Some Neutrosophic Uncertain Linguistic Number Heronian Mean Operators and Their Application to Multi-Attribute Group Decision Making

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Abstract: Heronian mean (HM) is a useful aggregation operator which is marked by catching the interrelations of the aggregated arguments and the neutrosophic uncertain linguistic set can be better to express the incomplete, indeterminate and inconsistent information. In this paper, we combine the Heronian mean and the neutrosophic uncertain linguistic set and proposed some Heronian mean operators based on neutrosophic uncertain linguistic numbers. Firstly, we introduce some definition and properties of uncertain linguistic numbers, the single valued neutrosophic set, and some heronian mean (HM) operators including the generalized weighted Heronian mean (GWHM) operator, the improved generalized weighted Heronian mean (IGWHM) operator, the improved generalized geometric weighted Heronian mean (IGGWHM) operator. Then, we propose the single valued neutrosophic uncertain linguistic set by combining the uncertain linguistic numbers and the single valued neutrosophic set. Further, the neutrosophic uncertain linguistic number improved generalized weighted Heronian mean (NULNIGWHM) operator and the neutrosophic uncertain linguistic number improved generalized geometric weighted Heronian mean (NLUNIGGWHM) operator are developed and the properties of them are analyzed. Furthermore, we develop the decision making methods for multi-attribute group decision making (MAGDM) problems with neutrosophic uncertain linguistic information and give the detail decision steps. At last, an illustrate example is given to show the process of decision making and the effectiveness of the proposed method.

Keywords: multiple attribute group decision making (MAGDM); Heronian mean; neutrosophic uncertain linguistic set; geometric Heronian mean

1. Introduction

Multiple attribute decision group making (MAGDM) problems exists extensively in many fields such as politics, economy, military and culture. The attribute values in the decision-making problems are usually incomplete, indeterminate and inconsistent due to the complexity and fuzziness of the real world. Zadeh [1] firstly proposed the fuzzy set (FS) theory which has a membership function. Based on fuzzy set theory, Atanassov [2, 3] proposed the intuitionistic fuzzy set (IFS) which added a non-membership function. The intuitionistic fuzzy set is composed of the membership (or called truth-membership) $T_A(x)$ and non-membership (or called falsity-membership) $F_A(x)$, and satisfies the conditions $T_A(x), F_A(x) \in [0,1]$ and $0 \le T_A(x) + F_A(x) \le 1$. However, IFSs can merely deal with incomplete information, but cannot do anything for the indeterminate information and inconsistent information. The indeterminacy (or called Hesitation degree) is $1 - T_A(x) - F_A(x)$ which is only given by default and cannot be solely expressed in IFSs. With respect to this situation, Smarandache[4] developed the neutrosophic set (NS) which consists of the truth-membership $T_A(x)$, falsity-membership $F_A(x)$ and indeterminacy-membership $I_A(x)$ and the three variables are independent completely. NS is a generalization of FS and IFS. Now there are many research achievements about NSs. Wang et al. [5] further proposed a single valued neutrosophic set (SVNS)which is an special instance of the neutrosophic set by changing the conditions to that $T_A(x), I_A(x), F_A(x) \in [0,1]$ and $0 \le T_A(x) + I_A(x) + F_A(x) \le 3$; Ye [6] proved that the cosine similarity degree is a special case of the correlation coefficient in SVNS; Wang et al. [7] proposed the definition of the interval neutrosophic sets (INSs) in which the truth-membership, indeterminacy-membership, and false-membership were extended to interval numbers, and discussed some properties of INSs. Ye [8] defined the similarity measures between INSs on the basis of the Hamming and Euclidean distances, and proposed a method for multi-criteria decision-making problems.

In real problems, sometimes we can use linguistic terms such as ‘good ’, ‘bad ’ to describe the state or performance of a car and cannot use some numbers to express some qualitative information. However, when we use the linguistic variables to express the qualitative information, it only means the membership degree belonged to a linguistic term is 1, and the non-membership degree or hesitation degree cannot be expressed. In order to overcome this shortcoming, Wang and Li [9] proposed the concept of intuitionistic linguistic set by combining intuitionistic fuzzy set and linguistic variables. For the above-mentioned example, we can give an evaluation value ‘good’ for the state of the car, however, for this evaluation, we have the certainty degree of 80 percent and negation degree of 10 percent, and then we can use the intuitionistic linguistic set to express the evaluation result. Furthermore, Wang and Li [9] proposed intuitionistic two-semantics and the Hamming distance between two intuitionistic two-semantics, and ranked the alternatives by calculating the comprehensive membership degree to the ideal solution for each alternative.

The information aggregation operators which are widely applied in multiple attribute group decision-making problems are a meaningful research scopes. Heronian mean (HM) is a useful aggregation operator which is marked by catching the interrelations of the aggregated arguments. Beliakov [10] had firstly proved that Heronian mean was an aggregation operator. On the basis of this, Skora [11,12] further extended to the generalized Heronian means and discussed two special cases of them. Yu and Wu [13] proposed a generalized interval-valued intuitionistic fuzzy Heronian mean (GIIFHM) and a generalized interval-valued intuitionistic fuzzy weighted Heronian mean (GIIFWHM) which extended Heronian mean from dealing with crisp numbers to intuitionistic fuzzy numbers, and some desirable properties and special cases of these operators were discussed. Yu [14] proposed some intuitionistic fuzzy aggregation operators based on HM, including the intuitionistic fuzzy geometric Heronian mean (IFGHM) operator and the intuitionistic fuzzy geometric weighed Heronian mean (IFGWHM) operator, and the properties of these operators were studied.

As mentioned above, the interactions among the attribute values are common in the real decision making problems. Because Heronian mean operator can cope with the interactions among the attribute values and the neutrosophic set can be better to express the incomplete, indeterminate and inconsistent information. However, there is no research on the HM operator under neutrosophic uncertain linguistic environment. Hence, in this paper, we will extend the HM operator to the neutrosophic uncertain linguistic set, and propose some Heronian mean operators based on neutrosophic uncertain linguistic numbers, including the neutrosophic uncertain linguistic number improved generalized weighted Heronian mean (NULNIGWHM) operator and the neutrosophic uncertain linguistic number improved generalized geometric weighted Heronian mean (NLUNIGGWHM) operator, then applied them to multi-attribute group decision-making problems.

