The special unit graph of V is tree which as follows:



V is a tree with six vertices.

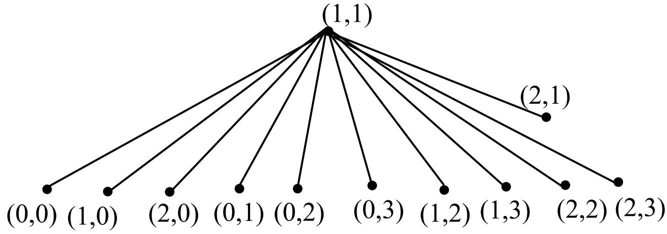
Example 3.16: Let V = $\{Z_2 \times Z_4 = (a, b) \mid a \in Z_2 \text{ and } b \in Z_4\}$ be a semigroup.

The special unit graph is a tree which is as follows:



Example 3.17: Consider the semigroup

S = $\{Z_3 \times Z_4 = (a, b) \mid a \in Z_3, b \in Z_4\}$. The special unit graph of S is as follows:



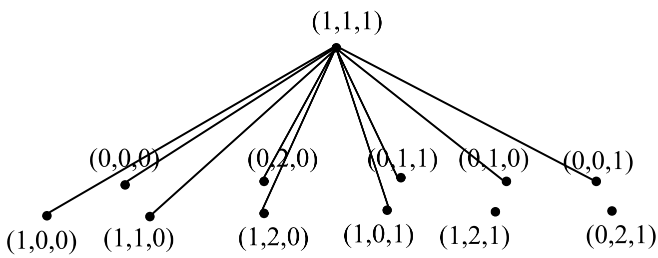
We see the special unit graph of S is tree with 12 vertices.

Example 3.18: Let S = $\{Z_4 \times Z_4 = (a, b) \mid a, b \in Z_4\}$ be a semigroup. The special unit graph of S is a semigroup which is tree with 16 vertices.

Example 3.19: Let S = $\{(a, b) \mid a, b \in Z_p$, p a prime$\}$ be a semigroup. The special unit graph of S is not a tree (p > 2).

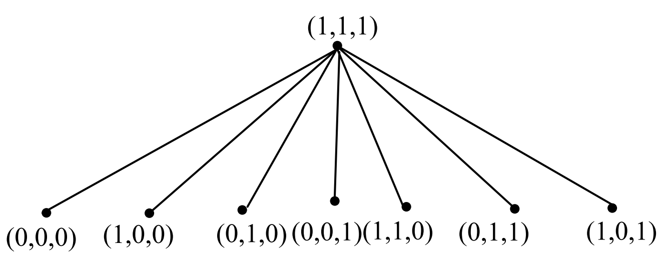
The special unit graph of the semigroup

S = $\{Z_2 \times Z_3 \times Z_2 = (a, b, c) \mid a, c \in Z_2$ and $b \in Z_3\}$ is as follows.



Clearly the graph is a tree.

Example 3.20: Let S = $\{Z_2 \times Z_2 \times Z_2 = (a, b, c) \mid a, b, c \in Z_2\}$ be a semigroup under product.

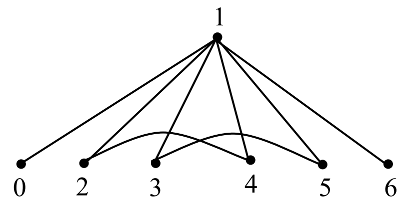


The special unit graph of S is a tree with 8 vertices.

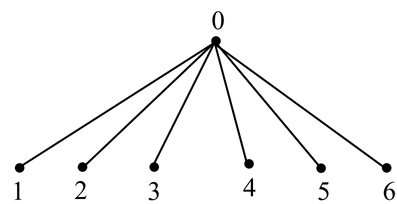
Now we find the zero divisor graph and the special unit graph of a semigroup of finite order.

Example 3.21: Let S = Z_7 be a finite semigroup.

The special unit graph of Z_7 is

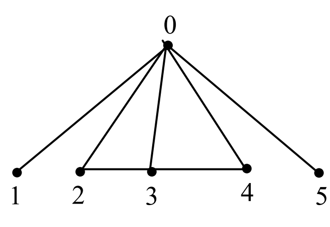


The zero divisor graph of Z_7 is a tree.

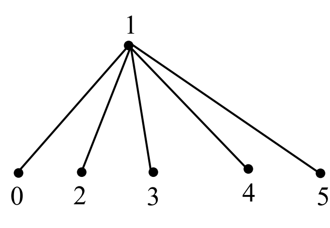


We see the both are distinct.
We see one is tree and the other is not a tree.

Example 3.22: Let S = Z_6 be the semigroup. The zero divisor graph of S is



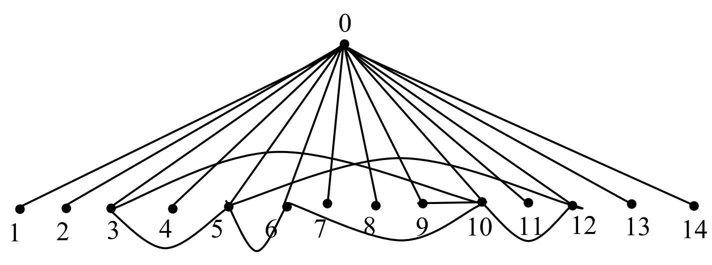
The special unit graph of S is a tree



We see the zero divisor graph of Z_6 is not a tree where as the special unit graph of Z_6 is a tree.

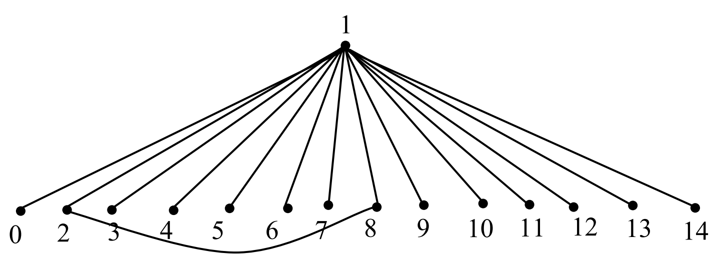
We see if in Z_n, n is a prime the zero divisor graph is a tree where as the special unit graph is not a tree where as if n is a composite number than the zero divisor graph is not a tree and the special unit graph may be a tree or may not be a tree. This is evident from the following examples.

Example 3.23: Let S = Z_{15} be the semigroup under multiplication modulo 15.



Clearly the zero divisor graph is not a tree.

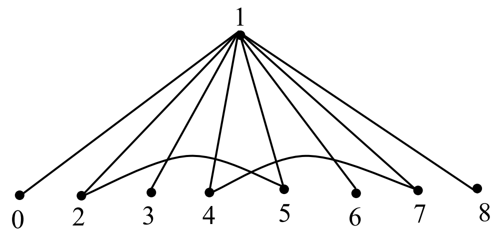
Now consider the special unit graph of Z_{15}.



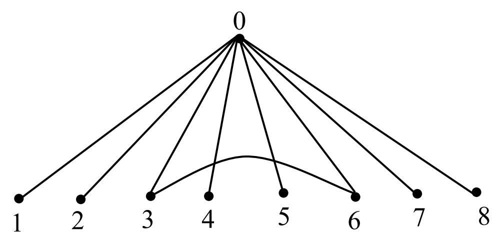
We see the special unit graph of Z_{15} is not a tree.

Example 3.24: Let Z = S be the semigroup under multiplication modulo 9.

The special unit graph of Z_9 is as follows.



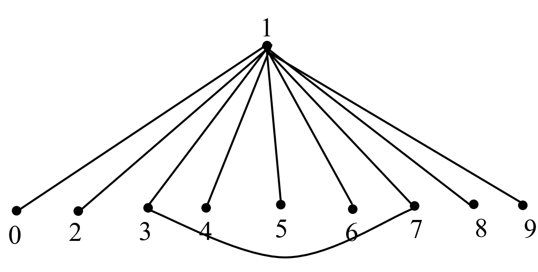
Clearly the graph is not a tree. Consider the zero divisor graph of Z_9.



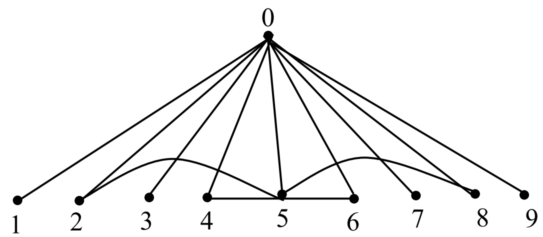
This is also not a tree.

In case of Z_9 and Z_{15} we see both the zero graph and the special unit graphs are not trees where as in Z_6 the zero divisor graph is not a tree where as the special unit graph is a tree.

We see the special unit graph and the zero divisor graph of Z_{10}. The special unit graph of Z_{10} is

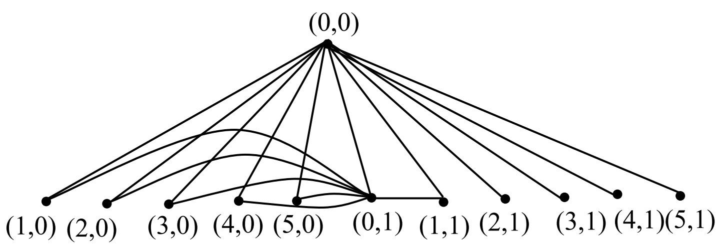


The zero divisor graph of Z_{10} is as follows:

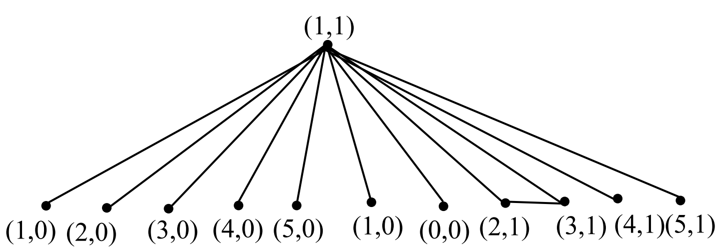


This is not a tree.

Example 3.25: Let S = $Z_6 \times Z_2$ be a semigroup under product with 12 elements.



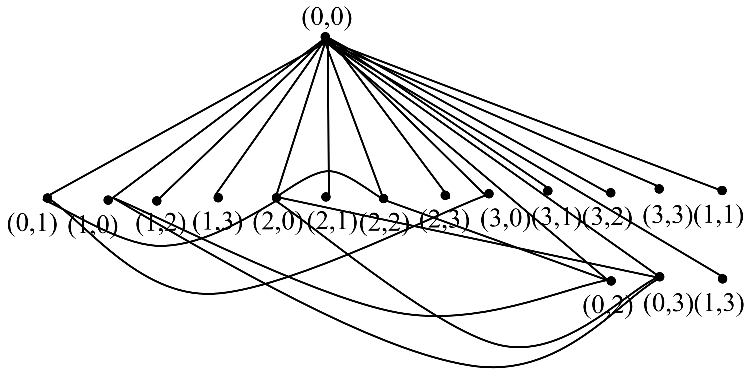
The zero divisor graph is not a tree.
The special unit graph of S is as follows:



Clearly it is not a tree with 12 vertices.

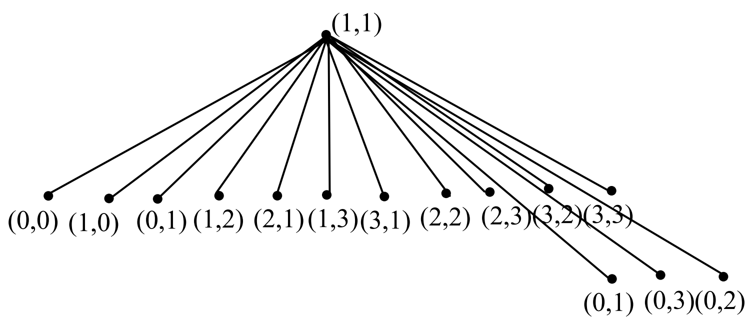
Example 3.26: Let M = $Z_4 \times Z_4$ be a semigroup under product.

The zero divisor graph of M is as follows:



This is not a tree.

The special unit graph of M is as follows:



Clearly M is a tree.

We are not in a position to obtain connected special features for zero divisor graph and the special unit graph.

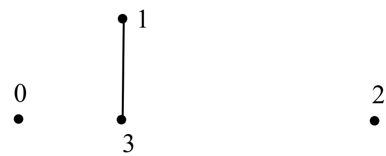
Next we proceed onto define Smarandache unit graph of a semigroup, zero divisor graph and unit graph of a semigroup.

Let S be a semigroup. If there exist $G \subseteq S$ such that G is a group of S. G a proper subset of S then we know S is a Smarandache semigroup. The Smarandache unit graph is nothing but the identity graph of G and rest of the elements of the semigroup S stand as vertices which are not connected.

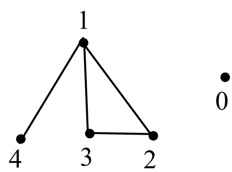
We will illustrate this situation by an example.

Example 3.27: Let S = Z_4 be a semigroup under multiplication modulo four. G = {1, 3} $\subseteq$ S is a group.