To achieve the purpose, the rest of this paper is organized as follows. In Section 2, we introduces the definition of uncertain linguistic numbers, the single valued neutrosophic set, and some operators based on heronian mean(HM) operator including the improved generalized weighted Heronian mean (IGWHM) operator and the generalized geometric Heronian mean (GGHM) operator. In Section 3, on the basis of uncertain linguistic numbers and the single valued neutrosophic set, we develop the single valued neutrosophic uncertain linguistic set and operational rules of it. Section 4 proposes some Heronian mean operators for the single neutrosophic uncertain linguistic numbers, such as the neutrosophic uncertain linguistic number improved generalized weighted Heronian mean (NULNIGWHM) operator and a neutrosophic uncertain linguistic number improved generalized geometric weighted Heronian mean (NLUNIGGWHM) operator and introduces some properties and special cases of them. In Section 5, we propose the decision-making methods based on the NULNIGWHM and NLUNIGGWHM operators. Section 6 shows a numerical example according to our approach. Section 7 summarizes the main conclusion of this paper.

2. Preliminaries

2.1 The linguistic set and uncertain linguistic numbers

The linguistic set is regarded as a good tool to express the qualitative information, we can express the linguistic set by $S=(s_0,s_1,\cdots,s_{l-1})$,and $s_\theta(\theta=1,2,\cdots,l-1)$ can be called an linguistic number, l is an odd value which can be the values of 3,5,7,9,etc. For example, when $l=9$, $S=(s_0,s_1,s_2,s_3,s_4,s_5,s_6,s_7,s_8)$ =(extremely poor, very poor, poor, slightly poor, fair, slightly good, good, very good, extremely good).

Let s_i and s_j be any two linguistic numbers in linguistic set S , they have the following characteristics [15,16]:

(i) If $i>j$, then $s_i \succ s_j$;

(ii) There exists negative operator: $neg(s_i)=s_j$, where $j=l-1-i$;

(iii) If $s_i \ge s_j$, $\max(s_i,s_j)=s_i$;

(iv) If $s_i \le s_j$, $\min(s_i,s_j)=s_i$.

The continuous linguistic set $\bar{S}=\{s_\alpha \mid \alpha\in R^+\}$, which can overcome the weakness of the loss of information in the process of calculations, is the extension of original discrete linguistic set $S=(s_0,s_1,\cdots,s_{l-1})$, and $\bar{S}=\{s_\alpha \mid \alpha\in R^+\}$ meets the strictly monotonically increasing condition [21, 22]. Some operational rules are defined as follows[15,16].

(1) $\beta s_i=s_{\beta\times i}\quad \beta\ge 0$ (1)

(2) $s_i\oplus s_j=s_{i+j}$ (2)

(3) $s_i\otimes s_j=s_{i\times j}$ (3)

(4) $(s_i)^n=s_{i^n}\quad n\ge 0$ (4)

Definition 1 [17]. Suppose $\tilde{s}=[s_a,s_b]$, $s_a,s_b\in\bar{S}$ with $a\le b$ are the lower limit and the upper limit of $\tilde{s}$, respectively, then $\tilde{s}$ is called an uncertain linguistic variable.

Let $\tilde{S}$ be a set of all uncertain linguistic variables, $\tilde{s}_1=[s_{a1},s_{b1}]$ and $\tilde{s}_2=[s_{a2},s_{b2}]$ be any two uncertain linguistic variables, the operational rules are defined as follows [18,19]:

(1) $\tilde{s}_1\oplus\tilde{s}_2=[s_{a1},s_{b1}]\oplus[s_{a2},s_{b2}]=[s_{a1+a2},s_{b1+b2}]$ (5)

(2) $\tilde{s}_1\otimes\tilde{s}_2=[s_{a1},s_{b1}]\otimes[s_{a2},s_{b2}]=[s_{a1\times a2},s_{b1\times b2}]$ (6)

(3) $\lambda\tilde{s}_1 = \lambda[s_{a1}, s_{b1}] = [s_{\lambda * a1}, s_{\lambda * b1}], \lambda \geq 0$ (7)

(4) $(\tilde{s}_1)^{\lambda} = [s_{a1}, s_{b1}]^{\lambda} = [s_{a1^{\lambda}}, s_{b1^{\lambda}}], \lambda \geq 0$ (8)

2.2. The single valued neutrosophic set

Definition 2 [5]. Let X be a universe of discourse, with a generic element in X denoted by x. A single valued neutrosophic set A in X is

$$A = \{x | (T_A(x), I_A(x), F_A(x)) | x \in X\} \tag{9}$$

where $T_A(x)$, $I_A(x)$ and $F_A(x)$ are the truth-membership, indeterminacy-membership and falsity-membership function, separately. For each point x in X, we have that $T_A(x),\ I_A(x),\ F_A(x) \in [0,1]$, and $0 \leq T_A(x) + I_A(x) + F_A(x) \leq 3$.

Definition 3 [20]. Suppose $A = \{x | (T_A(x), I_A(x), F_A(x)) | x \in X\}$ and $B = \{x | (T_B(x), I_B(x), F_B(x)) | x \in X\}$ are two NSs. If and only if $T_A(x) \leq T_B(x)$, $I_A(x) \geq I_B(x)$, $F_A(x) \geq F_B(x)$ for all x in X, then $A \leq B$.

2.3. Some operators based on heronian mean(HM) operator

Heronian mean (HM) is a useful aggregation operator which is marked by catching the interrelations of the aggregated arguments [21,22] and can be defined as follows.

Definition 4 [22]: Let $I = [0,1]$, $H : I_n \to I$, if

$$H(x_1, x_2, \cdots, x_n) = \frac{2}{n(n+1)} \sum_{i=1}^{n} \sum_{j=i}^{n} \sqrt{x_i x_j} \tag{10}$$

then $H(x_1, x_2, \cdots, x_n)$ is called the Heronian mean (HM) operator.

Definition 5 [21,22]. A GHM operator of dimension n is a mapping, $GHM : I^n \to I$, so that,

$$GHM^{p,q}(x_1, x_2, \cdots, x_n) = \left(\frac{2}{n(n+1)} \sum_{i=1}^{n} \sum_{j=i}^{n} x_i^p x_j^q \right)^{\frac{1}{p+q}} \tag{11}$$

where $p, q \geq 0$ and $I = [0,1]$. Then $GHM^{p,q}$ is called the generalized Heronian mean (GHM) operator.