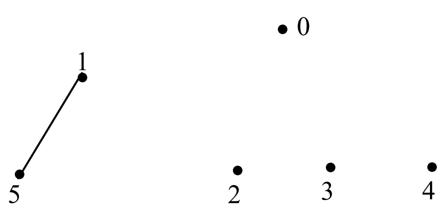
The Smarandache unit graph of S is



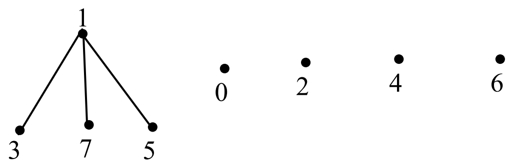
Example 3.28: Let S = Z_5 be a S-semigroup under multiplication modulo 5. The Smarandache unit graph of S is



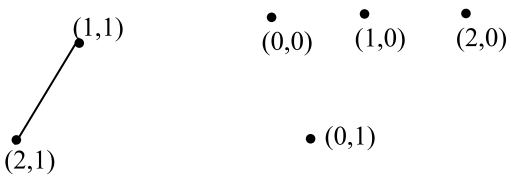
Example 3.29: Let P = (Z_6, ×) be a S-semigroup. The Smarandache unit graph of P is



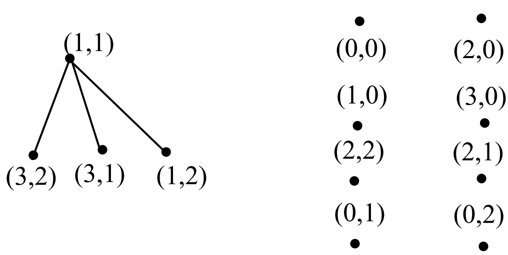
Example 3.30: The Smarandache unit graph of S = $(Z_8, \times)$ is as follows:



Example 3.31: Let S = $\{Z_3 \times Z_2 = (a, b) \mid a \in Z_3 \text{ and } b \in Z_2, \times\}$ be a S-semigroup. The Smarandache unit graph of S is



Example 3.32: Let M = $\{Z_4 \times Z_3 = (a, b) \mid a \in Z_4 \text{ and } b \in Z_3\}$ be a S-semigroup under $\times$. The Smarandache unit graph of M is given below.



We see almost all S-semigroups which are zero divisors and units have a nontrivial S-unit graph.

Example 3.33: Let
M = $\{(2Z_8, 2Z_6) = (a, b) \mid a \in 2Z_8 = \{0, 2, 4, 6\} \text{ and } b \in 2Z_6 = \{0, 2, 4\}\}$ be a semigroup. Clearly M is not a S-semigroup. Hence the question of S-unit graph does not arise.

Thus we have class of S-semigroups for which S unit graphs do not exists.

THEOREM 3.4: *Let $S = \{p_i Z_n \mid p_i \mid n; 1 \leq i \leq t$ where $n = p_1, \ldots, p_t, p_1, p_2, \ldots, p_1$ are distinct primes$\}$ be a semigroups under product$\}$. Only those semigroups which are S-semigroups have S-unit graphs associated with them even though S does not contain 1.*

This proof will be understood by the reader once he/she understands the following examples.

Example 3.34: Let S = $\{10Z_{30} = \{0, 10, 20\}\}$ be a semigroup under product. S is a S-semigroup with 10 as its unit G = {10, 20} is a group given by the following table.

×	10	20
10	10	20
20	20	10

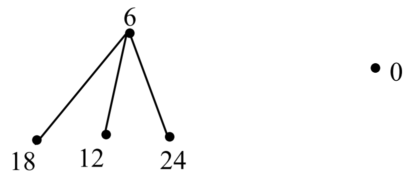
The S-unit graph is S is

Consider M = {0, 6, 12, 18, 24} be a semigroup under multiplication modulo 30.

This has G = {6, 12, 18, 24} to be a group given by the following table.

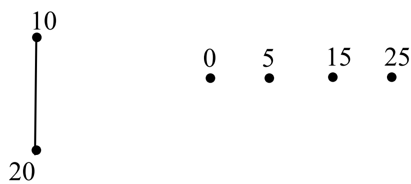
×	6	12	18	24
6	6	12	18	24
12	12	24	6	18
18	18	6	24	12
24	24	18	12	6

This is a group with 6 as the identity.
The S-unit graph of M is as follows:



Take W = {0, 15}, a semigroup under product. W is not a S-semigroup.

Let T = {0, 5, 10, 15, 20, 25}. M = {10, 20} is a group. Hence T is a S-semigroup and the unit S-unit graph of T.

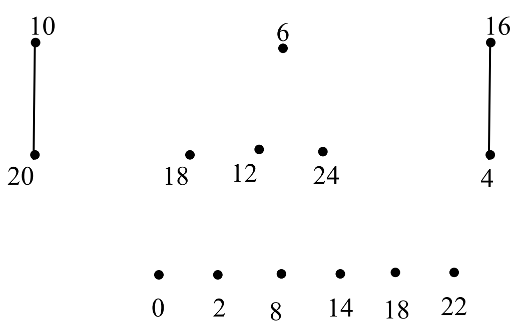


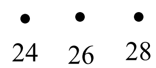
Consider

L = {0, 2, 4, 6, 8, 10, 12, 14, 16, 18, 20, 22, 24, 26, 28}

be a semigroup L is a S-semigroup as {10, 20} in L is a group with 10 as identity. R = {6, 12, 18, 24} is also a S-semigroup with 6 as identity V = {4, 16} is also a group of L.

This has S-unit graphs, two graphs associated with it.





Thus we see from this example a semigroup can have several S-unit graphs. Consider the

Y = {0, 3, 6, 9, 12, 15, 18, 21, 24, 27} $\subseteq Z_{30}$; to be a semigroup under product modulo 30. G = {9, 21} $\subseteq$ Y is a group given by the following table.

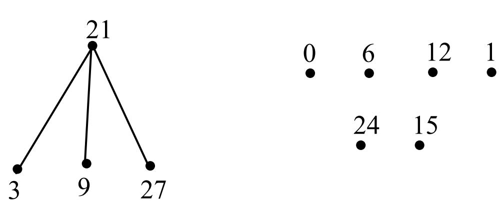
×	9	21
9	21	9
21	9	21

The set H = {3, 9, 21, 27} is a group with 21 as the multiplicative identity.

The table of H is as follows:

×	3	9	21	27
3	9	27	3	21
9	27	21	9	3
21	3	9	21	27
27	21	27	3	9

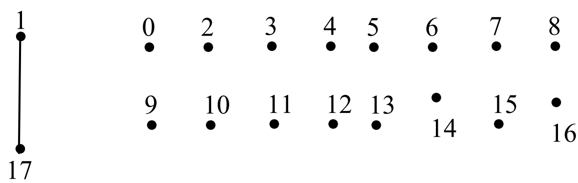
The S-unit graph of Y is as follows.



Example 3.35: Let S = $(Z_{18}, \times)$ be a semigroup. P_1 = {0, 2, 4, 6, 8, 10, 12, 14, 16} $\subseteq$ S, P_2 = {0, 3, 6, 9, 12, 15} $\subseteq$ S and P_3 = {0, 6, 12} $\subseteq$ S and P_4 = {0, 9} $\subseteq$ S be semigroups M_1 = {0, 1, 17} is a S-semigroup of S.

P_4 is not a S-semigroup. M_2 = {0, 1, 10} is not a S-semigroup for $10^2 = 10$. P_2 is not a S-semigroup. However S is a S-semigroup.

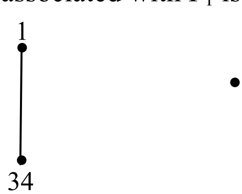
The S-unit graph of S is as follows:



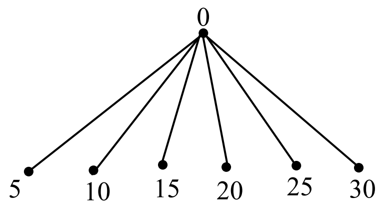
From this example we see nZ_{18} for n = 2, 3, 6, 12, 9 is a S-semigroup so the S-unit graph does not exist for nZ_{18} but S-unit graph exist for Z_{18}.

Example 3.36: Let M = $(Z_{35}, \times)$ be a semigroup under product $\times$ modulo 35. Take P_1 = {34, 1, 0} $\subseteq$ M; P_1 is a subsemigroup. Infact P_1 is a S-semigroup. P_2 = {0, 5, 25, 20, 30, 10, 15} $\subseteq$ M is a semigroup.

The unit graph associated with P_1 is

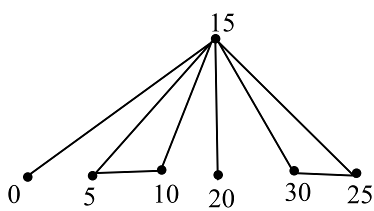


The zero divisor graph associated with P_2 is



We see P_2 cannot have a unit graph associated with it as $P_2 \subseteq Z_{35}$ and 1 is its unit.

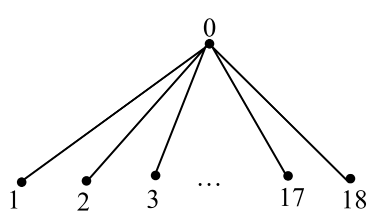
If we take P_2 as a S-semigroup then 15 acts as the unit so the Smarandache unit graph of the S-semigroup P_2 is as follows:



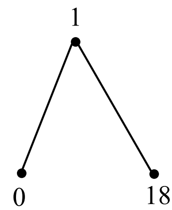
Clearly the S-unit graph of P_2 is not a tree. However the S-zero divisor graph of P_2 is a tree.

Example 3.37: Let us consider the semigroup $S = \{Z_{19}, \times\}$.

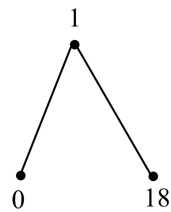
Clearly the zero divisor graph of S is a tree with 19 vertices given by



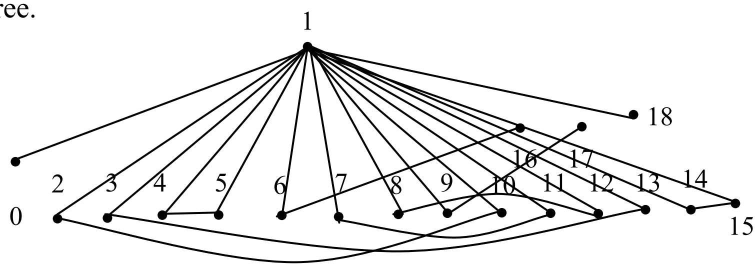
Clearly S is a S-semigroup for P = {0, 1, 18} is also a S-semigroup as {1, 19} = G is a group. Now the S zero divisor graph of P is also a tree with 3 vertices.



Now the S-unit graph of P is



is also a tree with 3 vertices. Now the unit graph of S is not a tree.



In view of this we have the following theorem.

THEOREM 3.5: *Let S = {Z_p, ×} be a semigroup; p a prime.*

(i) The unit graph of S is not a tree.
(ii) The zero divisor graph of S is a tree
(iii) S has a S-unit graph which is a tree with 3 vertices.

Proof is direct and hence is left as an exercise to the reader.

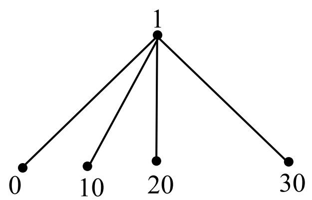
Example 3.38: Let S = {Z_{40}, ×} be a semigroup.

Take $P_1 = \{0, 2, 4, \ldots, 38\}$ to be a subsemigroup of S. The zero divisor graph of P_1 is not a tree.

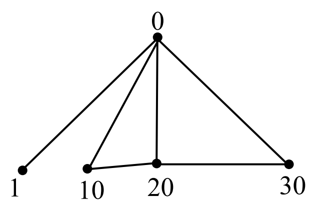
However the unit graph of P is a tree.

Consider $P_2 = \{1, 10, 20, 30, 0\}$ a subsemigroup of S.

The unit graph of P_2 is

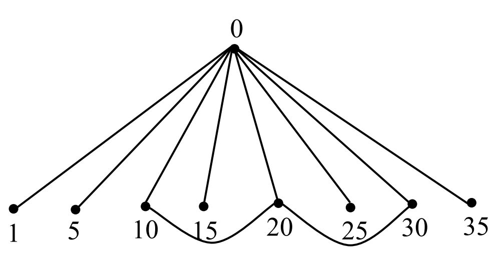


The zero divisor graph of P_2 is



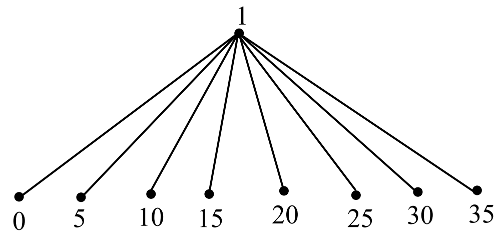
However the zero divisor graph of P_2 is not a tree.

Take $P_3 = \{0, 5, 10, 15, 20, 25, 30, 35, 1\}$ be the subsemigroup of S. The zero divisor graph of P_3 is

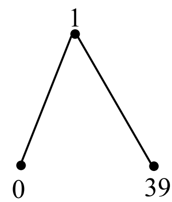


Clearly it is not a tree.

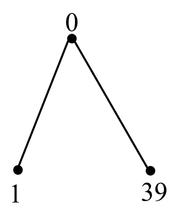
However the unit graph of P_3 is a tree given in the following.



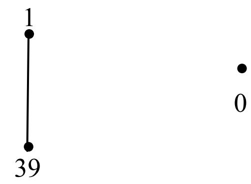
Consider $P_4 = \{0, 1, 39\}$ a subsemigroup of S. The unit graph of P_4 is



The zero divisor graph of P_4 is



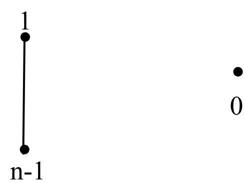
The S-unit graph of P_4 is



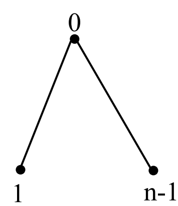
In view of all this we have the following result.