It is easy to prove that the GHM operator has the following properties [23].

Theorem 1 (Idempotency)

Let $x_i = x$ for all $i = 1,2,\cdots,n$, then

$$GHM^{p,q}(x_1, x_2, \cdots, x_n) = x. \tag{12}$$

Theorem 2 (Monotonicity)

Let $(x_1, x_2, \cdots, x_n)$ and $(y_1, y_2, \cdots, y_n)$ be two collections of the nonnegative numbers, if $x_i \leq y_i$ for all $i = 1,2,\cdots,n$, then

$$GHM^{p,q}(x_1, x_2, \cdots, x_n) \leq GHM^{p,q}(y_1, y_2, \cdots, y_n). \tag{13}$$

Theorem 3 (Boundary)

GHM operator lies between the max and min operators, $a_{\min} = \min(x_1, x_2, \cdots, x_n), a_{\max} = \max(x_1, x_2, \cdots, x_n)$, i.e.

$$a_{\min} \leq GHM^{p,q}(x_1, x_2, \cdots, x_n) \leq a_{\max}(x_1, x_2, \cdots, x_n). \tag{14}$$

Definition 6 [13] Let $p,q \geq 0$, and $x_i(i=1,2,\cdots,n)$ be a collection of nonnegative numbers. $W=(w_1,w_2,\cdots,w_n)^T$ is the weight vector of $x_i(i=1,2,\cdots,n)$, and satisfies $w_i \geq 0$, $\sum_{i=1}^{n} w_i = 1$. If

$$GWHM^{p,q}(x_1,x_2,\cdots,x_n)=\left(\frac{2}{n(n+1)}\sum_{i=1}^{n}\sum_{j=i}^{n}(w_i x_i)^p (w_j x_j)^q\right)^{\frac{1}{p+q}} \tag{15}$$

then $GWHM^{p,q}$ is called the generalized weighted Heronian mean (GWHM) operator.

Definition 7 [24] Let $p,q \geq 0$, and $x_i(i=1,2,\cdots,n)$ be a collection of nonnegative numbers. $W=(w_1,w_2,\cdots,w_n)^T$ is the weight vector of $x_i(i=1,2,\cdots,n)$, and satisfies $w_i \geq 0$, $\sum_{i=1}^{n} w_i = 1$. If

$$IGWHM^{p,q}(x_1,x_2,\cdots,x_n)=\frac{\left(\sum_{i=1}^{n}\sum_{j=i}^{n} w_i w_j x_i^p x_j^q\right)^{\frac{1}{p+q}}}{\left(\sum_{i=1}^{n}\sum_{j=i}^{n} w_i w_j\right)^{\frac{1}{p+q}}} \tag{16}$$

then $IGWHM^{p,q}$ is called the improved generalized weighted Heronian mean (IGWHM) operator.

Similar to Theorems 1-3, it is easy to prove the $IGWHM^{p,q}$ operator has these properties [24].

Theorem 4 (Idempotency)

Let $x_i = x$ for all $j=1,2,\cdots,n$, then

$$IGWHM^{p,q}(x_1,x_2,\cdots,x_n)=x \tag{17}$$

Theorem 5 (Monotonicity)

Let $(x_1,x_2,\cdots,x_n)$ and $(y_1,y_2,\cdots,y_n)$ be two collections of the nonnegative numbers, if $x_j \leq y_j$ for all $j=1,2,\cdots,n$, then

$$IGWHM^{p,q}(x_1,x_2,\cdots,x_n) \leq IGWHM^{p,q}(y_1,y_2,\cdots,y_n). \tag{18}$$

Theorem 6 (Boundary)

$IGWHM^{p,q}$ operator lies between the max and min operators, $a_{\min}=(x_1,x_2,\cdots,x_n), a_{\max}=(x_1,x_2,\cdots,x_n)$ i.e.

$$a_{\min}(x_1,x_2,\cdots,x_n) \leq IGWHM^{p,q}(x_1,x_2,\cdots,x_n) \leq a_{\max}(x_1,x_2,\cdots,x_n). \tag{19}$$

In the following, we can analyze some special cases of the IGWHM operator

(1) When $q=0$, then

$$IGWHM^{p,0}(x_1,x_2,\cdots,x_n)=\frac{\left(\sum_{i=1}^{n}\sum_{j=i}^{n}w_iw_jx_i^p\right)^{\frac{1}{p}}}{\left(\sum_{i=1}^{n}\sum_{j=i}^{n}w_iw_j\right)^{\frac{1}{p}}}. \tag{20}$$

Further, when $p=1$, there is

$$IGWHM^{1,0}(x_1,x_2,\cdots,x_n)=\frac{\sum_{i=1}^{n}\sum_{j=i}^{n}w_iw_jx_i}{\sum_{i=1}^{n}\sum_{j=i}^{n}w_iw_j}. \tag{21}$$

(2) When $p=0$, then

$$IGWHM^{0,q}(x_1,x_2,\cdots,x_n)=\frac{\left(\sum_{i=1}^{n}\sum_{j=i}^{n}w_iw_jx_j^q\right)^{\frac{1}{q}}}{\left(\sum_{i=1}^{n}\sum_{j=i}^{n}w_iw_j\right)^{\frac{1}{q}}}. \tag{22}$$

From here we see that the parameters p and q don't have the interchangeability.

(3) When $p=q=1$, then

$$IGWHM^{1,1}(x_1,x_2,\cdots,x_n)=\frac{\left(\sum_{i=1}^{n}\sum_{j=i}^{n}w_iw_jx_ix_j\right)^{\frac{1}{2}}}{\left(\sum_{i=1}^{n}\sum_{j=i}^{n}w_iw_j\right)^{\frac{1}{2}}}. \tag{23}$$

Based on HM and GHM operators, Yu [14] propose the generalized geometric Heronian mean (GGHM) operator shown as follows.

Definition 8 [14] Let $p,q\geq 0$, and $x_i(i=1,2,\cdots,n)$ be a collection of nonnegative numbers. If

$$GGHM^{p,q}(x_1,x_2,\cdots,x_n)=\frac{1}{p+q}\prod_{i=1}^{n}\prod_{j=i}^{n}(px_i+qx_j)^{\frac{2}{n(n+1)}}. \tag{24}$$

then $GGHM^{p,q}$ is called the generalized geometric Heronian mean (GGHM) operator.

Similar to GHM operator, the GGHM operator also only takes the correlations of the aggregated arguments into account and ignores their own weights.

Definition 9 [14] Let $p,q\geq 0$, and $x_i(i=1,2,\cdots,n)$ be a collection of nonnegative numbers. $W=(w_1,w_2,\cdots,w_n)^T$ is the weight vector of $x_i(i=1,2,\cdots,n)$, and satisfies $w_i\geq 0$, $\sum_{i=1}^{n}w_i=1$.If

$$IGGWHM^{p,q}(x_1,x_2,\cdots,x_n)=\frac{1}{p+q}\prod_{i=1}^{n}\prod_{j=i}^{n}(px_i+qx_j)^{\frac{2(n+1-i)}{n(n+1)}\frac{w_j}{\sum_{k=i}^{n}w_k}}. \tag{25}$$

then $IGGWHM^{p,q}$ is called the improved generalized geometric weighted Heronian mean (IGGWHM) operator. Similar to Theorems 1-3, it is easy to prove the $IGGWHM^{p,q}$ operator has these properties [24].

Theorem 7 (Reducibility).

Let $W=\left(\frac{1}{n},\frac{1}{n},\cdots,\frac{1}{n}\right)^T$,then

$$IGGHM^{p,q}(x_1,x_2,\cdots,x_n)=GGHM^{p,q}(x_1,x_2,\cdots,x_n) \tag{26}$$

Theorem 8 (Idempotency)

Let $x_i=x$ for all $i=1,2,\cdots,n$, then

$$IGGWHM^{p,q}(x_1,x_2,\cdots,x_n)=x\,. \tag{27}$$

Theorem 9 (Monotonicity)

Let $(x_1,x_2,\cdots,x_n)$ and $(y_1,y_2,\cdots,y_n)$ be two collections of the nonnegative numbers, if $x_i\le y_i$ for all $i=1,2,\cdots,n$, then

$$IGGWHM^{p,q}(x_1,x_2,\cdots,x_n)\le IGGWHM^{p,q}(y_1,y_2,\cdots,y_n). \tag{28}$$

Theorem 10 (Boundary)

The $IGGHM^{p,q}$ operator lies between the max and min operators, $a_{\min}=(x_1,x_2,\cdots,x_n), a_{\max}=(x_1,x_2,\cdots,x_n)$ i.e.,

$$a_{\min}(x_1,x_2,\cdots,x_n)\le IGGWHM^{p,q}(x_1,x_2,\cdots,x_n)\le a_{\max}(x_1,x_2,\cdots,x_n) \tag{29}$$

In the following, we can analyze some special cases of the $IGGWHM^{p,q}$ operator

(1) When $q=0$, then

$$IGGHM^{p,0}(x_1,x_2,\cdots,x_n)=\prod_{i=1}^{n}\prod_{j=i}^{n}(x_i)^{\frac{2(n+1-i)}{n(n+1)}\frac{w_j}{\sum_{k=i}^{n}w_k}}\,. \tag{30}$$

From here we see that $IGGHM^{p,0}$ does not have any relationship with p .

(2) When $p=0$, then

$$IGGHM^{0,q}(x_1,x_2,\cdots,x_n)=\prod_{i=1}^{n}\prod_{j=i}^{n}(x_j)^{\frac{2(n+1-i)}{n(n+1)}\frac{w_j}{\sum_{k=i}^{n}w_k}}\,. \tag{31}$$

Similarly, $IGGHM^{0,q}$ does not have any relationship with q .

(3) When $p=q=1$, then

$$IGGHM^{1,1}(x_1,x_2,\cdots,x_n)=\frac{1}{2}\prod_{i=1}^{n}\prod_{j=i}^{n}(x_i+x_j)^{\frac{2(n+1-i)}{n(n+1)}\frac{w_j}{\sum_{k=i}^{n}w_k}}\,. \tag{32}$$

3. The single valued neutrosophic uncertain linguistic set

Definition 10. Let $[s_{\theta(x)},s_{\tau(x)}]\in\tilde{S}$, and X be the given discourse domain, then

$$A=\{\langle x\,|\,[s_{\theta(x)},s_{\tau(x)}],\left(\tilde{t}(x),\tilde{i}(x),\tilde{f}(x)\right)\rangle\,|\,x\in X\}\ . \tag{33}$$

is called a single valued neutrosophic uncertain linguistic set (SVNULS) where $s_{\theta(x)},s_{\tau(x)}\in\bar{S}$, and $\tilde{t}(x)$, $\tilde{i}(x)$ and $\tilde{f}(x)$ are three sets of some single value in real unit interval $[0,1]$ which express the truth-membership, indeterminacy-membership and falsity-membership function of the element x to A separately.

Definition 11. Let $A=\{\langle x\,|\,[s_{\theta},s_{\tau}],\left(\tilde{t},\tilde{i},\tilde{f}\right)\rangle\,|\,x\in X\}$ be a single valued neutrosophic uncertain linguistic set, and $\tilde{a}=\left\langle[s_{\theta},s_{\tau}],\left(\tilde{t},\tilde{i},\tilde{f}\right)\right\rangle$ is called a single valued neutrosophic uncertain linguistic number (SVNULN).

Suppose $\tilde{a}_1=\left\langle[s_{\theta_1},s_{\tau_1}],\left(\tilde{t}_1,\tilde{i}_1,\tilde{f}_1\right)\right\rangle$ and $\tilde{a}_2=\left\langle[s_{\theta_2},s_{\tau_2}],\left(\tilde{t}_2,\tilde{i}_2,\tilde{f}_2\right)\right\rangle$ are any two single valued neutrosophic uncertain linguistic numbers, the operational laws are defined as follows:

(1) $\tilde{a}_1+\tilde{a}_2=\left\langle\left[s_{\theta_1+\theta_2},s_{\tau_1+\tau_2}\right],\left(\tilde{t}_1+\tilde{t}_2-\tilde{t}_1\tilde{t}_2,\tilde{i}_1\tilde{i}_2,\tilde{f}_1\tilde{f}_2\right)\right\rangle$; (34)