THEOREM 3.6: *Let $S = \{Z_n, \times\}$ be a semigroup. S is a S-semigroup with a special S-unit subgraph containing a point and a tree with 2 vertices and a special zero subgraph which is tree with 3 vertices.*

Proof: Take $P = \{0, n–1, 1\} \subseteq S$; P is a S-semigroup. The S-unit graph associated with P is



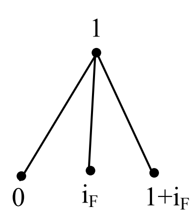
and the special zero divisor graph of P is



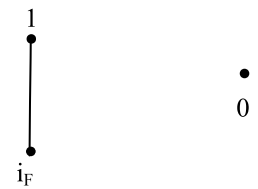
a tree with 3 vertices.

Next we proceed onto describe the unit graph of complex modulo integers.

Example 3.39: Let $C(Z_2) = \{0, 1, i_F, 1+i_F\}$ be a semigroup of complex modul integers under product. The unit graph of $C(Z_2)$ is a tree with four vertices.



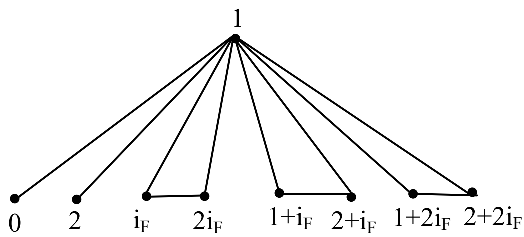
This is a S-semigroup and the S-unit graph of $C(Z_2)$ is



Example 3.40: Let

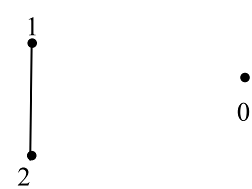
$S = \{C(Z_3), \times\} = \{0, 1, 2, i_F, 2i_F, 1+i_F, 2+i_F, 1+2i_F, 2+2i_F\}$ be the complex modulo integer semigroup.

The unit graph of S is as follows.

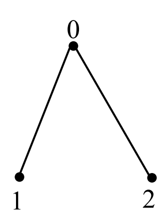


Clearly the unit graph of S is not a tree.

The S-unit subgraph of $P = \{0, 1, 2\}$ is a tree with two vertices and a point.



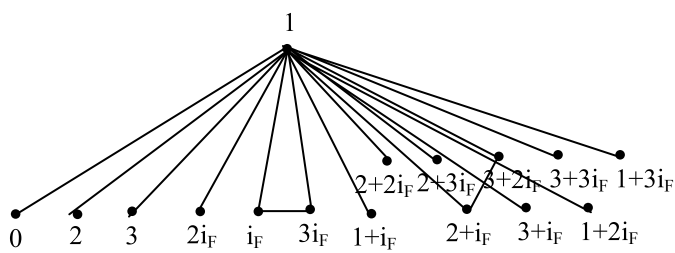
The special unit subgraph related with P is a tree with 3 vertices.



Example 3.41: Let M = $\{C(Z_4), \times\}$ be a semigroup of finite complex modulo integers.

M = $\{0, 1, 2, 3, i_F, 2i_F, 3i_F, 1+i_F, 1+2i_F, 1+3i_F, 2+i_F, 2+2i_F, 2+3i_F, 3+i_F, 3+2i_F, 3+3i_F\}$.

Now we give the unit graph of M.



Clearly the unit graph of M is not a tree.

Example 3.42: Let N = $\{C(Z_5), \times\}$ be a semigroup of finite complex modulo integers. The unit graph of $C(Z_5)$ is not a tree. Follows from the simple fact $2.3 \equiv 1 \bmod (5)$.

In view of this we have the following result the proof of which is simple and direct.

THEOREM 3.7: *Let P = $\{C(Z_p), \times\}$ (p a prime) be a semigroup of finite complex modulo integers. The unit graph of P is not a tree.*

Example 3.43: Let S = $\{C(Z_{28}), \times\}$ be a semigroup of finite complex modulo integers. The unit graph of S is not a tree.

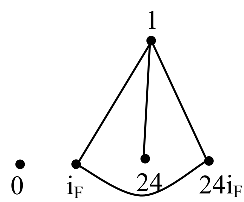
Consider i_F, $27i_F$ is S we see $i_F \times 27i_F = 27\,i_F^2 = (27)^2 = 1$ so the unit graph is not a tree.

In view of this we have the following theorem.

THEOREM 3.8: *Let S = {C (Z_n), ×, where n is a composite number} be a semigroup under ×. The unit graph of S is not a tree.*

Proof: Follows from the simple fact i_F, $(n–1)i_F$ in S are such that $i_F \times (n–1)i_F = (n–1)\ i_F^2 = (n–1)^2 = 1$ as $i_F^2 = n–1$ in $C(Z_n)$. Hence the claim.

Example 3.44: Let S = {C (Z_{25}), ×} be a finite complex modulo integer semigroup. Consider the Smarandache subsemigroup P = {0, 1, i_F, 24, $24i_F$} ⊆ S. The Smarandache unit graph of P is as follows:

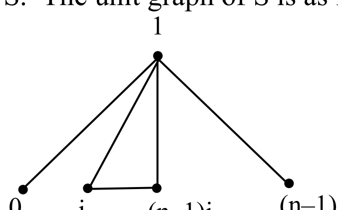


Clearly this not a tree.

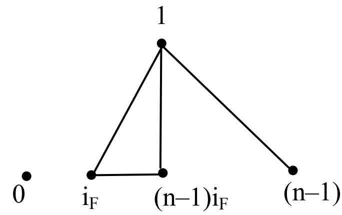
In view of this we have the following theorem.

THEOREM 3.9: *Let P = {C(Z_n), ×} be a finite complex modulo integer semigroup. P has a Smarandache subsemigroup S of order 5 and S has a group of order 4. The Smarandache unit graph of S is not a tree.*

Proof: Consider S = {0, i_F, 1, (n–1), $(n–1)i_F$} ⊆ P is a S-subsemigroup of S. The unit graph of S is as follows.



Clearly the unit graph is not a tree. The Smarandache unit graph of S is as follows:

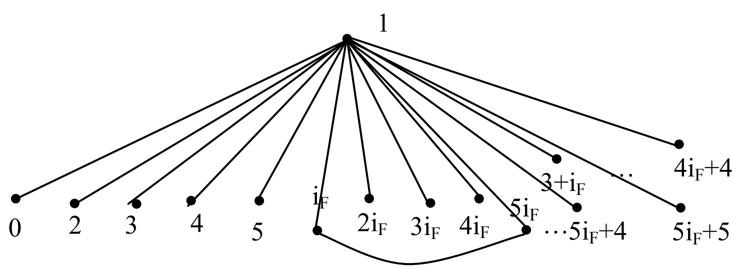


this is also not a tree.

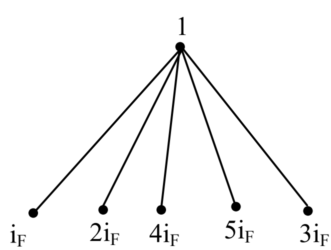
It is pertinent to mention that not all subgraphs of a unit graph correspond to a subsemigroup of the semigroup S. We call those unit subgraphs of the unit graph which corresponds to a subsemigroup as a special unit subgraph and the unit graph corresponding to a S-subsemigroup as special Smarandache unit subgraph.

We will only illustrate this situation by a few examples.

Example 3.45: Let S = $\{C(Z_6), \times\}$ be a semigroup of complex modulo integers. The unit graph associated with S is as follows.

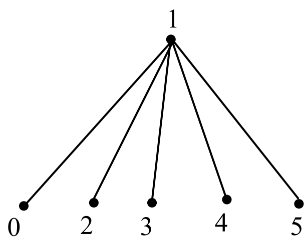


We see the unit graph of S is not a tree. Consider the subgraph of this unit graph.

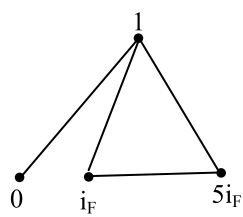


Clearly this is a tree but V = {1, i_F, $2i_F$, $4i_F$, $5i_F$, $3i_F$} $\subseteq$ S is not a subsemigroup as $3i_F \times 2i_F \equiv 0 \notin V$. However it is a subgraph of the graph.

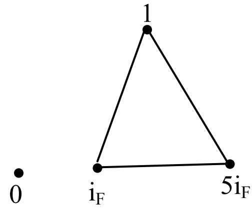
Consider the subsemigroup W = {0, 1, 2, 3, 4, 5} $\subseteq$ S. The unit graph associated with W is as follows:



Clearly it is a tree and a special unit subgraph of S. Consider V = {0, 1, i_F, $5i_F$} $\subseteq$ S; V is a Smarandache subsemigroup of S. The unit graph of V is as follows:



The Smarandache unit graph of V is



In view of this example we have the following theorem.

THEOREM 3.10: *Let S ={C(Z_n), ×} be a semigroup of finite modulo integers. Let G be the unit graph of S. S has both* *subsemigroup unit special subgraphs of G as well as subgraphs which are not associated with any subsemigroup of G.*

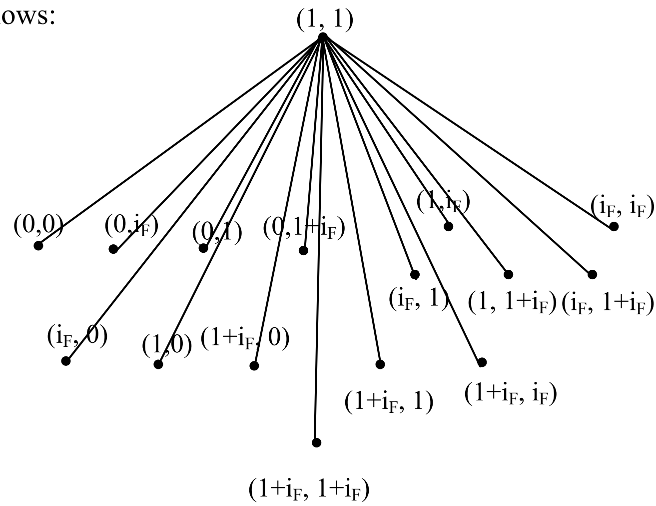
The proof is direct and hence left as an exercise to the reader.

We can also find the direct product of finite complex modulo integer semigroups.

We will illustrate these situations only by examples.

Example 3.46: Let S = C $(Z_2) \times C(Z_2) = \{(a, b) \mid a, b \in C(Z_2) = \{0, 1, i_F, 1+i_F)$ with $i_F^2 = 1\}$ be the semigroup of finite complex modulo integers.

Clearly S has 16 elements. The unit graph of S is as follows:



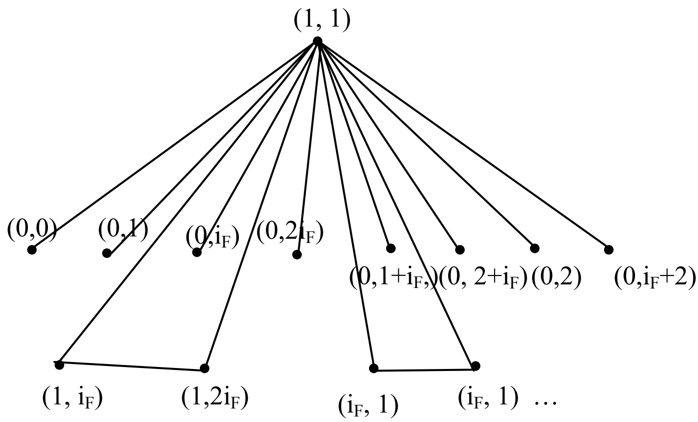
Clearly this is a tree with 16 vertices.

Example 3.47: Let

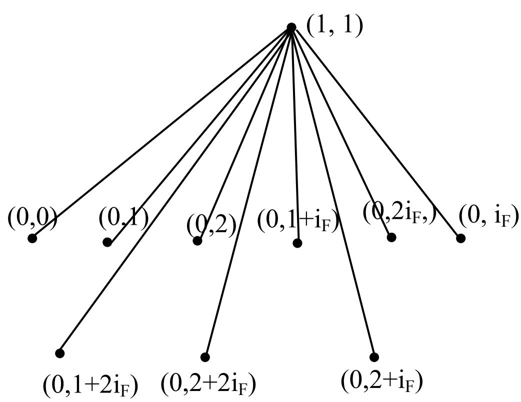
$$S = C(Z_2) \times C(Z_3) = \{(a, b) \mid a \in C(Z_2), b \in C(Z_3)\}$$

be a semigroup of finite complex modulo integers.

The number of elements in S is 36. The unit graph of S is as follows:

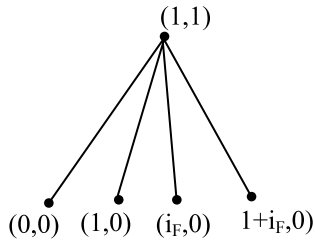


We see the unit graph of S is not a tree. This has subgraphs which are trees. For instance consider W = {(0, 1), (0, 2), (0, i_F), (0, 2i_F), (0, 0), (0, 1+i_F), (0, 1+2i_F), (0, 2+i_F), (1, 1) (0, 2+2i_F)} $\subseteq$ S; W is a subsemigroup. The unit graph associated with W is as follows:



Clearly the unit graph is a tree and this is the special unit subgraph of the unit graph associated with S.

Consider V = {(0, 0), (1, 1), (1,0), (i_F, 0), (1+i_F, 0)} $\subseteq$ S be the subsemigroup of S. The unit graph of V is as follows:



This is also a tree and this subgraph is the special unit subgraph of the unit graph of S.