(2) $\tilde{a}_1\otimes\tilde{a}_2=\left\langle\left[s_{\theta_1\theta_2},s_{\tau_1\tau_2}\right],\left(\tilde{t}_1\tilde{t}_2,\tilde{i}_1+\tilde{i}_2-\tilde{i}_1\tilde{i}_2,\tilde{f}_1+\tilde{f}_2-\tilde{f}_1\tilde{f}_2\right)\right\rangle$; (35)

(3) $\lambda\tilde{a}_1=\left\langle\left[s_{\lambda\theta_1},s_{\lambda\tau_1}\right],\left(1-\left(1-\tilde{t}_1\right)^{\lambda},\left(\tilde{i}_1\right)^{\lambda},\left(\tilde{f}_1\right)^{\lambda}\right)\right\rangle,\quad\lambda\geq 0;$ (36)

(4) $\tilde{a}_1^{\lambda}=\left\langle\left[s_{\theta_1^{\lambda}},s_{\tau_1^{\lambda}}\right],\left(\left(\tilde{t}_1\right)^{\lambda},1-\left(1-\tilde{i}_1\right)^{\lambda},1-\left(1-\tilde{f}_1\right)^{\lambda}\right)\right\rangle\ ,\lambda\geq 0$ (37)

Obviously, these operational results are still the single valued neutrosophic uncertain linguistic numbers.

Theorem 11 . Let $\tilde{a}_1=\left\langle[s_{\theta_1},s_{\tau_1}],\left(\tilde{t}_1,\tilde{i}_1,\tilde{f}_1\right)\right\rangle$ and $\tilde{a}_2=\left\langle[s_{\theta_2},s_{\tau_2}],\left(\tilde{t}_2,\tilde{i}_2,\tilde{f}_2\right)\right\rangle$ be any two single valued neutrosophic uncertain linguistic numbers, the operational laws have the following characteristics.

(1) $\tilde{a}_1+\tilde{a}_2=\tilde{a}_2+\tilde{a}_1$ (38)

(2) $\tilde{a}_1\otimes\tilde{a}_2=\tilde{a}_2\otimes\tilde{a}_1$ (39)

(3) $\lambda(\tilde{a}_1+\tilde{a}_2)=\lambda\tilde{a}_1+\lambda\tilde{a}_2,\lambda\geq 0$ (40)

(4) $\lambda_1\tilde{a}_1+\lambda_2\tilde{a}_1=(\lambda_1+\lambda_2)\tilde{a}_1,\ \lambda_1,\lambda_2\ \geq 0$ (41)

(5) $\tilde{a}_1^{\lambda_1}\otimes\tilde{a}_1^{\lambda_2}=(\tilde{a}_1)^{\lambda_1+\lambda_2},\ \lambda_1,\lambda_2\ \geq 0$ (42)

(6) $\tilde{a}_1^{\lambda_1}\otimes\tilde{a}_2^{\lambda_1}=(\tilde{a}_1\otimes\tilde{a}_2)^{\lambda_1},\ \lambda_1\geq 0$ (43)

Proof:

(1) Formula (38) is obviously right according to the operational rule (1) expressed by (34).

(2) Formula (39) is obviously right according to the operational rule (2) expressed by (35).

(3) For the left hand of (40), we have

$$\tilde{a}_1+\tilde{a}_2=\left\langle\left[s_{\theta_1+\theta_2},s_{\tau_1+\tau_2}\right],\left(\tilde{t}_1+\tilde{t}_2-\tilde{t}_1\tilde{t}_2,\tilde{i}_1\tilde{i}_2,\tilde{f}_1\tilde{f}_2\right)\right\rangle$$

then

$$\lambda\left(\tilde{a}_1+\tilde{a}_2\right)=\left\langle\left[s_{\lambda(\theta_1+\theta_2)},s_{\lambda(\tau_1+\tau_2)}\right],\left(1-\left(1-\tilde{t}_1+\tilde{t}_2-\tilde{t}_1\tilde{t}_2\right)^{\lambda},\left(\tilde{i}_1\tilde{i}_2\right)^{\lambda},\left(\tilde{f}_1\tilde{f}_2\right)^{\lambda}\right)\right\rangle$$

and for the right hand of (40), we have,

$$\lambda\tilde{a}_1=\left\langle\left[s_{\lambda\theta_1},s_{\lambda\tau_1}\right],\left(1-\left(1-\tilde{t}_1\right)^{\lambda},\left(\tilde{i}_1\right)^{\lambda},\left(\tilde{f}_1\right)^{\lambda}\right)\right\rangle,\quad \lambda\tilde{a}_2=\left\langle\left[s_{\lambda\theta_2},s_{\lambda\tau_2}\right],\left(1-\left(1-\tilde{t}_2\right)^{\lambda},\left(\tilde{i}_2\right)^{\lambda},\left(\tilde{f}_2\right)^{\lambda}\right)\right\rangle$$

then $\lambda\tilde{a}_1+\lambda\tilde{a}_2=\left\langle\left[s_{\lambda\theta_1+\lambda\theta_2},s_{\lambda\tau_1+\lambda\tau_2}\right],\left(1-\left(1-\tilde{t}_1\right)^{\lambda}+1-\left(1-\tilde{t}_2\right)^{\lambda}-\left(\left(1-\left(1-\tilde{t}_1\right)^{\lambda}\right)\left(1-\left(1-\tilde{t}_2\right)^{\lambda}\right)\right),\tilde{i}_1^{\lambda}\tilde{i}_2^{\lambda},\tilde{f}_1^{\lambda}\tilde{f}_2^{\lambda}\right)\right\rangle$

,i.e.

$$\lambda\tilde{a}_1+\lambda\tilde{a}_2=\left\langle\left[s_{\lambda(\theta_1+\theta_2)},s_{\lambda(\tau_1+\tau_2)}\right],\left(1-\left(1-\tilde{t}_1+\tilde{t}_2-\tilde{t}_1\tilde{t}_2\right)^{\lambda},\left(\tilde{i}_1\tilde{i}_2\right)^{\lambda},\left(\tilde{f}_1\tilde{f}_2\right)^{\lambda}\right)\right\rangle.$$

so, we have $\lambda\left(\tilde{a}_1+\tilde{a}_2\right)=\lambda\tilde{a}_2+\lambda\tilde{a}_1,\lambda\geq 0$. i.e., formula (40) is right.

(4) Similar to the proof of (40), it is easy to prove the formula (41) is right. The proof is omitted here.

(5) For the left hand of (42), we have

$$\tilde{a}_1^{\lambda_1}=\left\langle\left[s_{\theta_1^{\lambda_1}},s_{\tau_1^{\lambda_1}}\right],\left(\left(\tilde{t}_1\right)^{\lambda_1},1-\left(1-\tilde{i}_1\right)^{\lambda_1},1-\left(1-\tilde{f}_1\right)^{\lambda_1}\right)\right\rangle,\quad \tilde{a}_1^{\lambda_2}=\left\langle\left[s_{\theta_1^{\lambda_2}},s_{\tau_1^{\lambda_2}}\right],\left(\left(\tilde{t}_1\right)^{\lambda_2},1-\left(1-\tilde{i}_1\right)^{\lambda_2},1-\left(1-\tilde{f}_1\right)^{\lambda_2}\right)\right\rangle$$

then, $\tilde{a}_1^{\lambda_1}\otimes\tilde{a}_1^{\lambda_2}=\Big\langle\left[s_{\theta_1^{\lambda_1+\lambda_2}},s_{\tau_1^{\lambda_1+\lambda_2}}\right],\Big(\left(\tilde{t}_1\right)^{\lambda_1+\lambda_2},1-\left(1-\tilde{i}_1\right)^{\lambda_1}+1-\left(1-\tilde{i}_1\right)^{\lambda_2}-\left(\left(1-\left(1-\tilde{i}_1\right)^{\lambda_1}\right)\left(1-\left(1-\tilde{i}_2\right)^{\lambda_2}\right)\right),$

$$1-\left(1-\tilde{f}_1\right)^{\lambda_1}+1-\left(1-\tilde{f}_1\right)^{\lambda_2}-\left(\left(1-\left(1-\tilde{f}_1\right)^{\lambda_1}\right)\left(1-\left(1-\tilde{f}_1\right)^{\lambda_2}\right)\right)\Big)\Big\rangle$$

i.e.

$$\tilde{a}_1^{\lambda_1}\otimes\tilde{a}_1^{\lambda_2}=\left\langle\left[s_{\theta_1^{\lambda_1+\lambda_2}},s_{\tau_1^{\lambda_1+\lambda_2}}\right],\left(\left(\tilde{t}_1\right)^{\lambda_1+\lambda_2},1-\left(1-\tilde{i}_1\right)^{\lambda_1+\lambda_2},1-\left(1-\tilde{f}_1\right)^{\lambda_1+\lambda_2}\right)\right\rangle.$$

and for the right hand of (42), we have,

$$\tilde{a}_1^{\lambda_1+\lambda_2}=\left\langle\left[s_{\theta_1^{\lambda_1+\lambda_2}},s_{\tau_1^{\lambda_1+\lambda_2}}\right],\left(\left(\tilde{t}_1\right)^{\lambda_1+\lambda_2},1-\left(1-\tilde{i}_1\right)^{\lambda_1+\lambda_2},1-\left(1-\tilde{f}_1\right)^{\lambda_1+\lambda_2}\right)\right\rangle.$$

So, we have $\tilde{a}_1^{\lambda_1}\oplus_{\varepsilon}\tilde{a}_1^{\lambda_2}=\tilde{a}_1^{\lambda_1+\lambda_2},\lambda_1\geq 0,\lambda_2\geq 0$. i.e., formula (42) is right.

(6) Similar to the proof of (42), it is easy to prove the formula (43) is right. The proof is omitted here.

Definition 12. Suppose $\tilde{a}_1=\left\langle[s_{\theta_1},s_{\tau_1}],\left(\tilde{t}_1,\tilde{i}_1,\tilde{f}_1\right)\right\rangle$ is a single valued neutrosophic uncertain linguistic number, then the expectation value $E(\tilde{a}_1)$ of $\tilde{a}_1$ can be defined as follows.

$$E(\tilde{a}_1)=\frac{1}{3}\times(2+\tilde{t}_1-\tilde{i}_1-\tilde{f}_1)\times s_{(\theta_1+\tau_1)/2}=s_{(\theta_1+\tau_1)\times(2+\tilde{t}_1-\tilde{i}_1-\tilde{f})/6} \tag{44}$$

Definition 13. Suppose $\tilde{a}_1=\left\langle[s_{\theta_1},s_{\tau_1}],\left(\tilde{t}_1,\tilde{i}_1,\tilde{f}_1\right)\right\rangle$ is a single valued neutrosophic uncertain linguistic number, then the accuracy function $H(\tilde{a}_1)$ of $\tilde{a}_1$ can be defined as follows.

$$H(\tilde{a}_1)=(\tilde{t}_1+\tilde{i}_1+\tilde{f}_1)\times s_{(\theta_1+\tau_1)/2}=s_{(\theta_1+\tau_1)\times(\tilde{t}_1+\tilde{i}_1+\tilde{f}_1)/2} \tag{45}$$

Definition 14. Let $\tilde{a}_1 = \left\langle [s_{\theta_1}, s_{\tau_1}], \left(\tilde{t}_1, \tilde{i}_1, \tilde{f}_1\right) \right\rangle$ and $\tilde{a}_2 = \left\langle [s_{\theta_2}, s_{\tau_2}], \left(\tilde{t}_2, \tilde{i}_2, \tilde{f}_2\right) \right\rangle$ be any two single valued neutrosophic uncertain linguistic numbers, then

(1) if $E(\tilde{a}_1) > E(\tilde{a}_2)$, then $\tilde{a}_1 \succ \tilde{a}_2$;

(2) if $E(\tilde{a}_1) = E(\tilde{a}_2)$, then

if $H(\tilde{a}_1) > H(\tilde{a}_2)$, then $\tilde{a}_1 \succ \tilde{a}_2$;

If $H(\tilde{a}_1) = H(\tilde{a}_2)$, then $\tilde{a}_1 = \tilde{a}_2$.

4. Some Heronian mean operators based on the single valued neutrosophic uncertain linguistic variables

In this section, we will extend the IGWHM and IGGWHM operators to aggregate the single neutrosophic uncertain linguistic variables, and propose a neutrosophic uncertain linguistic number improved generalized weighted Heronian mean (NULNIGWHM) operator and a neutrosophic uncertain linguistic number improved generalized geometric weighted Heronian mean (NLUNIGGWHM) operator which can be described as follows.