In view of this we have the following interesting theorem.

THEOREM 3.11: *Let S = (C $(Z_n) \times C(Z_m)$ | $C(Z_n)$ and $C(Z_m)$ are finite complex modulo intergers} = {(a, b) | a $\in C(Z_n)$ and b $\in C(Z_m)$} be a complex modulo integer semigroup. The unit graph is not a tree but S has two subsemigroups whose special unit subgraphs are trees with (m+1) and (n+1) vertices.*

Proof: We know the unit graph associated with S is a not a tree. For take $(1, i_F)$ and $(1, (m-1)i_F) \in S$; these two elements are inverses of each other; for

$$\begin{aligned}(1, i_F) \times (1, (m-1)i_F) &= (1, (m-1)\ i_F^2) \\ &= (1, (m-1)^2) \\ &= (1,1) \text{ as } i_F^2 = m-1 \text{ and } (m-1)^2 = 1.\end{aligned}$$

Similarly $((n-1)i_F, 1)$ and $(i_F, 1)$ in S are such that

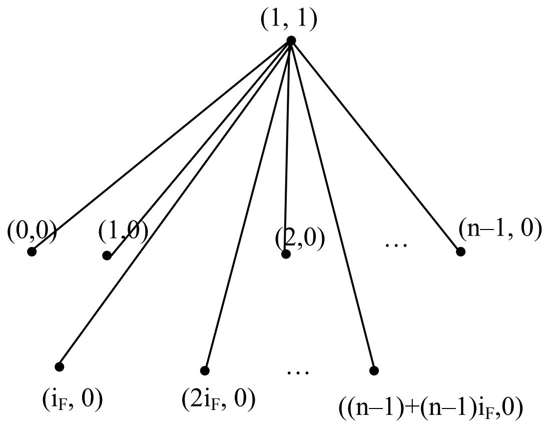
$$((n-1)i_F, 1) \times (i_F, 1) = ((n-1)\ i_F^2, 1) = ((n-1)^2, 1) = (1, 1);$$

using the fact $(n-1)^2 = 1$ and $i_F^2 = n-1$.

Thus the unit graph of S is not a tree.

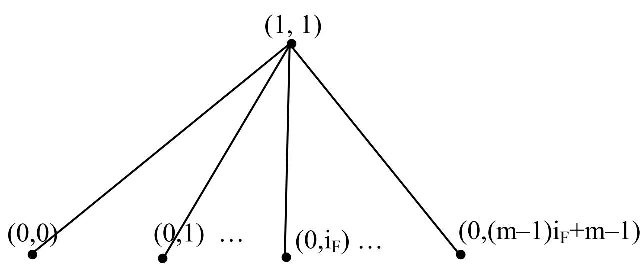
Now consider

P = {(1,1), (a,0) | a $\in C(Z_n) \times \{0\}$) that is a $\in Z_n\} \subseteq$ S, P is a semigroup with unit (1,1). The unit graph of P is a tree with (n+1) vertices. Infact this graph is a special unit subgraph of the unit graph associated with S.



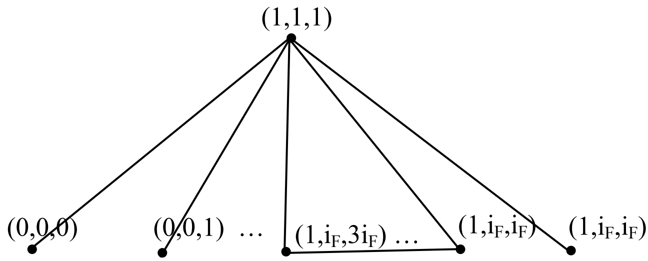
Consider

M = {(1,1), (0, a) | a ∈ Z_m or (0, a) ∈ {{0} ∈ C (Z_m)}} ⊆ S, M is a subsemigroup with unit (1,1). Clearly the unit graph associated with M is

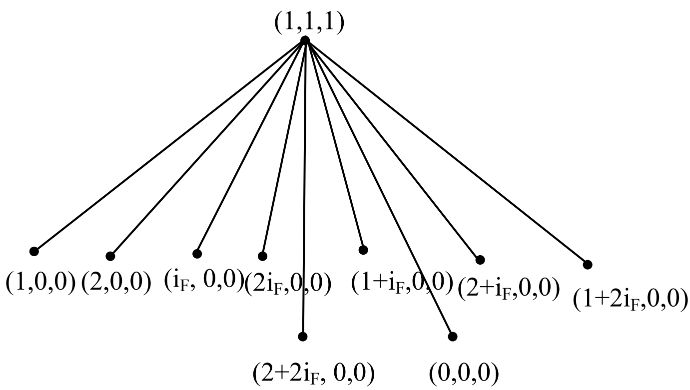


tree with (m+1) vertices. Thus the unit graph is the special unit subgraph of the unit graph associated with S.

Example 3.48: Let S = C(Z_3) × C(Z_2) × C(Z_4) = {(a, b, c) | a ∈ C(Z_3), b ∈ C(Z_2) and c ∈ C(Z_4)} be a semigroup of complex modulo finite integers. The unit graph G of S is as follows:

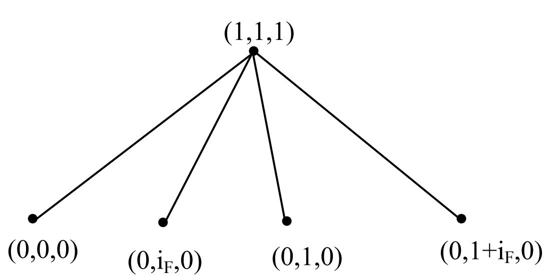


Clearly G is not a tree. Consider H = {(a, 0, 0), (1, 1, 1) | a ∈ C(Z_3)} ⊆ S, H is a subsemigroup of S. The unit graph G_1 of H is as follows:



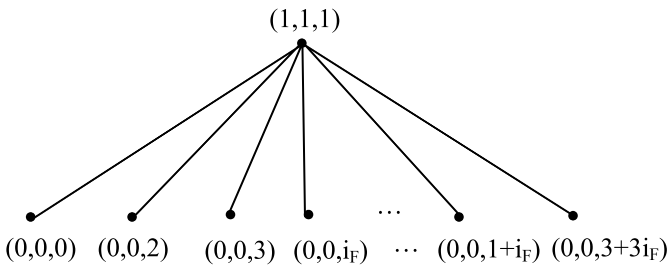
G_1 is a tree with 10 vertices and G_1 is a special unit subgraph of the unit graph G.

Take P = {(0, a, 0) and (1, 1, 1) | a ∈ C(Z_2)} ⊆ S, P is a subsemigroup with unit (1, 1, 1) of S. The unit graph G_2 associated with the subsemigroup P is as follows.



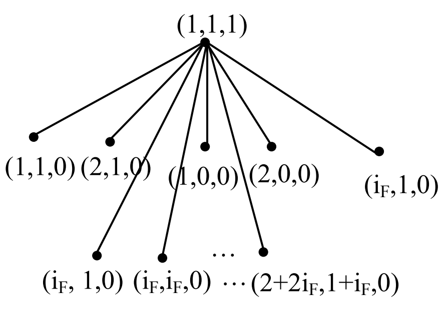
Clearly the unit subgraph G_2 is a special unit subgraph of G and is a tree with five vertices.

Consider V = {(0, 0, a) and (1, 1, 1) | a ∈ $C(Z_4)$} ⊆ S, V is a subsemigroup of S. The unit graph G_3 associated with V is as follows:



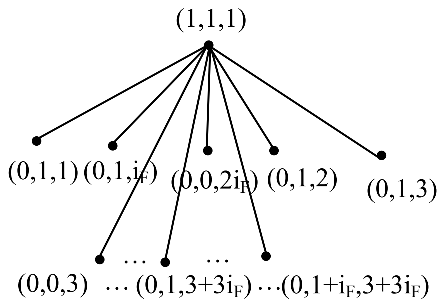
G_3 is a special unit subgraph of G which is a tree with 17 vertices.

Let
W={(a, b, 0) and (1, 1, 1) | (a, b, 0) ∈ $C(Z_3) \times C(Z_2) \times \{0\}$} ⊆ S; W is a subsemigroup with unit of the semigroup S. The unit graph G_4 associated with W is as follows.



Clearly G_4 is the special unit subgraph of G which is tree with 37 vertices.

Consider
B = {(0, a, b), (1, 1, 1) | (0, a, b) ∈ $\{0\} \times C(Z_2) \times C(Z_4)$} ⊆ S; B is a subsemigroup with unit of S. The unit graph G_5 associated with B is as follows:

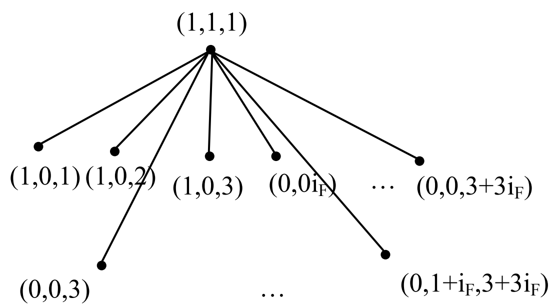


The unit graph G_5 of G is a special unit subgraph of G which is a tree with 65 vertices.

Finally let

$C = \{(a, 0, b), (1, 1, 1) \mid (a, 0, b) \in C(Z_3) \times \{0\} \times C(Z_4)\} \subseteq S$;

be a subsemigroup of S. The unit graph G_6 of C is as follows:



Clearly G_6 is a unit graph and is a special unit subgraph of G which is a tree with 145 vertices. We see G has atleast 6 special unit subgraphs which are trees. Inview of this we have the following theorem.

THEOREM 3.12: *Let $S = (C(Z_n) \times C(Z_n) \times \ldots \times C(Z_{n_t}) = \{(a_1, a_2, \ldots, a_t) \mid a_i \in C(Z_{n_i}); 1 \leq i \leq t\}$ be a semigroup of complex modulo integers.*

(1) The unit graph G associated with S is not a tree (all $n_i \neq 2$)

(2) G has at least $t + {}_tC_2 + {}_tC_3 + \ldots + {}_tC_{t-1}$ number of special unit subgraphs which are trees.

Proof follows using simple number theoretic techniques. Next we proceed onto define Smarandache units in semigroups.

DEFINITION 3.2: *Let (S, .) be semigroup with unit. We say $x \in S \setminus \{1\}$ is a Smarandache unit (S-unit) if there exists $y \in S$ with*

(i) $x.y = 1$

(ii) There exists $a, b \in S \setminus \{x, y, 1\}$

(a) $xa = y$ or $ax = y$ or

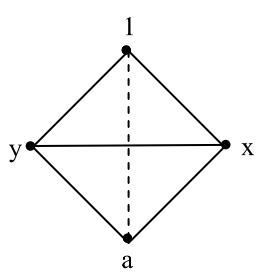
(b) $yb = x$ or $by = x$ and

(c) $ab = 1$

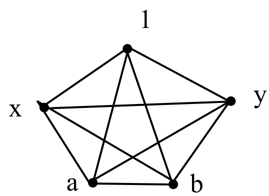
(ii)a or (ii)b is satisfied it is enough to make x a S unit.

DEFINITION 3.3: *Let S be a semigroup with unit. Let $x \in S \setminus \{0\}$ be such that $x.y = 1$ and $ax = y$ and $ay = x$, $a \in S$ then the special S-unit graph of the set $\{1, x, y, a\}$ as its vertices and the edges are 1 to x, 1 to y x to y and the edge 1 to a is drawn as a dotted edge; or can be usual edge.*

The Special S-unit graph of $\{1, x, y, a\}$ is as follows:

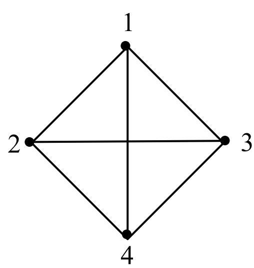


Suppose we have instead of the set {1, x, y, a} the set $\{1, x, y, a, b, a \neq b; ab = 1\}$ then the special S-unit graph of the set {1, x, y, a, b} as its vertices is as follows:



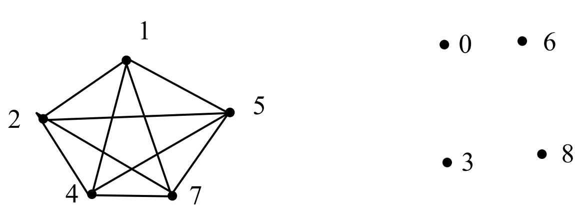
The graphs are self explanatory. First we will illustrate this situation by some examples.

Example 3.49: Let S = $\{Z_5, \times\}$ be a semigroup with unit. Clearly $3 \in S$ is a S-unit for $2.3 \equiv 1 \pmod 5$ and $4 \in Z_5$ is such that $4.2 \equiv 3 \pmod 5$ and $4.3 \equiv 2 \pmod 5$. The special S-unit graph of {1, 2, 3, 4} is as follows:

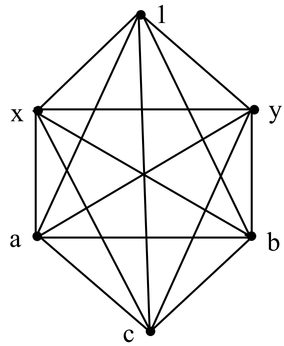


Example 3.50: Let S = $\{Z_9, \times\}$ be the semigroup with unit. Let $2 \in Z_9$ is a S-unit of S as $5 \in Z_9$ is such that $2.5 \equiv 1 \pmod 9$ and $7, 4 \in Z_9$ satisfies $2.7 \equiv 5 \pmod 9$ and $4.5 \equiv 2 \pmod 9$.

The special S-unit graph of {2, 5, 7, 4, 1} is as follows:

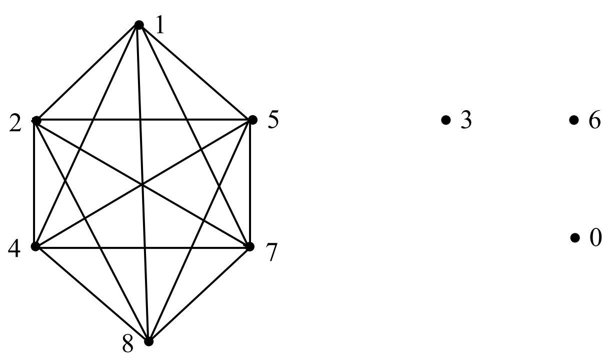


DEFINITION 3.4: *Let S be a semigroup with unit. An element x $\in S \setminus \{1\}$ is said to be a super Smarandache unit of S if there is a y $\in$ S with xy = 1 (y $\neq$ x) and a, b $\in S \setminus \{x, y, 1\}$ (a $\neq$ b) with {a, b, x, y, 1} as a S-unit. If there exist a, c $\in S \setminus \{a, b, x, y, 1\}$ with c^2 = 1 and ca = b and cb = a then we call x to be a Smarandache super unit in S. We can get the super special Smarandache unit graph of S given by the set = {1, x, y, a, b, c} is as follows:*



We will first illustrate this situation by some examples.

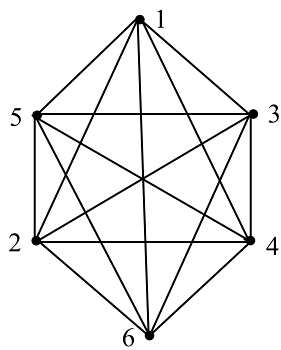
Example 3.51: Let S = $\{Z_9, \times\}$ be a semigroup. Consider 2 $\in$ S, we see 2 $\times$ 5 $\equiv$ 1 (mod 9); so 2 is a unit. For we have 4 and 7 in S such that 4.7 $\equiv$ 1 (mod 9), and 2.7 $\equiv$ 5 (mod 9) and 5.4 $\equiv$ 2 (mod 9). Finally 8 $\in$ S is such that 8.2 $\equiv$ 7 (mod 9) and 8.5 $\equiv$ 4 (mod 9). Thus 2 is a Super Smarandache special unit of S. The Super special S-unit graph of 2 is as follows:



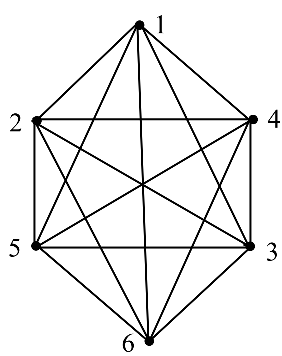
Example 3.52: Let S = {Z_5, ×} be a semigroup S has no super special S-unit. For if a semigroup with unit should have a super special S-unit the order of S must be atleast six or greater than six. Since order of S is five S has no super special S-unit.

Example 3.53: Let S = {Z_7, ×} be a semigroup with unit. Let 4, 2 ∈ $Z_7 \setminus \{0\}$ be such that $2.4 \equiv 1 \pmod 7$ we see 5 and 3 in S are such that $5.3 = 1 \pmod 7$ and $3.4 \equiv 5 \pmod 7$ and $5.2 \equiv 3 \pmod 7$, $6 \equiv Z_7$ is such that 5 (or 3) is a super special S-unit of S.

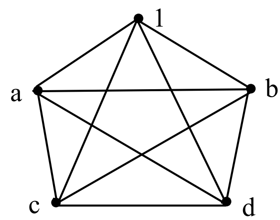
For the super special S-unit graph associated with the set {1, 3, 5, 2, 4, 6} is as follows:



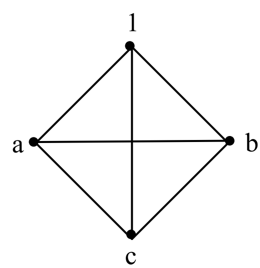
The natural question is 5 and 3 are super S-units will 2 and 4 be super S-units. The answer is yes we see 2 and 4 are also super S-units for consider the set {1, 2, 4, 5, 3, 6} the super special Smarandache unit graph is as follows:



We see we do not have a S-unit graph of the form.



However these does not exist a S-unit graph of the form in $S = \{Z_{11}, \times\}$.



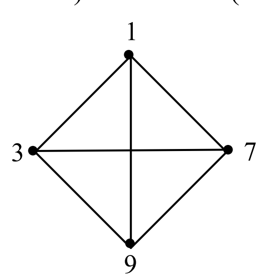
Now we give some more examples.

Example 3.54: Let $S = \{Z_6, \times\}$ be a semigroup. We see the only invertible element of S is 5. So S has no S-unit.

Example 3.55: Let $S = \{Z_8, \times\}$ be a semigroup with unit. The elements in S which are invertible are 7, 3 and 5 with one.

Clearly $S = \{Z_8, \times\}$ does not have S-unit or super S-units.

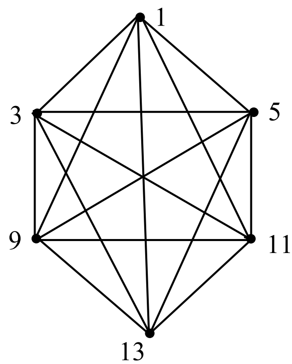
Example 3.56: Let $S = \{Z_{10}, \times\}$ be semigroup with unit. 3 is a S unit and the Special Smarandache unit of 3 is and $3.7 \equiv 1 \pmod{10}$, $9.3 \equiv 7 \pmod{10}$ and $7.9 \equiv 3 \pmod{10}$.



Clearly S has no Super Smarandache units.

Example 3.57: Let S = $\{Z_{12}, \times\}$ be a semigroup with unit. Clearly 5, 7 and 11 in S are units and all of them are self inversed element.

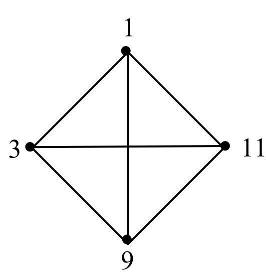
Example 3.58: Let S = $\{Z_{14}, \times\}$ be a semigroup with unit. $3 \in$ S is a super Smarandache unit of S. Take $\{1, 3, 5, 9, 11, 13\} \subseteq$ S, the super special Smarandache unit of S is as follows:



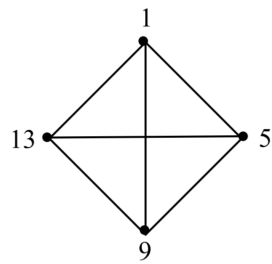
0, 2, 7, 4, 6, 8, 10 and 12 are zero divisors of S.

Example 3.59: Let S = $\{Z_{16}, \times\}$ be a semigroup with unit. 3 is a S-unit and the inverse of 3 is 11 for $3.11 \equiv 1 \pmod{16}$ and $9^2 \equiv 1 \pmod{16}$.

The S-unit graph of 3 is

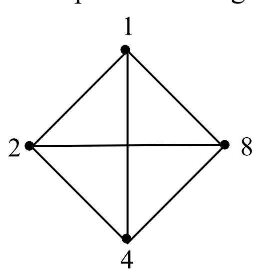


Also 13 is a S unit of S for 13.5 ≡ 1 (mod 16) and the S unit graph of 13 is as follows:

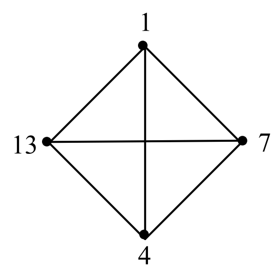


However S has no super S-unit.

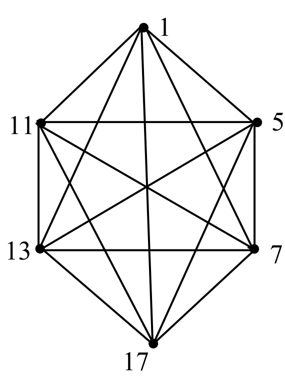
Example 3.60: Consider S = {Z_{15}, ×} be a semigroup with unit. Clearly S has no super S units. However S has two sets of S-units whose Special S-unit graph is as follows.



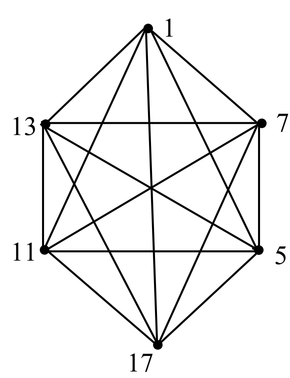
and



Example 3.61: Let S = {Z_{18}, ×} be a semigroup with unit. 11 and 13 are two super Smarandache super Smarandache special unit graphs.

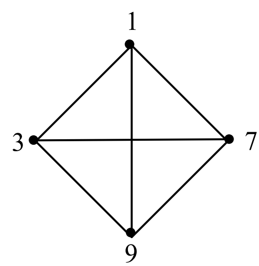


and

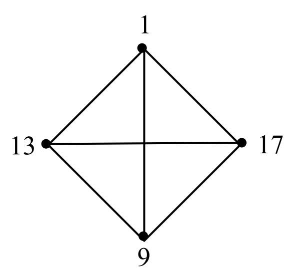


Example 3.62: Let S = {Z_{20}, ×} be a semigroup with unit. We see $3 \times 7 \equiv 1 \pmod{20}$, $9^2 \equiv 1 \pmod{20}$, $11^2 \equiv 1 \pmod{20}$, $19^2 \equiv 1 \pmod{20}$ and $13 \times 17 \equiv 1 \pmod{20}$. Clearly S has no super Smarandache units but S has S-units.

For 3 is a S-unit and the special S-unit graph of 3 is as follows:

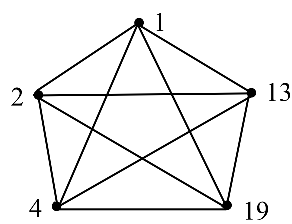


Also 13 is a S-unit and the special S-unit graph associated with 13 is as follows:

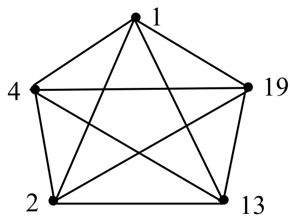


Next we study the following.

Example 3.63: Let S = {Z_{25}, ×} be a semigroup with unit. 2 is a S-unit of S the special S-unit graph of 2 is



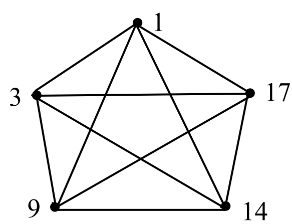
However 2 is not a super S-unit. 4 is also a S-unit and not a Super S-unit, the special S-unit graph of 4 is



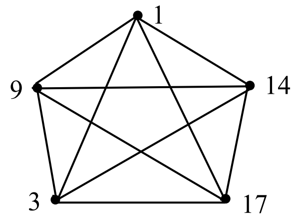
3 is also only a S-unit and not a super S-unit for $3 \times 17 = 1 \pmod{25}$ and 9, 14 in S are such that

$$3 \times 14 = 17 \pmod{25}$$
$$9 \times 17 = 3 \pmod{25}.$$

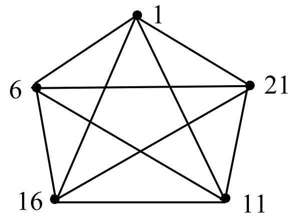
The S-unit graph associated with them are



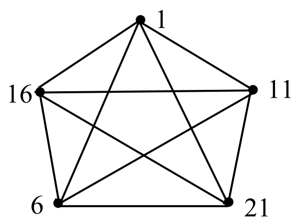
Similarly 9 is a S-unit and not a Super S-unit and the S unit graph of 9 is as follows:



6 is a S-unit of S and the special S-unit graph of 6 is as follows:

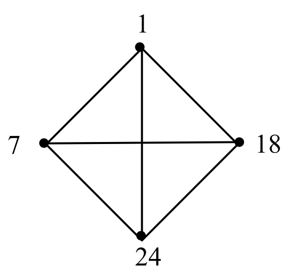


Similarly 16 is a S unit of S and the special S unit graph of 16 is



However both 16 and 6 are not super S units of S. Further 12 (23), and 8 (22) are not even S units they are only units.

However 7 is S-unit of the form



But 12 or 23 and 8 or 22 are not S units even of the above form.

Next we proceed onto define the notion of quasi Smarandache unit triple.

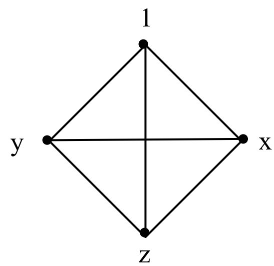
DEFINITION 3.5: *Let S = {Z_n, ×} be a semigroup with unit. If we have a triple {x, y, z} ∈ Z_n \ {1} such that*

$$x \times x = 1 \pmod{n} \quad y \times y \equiv 1 \pmod{n} \text{ and}$$
$$z \times z \equiv 1 \pmod{n}$$

and

$$x \times x = z \pmod{n}, \quad x \times z = y \pmod{n} \text{ and}$$
$$y \times z = x \pmod{n}$$

then we define the triple {x, y, z} to be a quasi Smarandache unit triple or quasi Smarandache triple unit or quasi triple Smarandache unit. The unit graph associated with the vertex set {1, x, y, z} will be known as the quasi S-unit triple graph and it will be as follows:



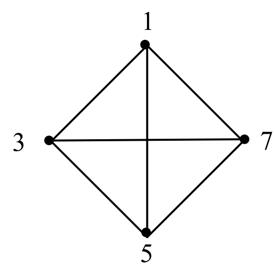
First we will illustrate this situation by some examples.