4.1 The NULNIGWHM operator

Definition 15. Let $p, q \geq 0$, and $\tilde{a}_i = \left\langle [s_{\theta_i}, s_{\tau_i}], \left(\tilde{t}_i, \tilde{i}_i, \tilde{f}_i\right) \right\rangle$ $(i = 1,2,\cdots,n)$ be a collection of the single valued neutrosophic uncertain linguistic numbers. $W = (w_1, w_2, \cdots, w_n)^T$ is the weight vector of $\tilde{a}_i (i = 1,2,\cdots,n)$, and satisfies $w_i \geq 0$, $\sum_{i=1}^{n} w_i = 1$.

If

$$NULNIGWHM^{p,q}(\tilde{a}_1, \tilde{a}_2, \cdots, \tilde{a}_n) = \left(\frac{\sum_{i=1}^{n} \sum_{j=i}^{n} w_i w_j \tilde{a}_i^p \tilde{a}_j^q}{\sum_{i=1}^{n} \sum_{j=i}^{n} w_i w_j} \right)^{\frac{1}{p+q}} \tag{46}$$

then $NULNIGWHM^{p,q}$ is called the neutrosophic uncertain linguistic number improved generalized weighted Heronian mean operator.

Theorem 12. Let $p, q \geq 0$, and $\tilde{a}_i = \left\langle [s_{\theta_i}, s_{\tau_i}], \left(\tilde{t}_i, \tilde{i}_i, \tilde{f}_i\right) \right\rangle$ $(i = 1,2,\cdots,n)$ be a collection of the single valued neutrosophic uncertain linguistic numbers. $W = (w_1, w_2, \cdots, w_n)^T$ is the weight vector of $\tilde{a}_i (i = 1,2,\cdots,n)$, and satisfies $w_i \geq 0$, $\sum_{i=1}^{n} w_i = 1$, then the result aggregated from Definition 15 is still a NULN, and

$$NULNIGWHM^{p,q}(\tilde{a}_1, \tilde{a}_2, \cdots, \tilde{a}_n) = \left(\frac{\sum_{i=1}^{n} \sum_{j=i}^{n} w_i w_j \tilde{a}_i^p \tilde{a}_j^q}{\sum_{i=1}^{n} \sum_{j=i}^{n} w_i w_j} \right)^{\frac{1}{p+q}} = \left\langle \left[s_{\left(\frac{\sum_{i=1}^{n} \sum_{j=i}^{n} w_i w_j \left(\theta_i^p + \theta_j^q\right)}{\sum_{i=1}^{n} \sum_{j=i}^{n} w_i w_j} \right)^{\frac{1}{p+q}}}, s_{\left(\frac{\sum_{i=1}^{n} \sum_{j=i}^{n} w_i w_j \left(\tau_i^p + \tau_j^q\right)}{\sum_{i=1}^{n} \sum_{j=i}^{n} w_i w_j} \right)^{\frac{1}{p+q}}} \right], \right.$$

$$\left\langle \left(\left(1-\left(\prod_{i=1}^{n}\prod_{j=i}^{n}\left(1-\tilde{t}_i^{\,p}\tilde{t}_j^{\,q}\right)^{w_iw_j}\right)^{\frac{1}{\sum_{i=1}^{n}\sum_{j=i}^{n}w_iw_j}}\right)^{1/(p+q)},\ 1-\left(1-\left(\prod_{i=1}^{n}\prod_{j=i}^{n}\left(1-\left(1-\tilde{i}_i\right)^{p}\left(1-\tilde{i}_j\right)^{q}\right)^{w_iw_j}\right)^{\frac{1}{\sum_{i=1}^{n}\sum_{j=i}^{n}w_iw_j}}\right)^{1/(p+q)},\right.\right.$$

$$\left.\left. 1-\left(1-\left(\prod_{i=1}^{n}\prod_{j=i}^{n}\left(1-\left(1-\tilde{f}_i\right)^{p}\left(1-\tilde{f}_j\right)^{q}\right)^{w_iw_j}\right)^{\frac{1}{\sum_{i=1}^{n}\sum_{j=i}^{n}w_iw_j}}\right)^{1/(p+q)}\right)\right\rangle \tag{47}$$

Proof

Since $\tilde{a}_i^{\,p}=\left\langle \left[s_{\theta_i^p}, s_{\tau_i^p}\right], \left(\left(\tilde{t}_i\right)^p, 1-\left(1-\tilde{i}_i\right)^p, 1-\left(1-\tilde{f}_i\right)^p\right)\right\rangle$; $\tilde{a}_j^{\,q}=\left\langle \left[s_{\theta_j^q}, s_{\tau_j^q}\right], \left(\left(\tilde{t}_j\right)^q, 1-\left(1-\tilde{i}_j\right)^q, 1-\left(1-\tilde{f}_j\right)^q\right)\right\rangle$

so, $w_iw_j\tilde{a}_i^{\,p}\tilde{a}_j^{\,q}=\left\langle \left[s_{w_iw_j\theta_i^p\theta_j^q}, s_{w_iw_j\tau_i^p\tau_j^q}\right], \left(1-\left(1-\left(\tilde{t}_i\right)^p\left(\tilde{t}_j\right)^q\right)^{w_iw_j},\right.\right.$

$$\left.\left.\left(1-\left(1-\tilde{i}_i\right)^p\left(1-\tilde{i}_j\right)^q\right)^{w_iw_j}, \left(1-\left(1-\tilde{f}_i\right)^p\left(1-\tilde{\eta}_j\right)^q\right)^{w_iw_j}\right)\right\rangle$$

then $\sum_{i=1}^{n}\sum_{j=1}^{n}w_iw_j\tilde{a}_i^{\,p}\tilde{a}_j^{\,q}=\left\langle \left[s_{\sum_{i=1}^{n}\sum_{j=i}^{n}w_iw_j\theta_i^p\theta_j^q}, s_{\sum_{i=1}^{n}\sum_{j=i}^{n}w_iw_j\tau_i^p\tau_j^q}\right], \left(1-\prod_{i=1}^{n}\prod_{j=i}^{n}\left(1-\left(\tilde{t}_i\right)^p\left(\tilde{t}_j\right)^q\right)^{w_iw_j},\right.\right.$