Example 3.64: Let S = {Z_8, ×} be a semigroup with unit. 7 ∈ Z is such that $7^2 \equiv 1 \pmod{8}$ 5 ∈ s is such that $5^2 \equiv 1 \pmod{8}$ and 3 ∈ S is such that $3^2 \equiv 1 \pmod{8}$. The other elements 4, 2 and 6 are zero divisors in S.

The triple {5, 3, 7} is a quasi S-unit triple for

$5 \times 3 \equiv 7 \pmod{8}$ $3 \times 7 \equiv 5 \pmod{8}$ and $7 \times 5 \equiv 3 \pmod{8}$.

The quasi S-unit triple graph is as follows:

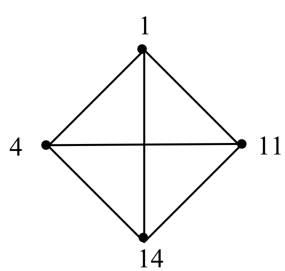


Example 3.65: Let S = $\{Z_6, \times\}$ be a semigroup with unit S has no quasi triple S-unit.

Example 3.66: Consider S = $\{Z_{15}, \times\}$ be a semigroup with unit. $\{11, 14, 4\} \in$ S is a quasi triple S-unit of S. For $4^2 \equiv 1 \pmod{15}$, $14 \equiv 1 \pmod{15}$ and $11^2 \equiv 1 \pmod{15}$.

Further $4 \times 11 = 14 \pmod{15}$, $4 \times 4 = 11 \pmod{15}$, and $11 \times 4 = 4 \pmod{15}$.

The quasi S-unit graph with {1, 4, 11, 14} as its vertices are as follows:



Example 3.67: Let S = $\{Z_{10}, \times\}$ be a semigroup with unit. Clearly S has no quasi S-unit.

Example 3.68: Let S = $\{Z_{14}, \times\}$ be a semigroup with unit. S has no quasi S-unit.

Example 3.69: Let S = $\{Z_{22}, \times\}$ be a semigroup wth unit. S has no quasi S-unit.

Inview of this we have the following theorem.

THEOREM 3.13: *Let $S = \{Z_{2p}, \times\}$ be a semigroup with unit. p a prime. S has no quasi S-unit.*

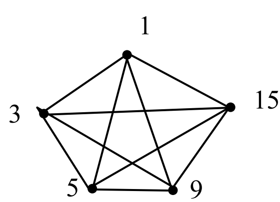
The proof is using only simple number theoretic methods. We can also have following corollary.

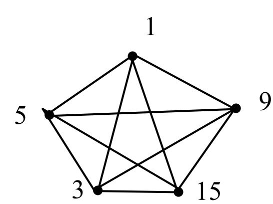
Corollary 3.1: Let $S = \{Z_{2p}, \times\}$ be a semigroup with unit (p a prime). All even numbers with p are zero divisors. Further odd numbers in Z_{2p} baring (2p–1) alone can result in the unit.

We now find whether S in example 3.67 has super S-units and S-units.

$$3 \times 15 \equiv 1 \pmod{22},$$
$$5 \times 9 \equiv 1 \pmod{22},$$
$$21^2 \equiv 1 \pmod{22},$$
$$19 \times 17 \equiv 1 \pmod{22} \text{ and}$$
$$13 \times 17 \equiv 1 \pmod{22},$$

3 and 5 in S are S-units but one not super S-units. For the S-unit graph attached with them are

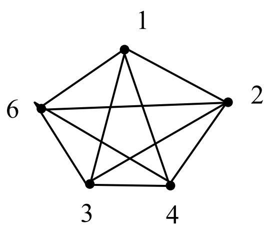


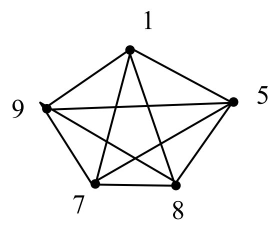


Clearly 19, 7, 3 and 15 are not S-units of S.

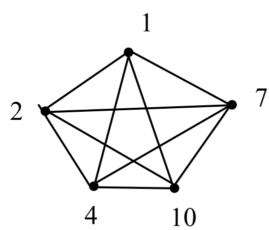
Example 3.70: Let $S = (Z_{11}, \times)$ be a semigroup with unit. Clearly S does not contain a quasi S-unit triple. However 6 (or z) is a S-unit, but 6 (or z) is a super S-unit. Also 9 (or 5) is a S-unit.

The special S-unit graph of them are as follows:





Example 3.71: Let S = (Z_{13}, ×) be a semigroup with unit. We see S has no quasi S-unit. 2 is a S-unit. For special S unit graph associated with 2 in S is as follows:



Other units in S are not S-unit.

In view of this we have the following theorem.

THEOREM 3.14: *Let S = {Z_p, ×} be a semigroup with unit, p a prime. S has no quasi S-unit triple.*

The proof is direct and uses only the fact Z_p is a field.

Now we leave the following as a open problem.

Problem 1: Let S = {Z_p, ×} be a semigroup with unit

(i) Can S have super S-unit?

(ii) How many S-unit can Z_p have?

Example 3.72: Let S(3) be the symmetric semigroup with unit.

Clearly $\begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}$ and $\begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}$ in S(3) are

such that $\begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix} \cdot \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}$,

$\begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}$ and

$\begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix} \cdot \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}$.

However this triple is not a quasi S-unit triple for

$$\begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix} \cdot \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}$$

and both the product is not $\begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}$.

Hence $\left\{ \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix} \right\}$ cannot be a quasi S unit triple. Also S(3) does not contain S-units.

Example 3.73: Let S(4) be the semigroup with unit,

$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 1 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 2 & 1 \end{pmatrix}$ are elements in S(4) such that

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \end{pmatrix} \cdot \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 1 & 2 \end{pmatrix} \cdot \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 1 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 2 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 2 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \end{pmatrix}$$

and

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \end{pmatrix} \cdot \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 1 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 2 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \end{pmatrix} \cdot \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 2 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 1 & 2 \end{pmatrix}$$

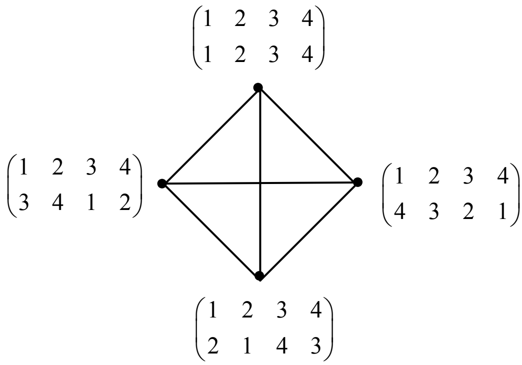
and

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 1 & 2 \end{pmatrix} \cdot \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 2 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \end{pmatrix}.$$

Thus S(4) has a quasi S-unit triple given by

$$\left\{ \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 1 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 2 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \end{pmatrix} \right\}$$

and the quasi S-unit graph is as follows:



Now we have the problem.

Problem 2: Can S(n) ($n \geq 5$) have quasi S-unit triple.

THEOREM 3.15: *Let S(n) be the symmetric group on n elements S(n) has several quasi S-unit triples.*

Proof: Let $x = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & \ldots & n \\ 2 & 1 & 4 & 3 & 5 & \ldots & n \end{pmatrix}$

$$y = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & \ldots & n \\ 3 & 4 & 1 & 2 & 5 & \ldots & n \end{pmatrix} \text{ and}$$

$$z = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & \ldots & n \\ 4 & 3 & 2 & 1 & 5 & \ldots & n \end{pmatrix} \text{ in S(n).}$$

We see {x, y, z} is a quasi S-unit triple.

Infact in (1, 2, 3, 4, …, n) if we fix n – 4 elements and permute the 4 elements by permuting two by two we get 3 permatations say a, b and c such that a.a = identity, b.b = identity and c.c = identity and

$$a.b = b.a = c,$$
$$a.c = c.a = b \text{ and}$$
$$b.c = c.b = a.$$

Hence the claim.

Thus we have a class of semigroups which contain quasi S-units.

Other open problems are as follows:

Problem 3: Can the symmetric semigroup S(n) have S-unit?

Problem 4: Can the symmetric semigroup S(n) have super S-units?

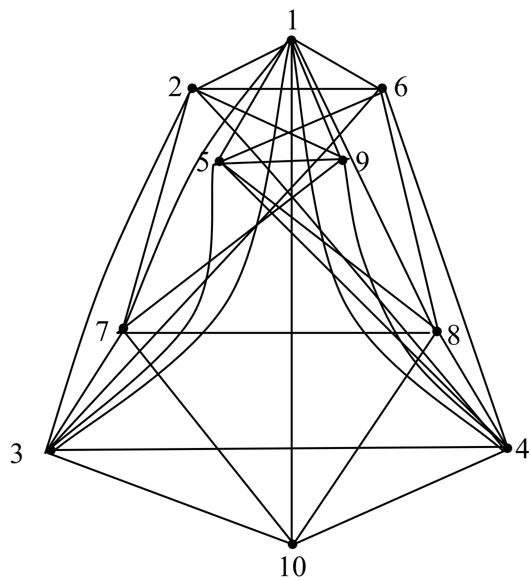
Now for even S(n) the symmetric semigroup contains S_n the symmetric group in S(n), so S(n) is a S-semigroup.

The unit graph of S(n) is the graph of the group S_n. For graphs of groups please refer [15].

Now we consider the semigroup S = (Z_{11}, $\times$). We see the units of Z_{11} are 1,

$10^2 = 1 \pmod{11}$; $2 \times 6 \equiv 1 \pmod{11}$, $3 \times 4 \equiv 1 \pmod{11}$
$5 \times 9 \equiv 1 \pmod{11}$, $7 \times 8 \equiv 1 \pmod{11}$.

We draw the S-unit graph of Z_{11} which is as follows:



We call this type of S-units as 4 tuple Super Super chain like S-units.

We now proceed onto define Super Super S-unit 4 tuple.

DEFINITION 3.6: *Let $S = \{Z_n, \times\}$ be a semigroup with unit. Suppose we have in S a subset*

$T = \{1, (n-1)^2 \equiv 1 \pmod n, (p, q), (r, s), (t, u)$ and $(a, b)\}$ such that

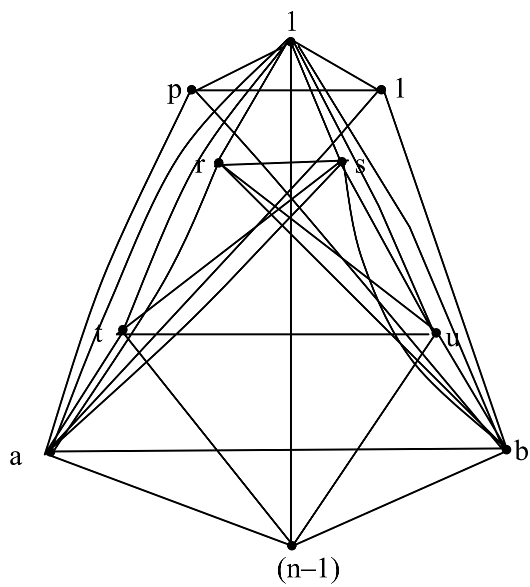
$$p \times q \equiv 1 \pmod n,$$
$$r \times s \equiv 1 \pmod n,$$
$$t \times u \equiv 1 \pmod n \text{ and}$$
$$a \times b \equiv 1 \pmod n.$$

Suppose we have (p, q), (r, s) are (t, u) are in T such that $p \times s \equiv t \pmod n$ and $q \times r = u$ and (a, b) in S such that $ra = b \pmod n$ and $sb = a \pmod n$. Further $t \times a \equiv (n-1) \pmod n$ and $u \times b = (n-1) \pmod n$.

Finally pa = q (mod n)

qb = p (mod n). Further ru = s (mod n)

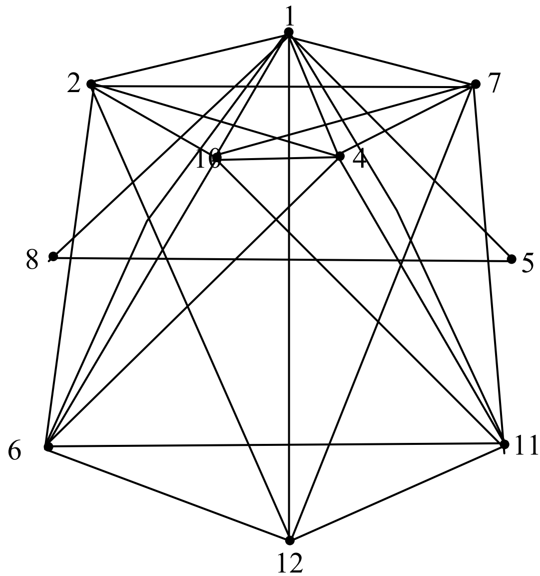
st = r (mod n) then we define {(p, q), (r, s), (t, u) and (a, b)} as a 4-tuple super super chain like S-unit. The graph is called the 4-tuple super super chain like S-unit graph with vertices {1, p, q, r, s, t, u, a, b and n–1}. The 4-tuple graph is as follows:



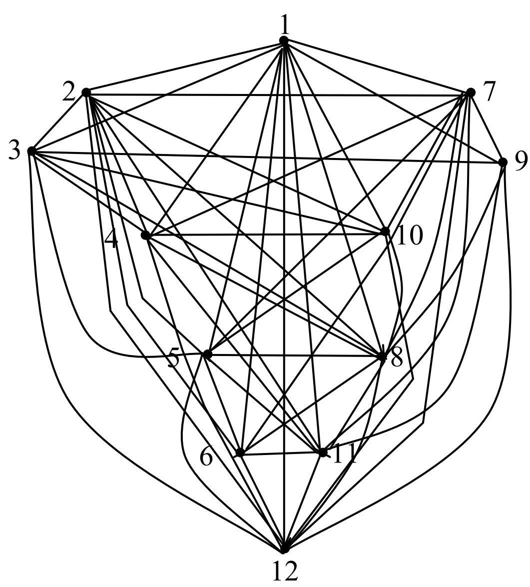
An important problem is will every $(Z_p, \times)$ with $p \geq 11$ have a 4-tuple super super chain like S-unit?

Example 3.74: Let S = $(Z_{13}, \times)$ be a semigroup with unit. Clearly $12 \times 12 \equiv 1 \pmod{13}$. The units of S are $2 \times 7 \equiv 1$ (mod 13), $3 \times 9 \equiv 1$ (mod 13), $4 \times 10 \equiv 1$ (mod 13), $5 \times 8 \equiv 1$ (mod 13), $6 \times 11 \equiv 1$ (mod 13) and $12^2 = 1$ (mod 13).

We can have a 4-tuple super super chain like S-unit.



We can also have a 5-tuple super super chain like S-unit, using the 5 pairs (3, 9), (2, 7), (4, 10), (5, 8), (6, 7) and 1 and 12. The graph is given in the following.



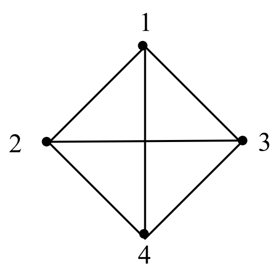
We will get a 5-tuple super super chain like S-units. We see this graph is also a complete graph with 12 vertices.

In view of this we have the following theorem.

THEOREM 3.16: *Let S = $\{Z_p, \times\}$ semigroup with unit (p a prime). S has (p–3)/2 sets of distinct units like a, b $\in$ S, a $\neq$ b but a $\times$ b $\equiv$ 1 (mod p) and $(p-1)^2 \equiv 1$ (mod p). S has a (p-3) tuple super super chain like S-units. Further the (p–1)/2 tuple super super chain like S-unit graph has (p–1) vertices and is a complete graph.*

The proof is direct and only involves simple number theoretic techniques.

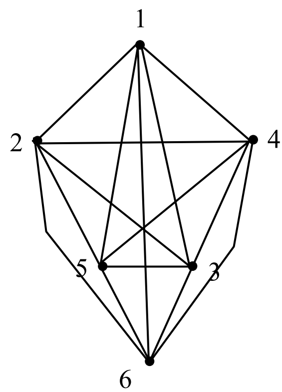
If p = 3 then S = $\{Z_3, \times\}$ has no S-unit.
If p = 5 then S = $\{Z_5, \times\}$ has units of the form.
The unit graph of {1, 4, 2, 3} is as follows:



Clearly S has a S-unit and the unit graph is a complete graph with four vertices.

If p = 7 then S = $\{Z_7, \times\}$ has units of the form $6^2 \equiv 1$ (mod 7), 5.3 $\equiv$ 1 (mod 7) $2 \times 4 \equiv 1$ (mod 7).

The unit graph with the vertices {1, 6, 2, 4, 5, 3} is as follows:



Clearly the S-unit is a special S-unit and the graph is a complete graph with six vertices.

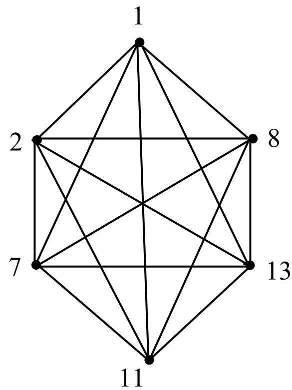
Now we wish to bring this to the notice of the reader.

If we consider Z_p, we see $\{Z_p \setminus \{0\}, \times\}$ is an abelian group and we can draw a special super graph of $Z_p \setminus \{0\}$ and that will be the complete graph with (p–1) vertices. It may be recalled we have the diherdral group has a complete graph related to the conjugacy classes. However here we cannot talk of such relation of basically $Z_p \setminus \{0\}$ is an abelian group.

Now another natural question is can $S = \{Z_n, \times\}$, n; not a prime have t-tuple super super S-units?

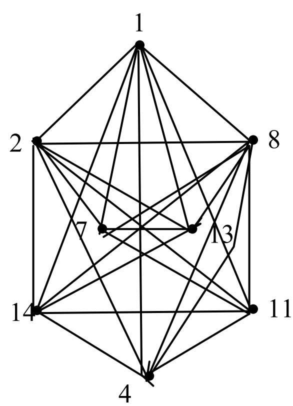
We study this in the first phase by some examples.

Example 3.75: Let $S = \{Z_{15}, \times\}$ be a semigroup with unit. 1, 14, $2 \times 8 \equiv 1 \pmod{15}$, $4^2 \equiv 1 \pmod{15}$, $11^2 \equiv 1 \pmod{15}$, $7 \times 13 \equiv 1 \pmod{15}$. Clearly $S = (Z_{15}, \times)$ cannot have super super S-unit. However it has a S-unit.



Example 3.76: Let S = $\{Z_{18}, \times\}$ be a semigroup with unit. The units of S are $\{1, 17^2 \equiv 1 \pmod{18}, 5 \times 11 \equiv 1 \pmod{18}, 7 \times 13 \equiv 1 \pmod{18}\}$. Clearly S has no super S-units.

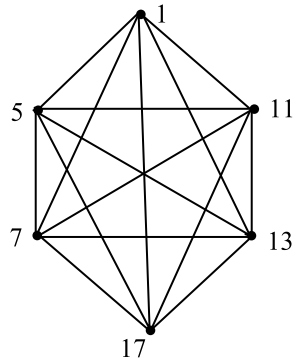
We now draw the unit graph of in example 3.75 and 3.76 respectively.



It is also a complete graph with 8 vertices.

Consider unit graph of the semigroup S = $\{Z_{18}, \times\}$ given in example 3.76.

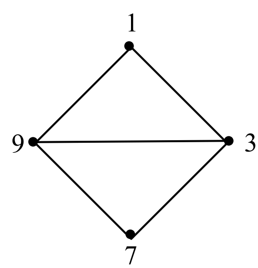
The set of units of Z_{18} is as follows: {1, 17, 5, 11, 7, 13}



The unit graph of Z_{18} is a complete graph with 6 vertices.

Example 3.77: Consider the semigroup Z_{21} under product. The set of units in Z_{21} are {1, $20^2 \equiv 1 \pmod{21}$, $2 \times 11 \equiv 1 \pmod{21}$, $8^2 \equiv 1 \pmod{21}$, $4 \times 16 \equiv 1 \pmod{21}$, $5 \times 17 \equiv 1 \pmod{21}$, $10 \times 19 \equiv 1 \pmod{21}$, $13^2 \equiv 1 \pmod{21}$}. We get a complete unit graph with 12 vertices.

Example 3.78: Let S = (Z_{10}, ×) be a semigroup with unit. The units of S are {1, 9, 3 × 7} has a S unit of the form.



Example 3.79: Consider the semigroup S = {Z_{33}, ×}. The units in S are {1, 32, $2 \times 7 \equiv 1 \pmod{33}$, $4 \times 25 \equiv 1 \pmod{33}$, $5 \times 20 \equiv 1 \pmod{33}$, $102 \equiv 1 \pmod{33}$, $29 \times 8 \equiv 1 \pmod{33}$, $19 \times 7 \equiv 1 \pmod{33}$ $(23)^2 \equiv 1 \pmod{33}$, $31 \times 16 \equiv 1 \pmod{33}$, $13 \times 28 \equiv 1 \pmod{33}$ and $14 \times 26 \equiv 1 \pmod{33}$}.

We see S is a 8 tuple super super chain like S-unit. The 8-tuple super super chain S-unit graph has 20 vertices which is a complete graph.

Example 3.80: Let S = $\{Z_{25}, \times\}$ be a semigroup with unit.

The units of S are $\{1, 24^2 \equiv 1 \pmod{25}$,
$13 \times 2 \equiv 1 \pmod{25}$, $3 \times 17 \equiv 1 \pmod{25}$,
$4 \times 19 \equiv 1 \pmod{25}$, $21 \times 6 \equiv 1 \pmod{25}$,
$7 \times 18 \equiv 1 \pmod{25}$, $8 \times 22 \equiv 1 \pmod{25}$,
$9 \times 14 \equiv 1 \pmod{25}$, $23 \times 12 \equiv 1 \pmod{25}$ and
$16 \times 11 \equiv 1 \pmod{25}\}$.

This semigroup has 9-tuple super super chain like S-unit and the graph associated with it is a complete graph with 20 vertices.

It is left as open problems.

Problem 5: If S = $\{ Z_{p^2}, \times$, p a prime$\}$ be a semigroup with unit. Can S have more than one element x such that $x \times x \equiv 1 \pmod{p^2}$ barring 1 and $(p^2-1)^2 \equiv 1 \pmod{p^2}$.

Study in this direction is innovative.

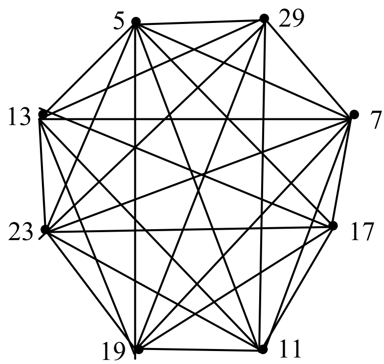
Problem 6: Can
S = $\{Z_{pq} \mid$ p and q are primes $p \neq q$, $p = 3$; q any other prime$\}$ have t-tuple super super chain like S-units?

Example 3.81: Let S = $(Z_{30}, \times)$ be a semigroup with unit.

The unit set of S is as follows:

T = $\{1, 29^2 \equiv 1 \pmod{30}$, $11^2 \equiv 1 \pmod{30}$,
$13 \times 7 \equiv 1 \pmod{30}$, $19^2 \equiv 1 \pmod{30}$ and
$23 \times 17 \equiv 1 \pmod{30}\}$.

The unit graph of T is as follows:



In view of all these examples we have the following theorem.

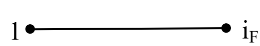
THEOREM 3.17: *Let S = {Z_n, ×} be a semigroup with unit. The set of units in S is a complete graph.*

The proof is direct and hence left as an exercise to the reader.

It is important to mention that in case of rings using Z_n the unit graph of all rings follows the same pattern as that of semigroups S = (Z_n, ×). Further the same hold good for Smarandache units in rings, however these unit set will not be a subring of the ring Z_n.

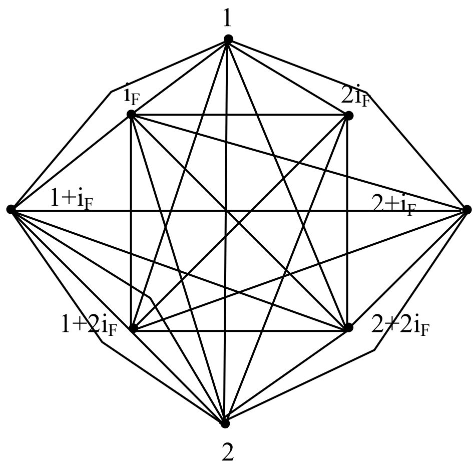
Now we proceed onto describe the S-units in C (Z_n).

Example 3.82: Let C(Z_2) = {0, 1, i_F, 1+i_F} be a complex modulo integer semigroup under has no S-unit. The unit graph of C(Z_2) is



Example 3.83: Let C (Z_3) = {0, 1, 2, i_F, $2i_F$, $1+i_F$, $1+2i_F$, $2+i_F$, $2+2i_F$} be a semigroup under product. The units of $C(Z_3)$ are 1, $i_F \times 2i_F = 1 \pmod 3$, $2 \times 2 \equiv 1 \pmod 3$, $(1+i_F)(2+i_F) \equiv 1 \pmod 3$ and $(1+2i_F)(2+2i_F) = 1 \pmod 3$}.

The unit graph of C (Z_3) is as follows:



Clearly the unit graph of $C(Z_3)$ is a complete graph with 8 vertices.

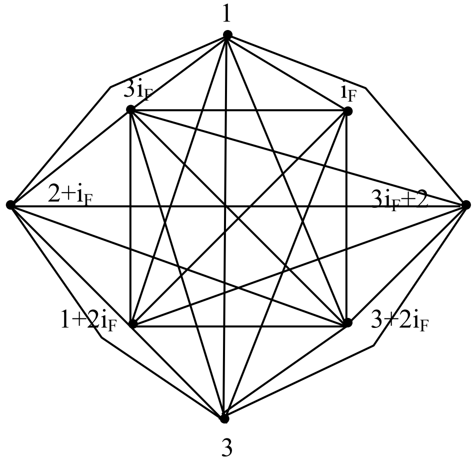
Clearly i_F is a S-unit; likewise Zi_F, $1+i_F$ and $2+i_F$ are S-units of $C(Z_3)$. Infact $C(Z_3)$ has a 3-tuple super super chain like S-unit.

Example 3.84: Let $C(Z_4) = \{a+ bi_F \mid a, b \in Z_4, i_F^2 = 3\}$ be a complex modulo integer semigroup under ×.

The units of $C(Z_4)$ are = {1, $3^2 \equiv 1 \pmod 4$, $3i_F \times i_F \equiv 1 \pmod 4$, $(1+2i_F)^2 = 1 \pmod 4$, $(2+i_F)(2+3i_F) \equiv 1 \pmod 4$ and $(3+2i_F)^2 = 1 \pmod 4$}.

For $C(Z_4)$ also the unit graph is a complete graph with 8 vertices.

However it not the same type as the unit graph given in example 3.83.



However this unit graph is not a 3-tuple super super chain like S-unit. Now this semigroup has S-unit.

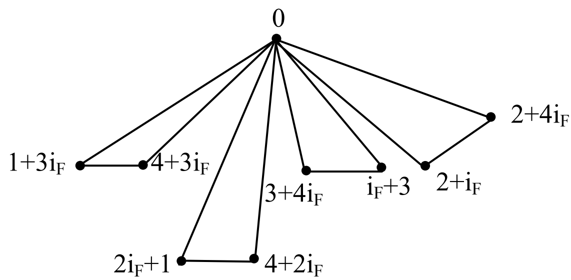
We can find the unit graph of $C(Z_5)$.

Example 3.85: Let S = $\{C(Z_5), \times\}$ be a semigroup. The units of S are as follows:

T = $\{1, 2 \times 3 \equiv 1 \pmod 5, 4, i_F \times 2i_F \equiv 1 \pmod 5, 3i_F \times 4i_F \equiv 1 \pmod 5, (1+i_F) \times (3 + 2i_F) \equiv 1 \pmod 5$ $(3i_F + 2)(4+4i_F) \equiv 1 \pmod 5, (3+3i_F)(1+4i_F) = 1 \pmod 5$ and $(4+i_F)(2+2i_F) = 1 \pmod 5\}$.

T has a unit graph which is a complete graph with 16 vertices. S has a 7-tuple super super chain like S-unit. It has only two elements such that $4^2 \equiv 1 \pmod 5$ and $1^2 \equiv 1 \pmod 5$.

Further the zero divisor graph of S is not a tree. However $C(Z_5)$ has no nilpotent elements and the zero divisor graph of S is as follows:



In view of all these we have the following results.

THEOREM 3.18: *Let $S = \{(C(Z_p), \times\}$ (p a prime) be a semigroup with unit and no zero divisors. The unit graph of S is a complete graph with $p^2 - 1$ vertices and S has a t-tuple super super chain like S-unit ($t \leq n$).*

Proof is direct and hence left as an exercise to the reader.

Suppose $S = \{C(Z_p), \times\}$ is a semigroup with unit and zero divisors. Can S have nilpotent elements? Suppose S has a m-tuple. What is the exact value of m? We see when p = 5 then S has a 7-tuple super super chain like S-unit.

When p = 3, we have S has a 3-tuple super super chain like S-unit.

Interested reader can study this sort of S-units in $C(Z_n)$, n prime as well as n-non prime.

Chapter Four

SUGGESTED PROBLEMS

In this chapter we suggest over 50 problems. It is important to mention we have also suggested some problems at the end of the theorem / examples in other chapters. Some problems are at research level.

1. Find the zero divisor graph of $S = \{Z_{20}, \times\}$.

2. Is the zero divisor graph of $S = \{Z_2 \times Z_2 \times Z_2 \times Z_2 \times Z_2 = (a, b, c, d, e) \mid a, b, c, d, e \in Z_2\}$ under product $\times$ modulo 2 a tree?

3. Find a zero divisor graph of $S = \{(Z_{20} \times Z_2), \times\}$.

4. Find some interesting properties associated with zero divisor graph of $S = \{Z_n \mid n = 2p$ where p is a prime$\}$.

5. Find the zero divisor graphs of $S = \{Z_n \mid n = pq; p \neq q\ p > 5\ q > 5$ with p and q primes$\}$.

6. Find the zero divisor graphs of S = $\{Z_{p^n}$ | p is a prime $n \geq 2\}$.

7. Characterize those semigroups which has its associated zero divisor graphs to be a tree.

8. Characterize those semigroups S = $\{Z_n, \times\}$ for which the zero divisor graph is not a tree.

9. Obtain some interesting properties about zero divisor graphs in semigroups.

10. Let S = $\left\{\begin{pmatrix} a & b & e \\ c & d & f \end{pmatrix} \middle| a, b, c, d, e, f \in Z_5\right\}$ be a semigroup under natural product. Find the zero divisor graph associated with S. Is it a tree?

11. Find the zero divisor graph of the finite complex modulo integer semigroup S = $\{C(Z_{12}), \times\}$.

12. Show that the zero divisor graph of the finite complex modulo integer semigroup S = $\{C(Z_{13}), \times\}$ is a tree.

13. Can the semigroup $C(Z_5)$ have nilpotent elements? Find the zero divisor graph of $C(Z_5)$ under product.

14. Characterize those semigroups S = $\{C(Z_n), \times\}$ which contains the zero divisor graphs of them.

15. Let S = $\left\{\begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{bmatrix} \middle| a_i \in C(Z_4), 1 \leq i \leq 4\right\}$ be a semigroup under natural product $\times_n$.

(i) Find the zero divisor graph of S.

(ii) Is it a tree?

(iii) Prove S has nilpotent elements in it.

16. Does there exist a semigroup S in which every subgraph of the zero divisor graph is also a special subgraph of the zero divisor graph?

17. Does there exist a semigroup S for which every special subgraph of the zero divisor graph G is a tree but G is not a tree? Justify your claim!

18. Find a finite complex modulo integer semigroup $S = \{C(Z_n), \times\}$ for which the zero divisor graph is a tree.

19. Find a semigroup S for which the zero divisor graph of S is such that no special subgraph of it is a tree.

20. Prove every finite complex modulo integer semigroup $S = \{C (Z_n), \times\}$ has a special zero divisor subgraph which is a tree with four vertices.

21. Obtain some special properties enjoyed by $S = \{C(Z_p), \times\ C(Z_q);\ p \neq q$ p and q not primes, $\times\}$ the complex modulo integer semigroup. What is the structure of the zero divisor graph of S?

22. Prove there exists semigroups which has no zero divisor graphs associated with it.

23. Find the zero divisor graph of $\{C(Z_{12}), \times\}$.

24. Can $S = \{C(Z_{17}), \times\}$ have 16 layered pseudo S-zero divisors?

25. Find $S = \{C(Z_p), \times\}$ to have (p-1) layered pseudo S-zero divisors?

26. Can $S = \{C(Z_n), \times\}$, n not a prime have layered pseudo S-zero divisors?

27. Find the zero divisor graph of $S = \{C(Z_{15}) \times C(Z_{12}), \times\}$.

28. Can S in problem 27 have t-layered pseudo S-zero divisors?

29. Find the zero divisor graph of $S = \{C(Z_7) \times C(Z_6), \times\}$.

30. Can $S = \{C(5) \times C(Z_{12}), \times\}$ have t-layered pseudo S-zero divisors?

31. Find the structure of the t-layered pseudo S-zero divisor graph.

32. Find the unit graph of the semigroup $S = \{Z_{35}, \times\}$.

    (i) Can S have S-units?

    (ii) Can S have super S-units?

33. Find the unit graph of the semigroup $S = \{Z_{19}, \times\}$. Can the graph be a tree?

34. Find some special properties enjoyed by unit graphs of the semigroup $S = \{Z_{2n}, \times;$ n is an odd number$\}$.

35. Let $S = \{Z_{17}, \times\}$ be a semigroup.

    (i) Find the unit graph of S.

    (ii) Can the unit graph of S be a tree?

    (iii) Can S have S-units?

    (iv) Can S have super S-units?

36. Let $S = \{Z_{20}, \times\}$ be a semigroup.

    (i) Can S have its unit graph to be a tree?

    (ii) Can S have S-units?

    (iii) Find the S-unit graph of S.

37. Let $S = \{C(Z_8), \times\}$ be a semigroup.

    (i) Find the S-unit graph of S.

    (ii) Can S have super S-unit?

38. Let $S = \{C\ (Z_{17}), \times\}$ be a semigroup.

    (i) Find the unit graph of S.

    (ii) Does S have S-unit?

    (iii) Can S have super S-unit?

    (iv) Can S have super super chain like S-units?

39. Let $S = \{C\ (Z_{23}), \times\}$ be a semigroup.

    (i) Find the unit graph of S.

    (ii) Can S have unit graphs?

40. Obtain a necessary and sufficient condition for the semigroup $S = \{C(Z_n), \times\}$ to have its unit graph to be a tree.

41. Obtain a necessary and sufficient condition for the semigroup $S = \{Z_n, \times\}$ to have a super super chain like S-zero divisors.

    (Study the same problem for $S = \{C(Z_n), \times\}$ also).

42. Let $S = \{Z_n, \times C(Z_m), \times, m \neq n)$ be a semigroup.

    (i) Find the unit graph of S.

    (ii) Can S have S-units?

    (iii) Can S have super super chain like S-zero divisors?

43. Let $S = \{C(Z_p \times Z_n) = \{(a + bi_F, c+di_F) \mid a, b \in Z_p, i_F^2 = p-1$ and $c, d \in Z_n, i_F^2 = n-1\}$ be a semigroup.

    (i) Find the unit graph of S.

    (ii) Find the S-unit graph of S.

    (iii) Find super S-unit graph of S.

44. Find the unit graph of S(7).

    (i) Can S(7) have S-units?

    (ii) Can S(7) have Super S units?

45. Find the unit graph of the semigroup

    $$S = \left\{ \begin{bmatrix} a \\ b \\ c \end{bmatrix} \middle| \, a, b, c \in Z_{13} \right\} \text{ under natural product } \times.$$

46. Let M = $\left\{\begin{pmatrix} a & b & c \\ d & e & f \end{pmatrix} \middle| \; a, b, c, d, e, f \in Z_{45}, \times_n\right\}$ be a semigroup.

    (i) Find the unit graph of M.

    (ii) Find the S-unit graph of M (if it exist).

    (iii) Can M have super S-unit?

47. Let T = $\left\{\begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ \vdots & \vdots & \vdots \\ a_{10} & a_{11} & a_{12} \end{bmatrix} \middle| \; a_i \in C(Z_6), 1 \leq i \leq 12, \times_n\right\}$ be a semigroup.

    (i) Find the unit graph of T.

    (ii) Can T have S-units?

    (iii) Can T have super super chain like S-unit?

48. Find some special properties enjoyed by those semigroups which has super super chain like S-units.

49. Find the special properties enjoyed by those semigroups whose unit graph is a tree.

50. Find the special properties enjoyed by semigroups which has the zero divisor graph to be a tree covering pseudo lattice.

51. Characterize those semigroups which has its zero divisor graph to be a t-layered zero divisor graph.

52. Characterize those semigroups which has its zero divisor graph to be a tree.

53. Characterize those semigroups which has S-zero divisors?

54. Can the symmetric semigroup S(n) have S-super super unit?

55. Characterize those semigroups which has quasi-S unit triple.

FURTHER READING

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Dr.W.B.Vasantha Kandasamy is an Associate Professor in the Department of Mathematics, Indian Institute of Technology Madras, Chennai. In the past decade she has guided 13 Ph.D. scholars in the different fields of non-associative algebras, algebraic coding theory, transportation theory, fuzzy groups, and applications of fuzzy theory of the problems faced in chemical industries and cement industries. She has to her credit 646 research papers. She has guided over 68 M.Sc. and M.Tech. projects. She has worked in collaboration projects with the Indian Space Research Organization and with the Tamil Nadu State AIDS Control Society. She is presently working on a research project funded by the Board of Research in Nuclear Sciences, Government of India. This is her 71st book.

On India's 60th Independence Day, Dr.Vasantha was conferred the Kalpana Chawla Award for Courage and Daring Enterprise by the State Government of Tamil Nadu in recognition of her sustained fight for social justice in the Indian Institute of Technology (IIT) Madras and for her contribution to mathematics. The award, instituted in the memory of Indian-American astronaut Kalpana Chawla who died aboard Space Shuttle Columbia, carried a cash prize of five lakh rupees (the highest prize-money for any Indian award) and a gold medal.
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Dr. Florentin Smarandache is a Professor of Mathematics at the University of New Mexico in USA. He published over 75 books and 200 articles and notes in mathematics, physics, philosophy, psychology, rebus, literature. In mathematics his research is in number theory, non-Euclidean geometry, synthetic geometry, algebraic structures, statistics, neutrosophic logic and set (generalizations of fuzzy logic and set respectively), neutrosophic probability (generalization of classical and imprecise probability). Also, small contributions to nuclear and particle physics, information fusion, neutrosophy (a generalization of dialectics), law of sensations and stimuli, etc. He got the 2010 Telesio-Galilei Academy of Science Gold Medal, Adjunct Professor (equivalent to Doctor Honoris Causa) of Beijing Jiaotong University in 2011, and 2011 Romanian Academy Award for Technical Science (the highest in the country). Dr. W. B. Vasantha Kandasamy and Dr. Florentin Smarandache got the 2011 New Mexico Book Award for Algebraic Structures. He can be contacted at smarand@unm.edu

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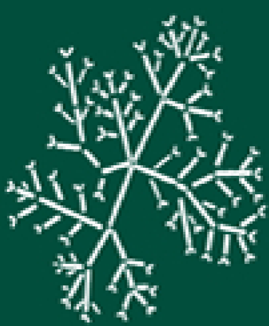
The zero divisor graph of semigroups of finite modulo integers, n under product is studied and characterized. If n is a non-prime, the zero divisor graph is not a tree. We introduce the new notion of tree covering pseudo lattice. When n is a even integer of the form 2p. p a prime, then the modulo integer zero divisor graph is a tree-covering pseudo lattice.



ISBN 9781599731919

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US $35.00

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