$$\left.\left.\prod_{i=1}^{n}\prod_{j=i}^{n}\left(1-\left(1-\tilde{i}_i\right)^p\left(1-\tilde{i}_j\right)^q\right)^{w_iw_j}, \prod_{i=1}^{n}\prod_{j=i}^{n}\left(1-\left(1-\tilde{f}_i\right)^p\left(1-\tilde{f}_j\right)^q\right)^{w_iw_j}\right)\right\rangle$$

$$\frac{1}{\sum_{i=1}^{n}\sum_{j=i}^{n}w_iw_j}\sum_{i=1}^{n}\sum_{j=i}^{n}w_iw_j\tilde{a}_i^{\,p}\tilde{a}_j^{\,q}=\left\langle \left[s_{\frac{1}{\sum_{i=1}^{n}\sum_{j=i}^{n}w_iw_j}\sum_{i=1}^{n}\sum_{j=i}^{n}w_iw_j\theta_i^p\theta_j^q}, s_{\frac{1}{\sum_{i=1}^{n}\sum_{j=i}^{n}w_iw_j}\sum_{i=1}^{n}\sum_{j=i}^{n}w_iw_j\tau_i^p\tau_j^q}\right],\right.$$

$$\left(1-\left(\prod_{i=1}^{n}\prod_{j=i}^{n}\left(1-\left(\tilde{t}_i\right)^p\left(\tilde{t}_j\right)^q\right)^{w_iw_j}\right)^{\frac{1}{\sum_{i=1}^{n}\sum_{j=i}^{n}w_iw_j}},\right.$$

$$\left.\left.\left(\prod_{i=1}^{n}\prod_{j=i}^{n}\left(1-\left(1-\tilde{i}_i\right)^p\left(1-\tilde{i}_j\right)^q\right)^{w_iw_j}\right)^{\frac{1}{\sum_{i=1}^{n}\sum_{j=i}^{n}w_iw_j}}, \left(\prod_{i=1}^{n}\prod_{j=i}^{n}\left(1-\left(1-\tilde{f}_i\right)^p\left(1-\tilde{f}_j\right)^q\right)^{w_iw_j}\right)^{\frac{1}{\sum_{i=1}^{n}\sum_{j=i}^{n}w_iw_j}}\right)\right\rangle,$$

and $\left(\frac{\sum_{i=1}^{n}\sum_{j=i}^{n} w_i w_j \tilde{a}_i^p \tilde{a}_j^q}{\sum_{i=1}^{n}\sum_{j=i}^{n} w_i w_j}\right)^{\frac{1}{p+q}} = \left\langle \left[s_{\left(\frac{\sum_{i=1}^{n}\sum_{j=i}^{n} w_i w_j \theta_i^p \theta_j^q}{\sum_{i=1}^{n}\sum_{j=i}^{n} w_i w_j}\right)^{1/p+q}}, s_{\left(\frac{\sum_{i=1}^{n}\sum_{j=i}^{n} w_i w_j \tau_i^p \tau_j^q}{\sum_{i=1}^{n}\sum_{j=i}^{n} w_i w_j}\right)^{1/p+q}} \right], \right.$

$$\left(\left(1-\left(\prod_{i=1}^{n}\prod_{j=i}^{n}\left(1-\left(\tilde{t}_i\right)^p\left(\tilde{t}_j\right)^q\right)^{w_i w_j}\right)^{\frac{1}{\sum_{i=1}^{n}\sum_{j=i}^{n} w_i w_j}}\right)^{1/p+q}, 1-\left(1-\left(\prod_{i=1}^{n}\prod_{j=i}^{n}\left(1-\left(1-\tilde{i}_i\right)^p\left(1-\tilde{i}_j\right)^q\right)^{w_i w_j}\right)^{\frac{1}{\sum_{i=1}^{n}\sum_{j=i}^{n} w_i w_j}}\right)^{1/p+q},\right.$$

$$\left.\left. 1-\left(1-\left(\prod_{i=1}^{n}\prod_{j=i}^{n}\left(1-\left(1-\tilde{f}_i\right)^p\left(1-\tilde{f}_j\right)^q\right)^{w_i w_j}\right)^{\frac{1}{\sum_{i=1}^{n}\sum_{j=i}^{n} w_i w_j}}\right)^{1/p+q}\right)\right\rangle$$

So, $NULNIGWHM^{p,q}\left(\tilde{a}_1, \tilde{a}_2, \cdots, \tilde{a}_n\right) = \left(\frac{\sum_{i=1}^{n}\sum_{j=i}^{n} w_i w_j \tilde{a}_i^p \tilde{a}_j^q}{\sum_{i=1}^{n}\sum_{j=i}^{n} w_i w_j}\right)^{\frac{1}{p+q}} =$

$$\left\langle \left[s_{\left(\frac{\sum_{i=1}^{n}\sum_{j=i}^{n} w_i w_j \left(\theta_i^p + \theta_j^q\right)}{\sum_{i=1}^{n}\sum_{j=i}^{n} w_i w_j}\right)^{\frac{1}{p+q}}}, s_{\left(\frac{\sum_{i=1}^{n}\sum_{j=i}^{n} w_i w_j \left(\tau_i^p + \tau_j^q\right)}{\sum_{i=1}^{n}\sum_{j=i}^{n} w_i w_j}\right)^{\frac{1}{p+q}}} \right], \left(\left(1-\left(\prod_{i=1}^{n}\prod_{j=i}^{n}\left(1-\tilde{t}_i^{\,p}\tilde{t}_j^{\,q}\right)^{w_i w_j}\right)^{\frac{1}{\sum_{i=1}^{n}\sum_{j=i}^{n} w_i w_j}}\right)^{1/p+q},\right.\right.$$

$$1-\left(1-\left(\prod_{i=1}^{n}\prod_{j=i}^{n}\left(1-\left(1-\tilde{i}_i\right)^p\left(1-\tilde{i}_j\right)^q\right)^{w_i w_j}\right)^{\frac{1}{\sum_{i=1}^{n}\sum_{j=i}^{n} w_i w_j}}\right)^{1/p+q},$$

$$\left.\left. 1-\left(1-\left(\prod_{i=1}^{n}\prod_{j=i}^{n}\left(1-\left(1-\tilde{f}_i\right)^p\left(1-\tilde{f}_j\right)^q\right)^{w_i w_j}\right)^{\frac{1}{\sum_{i=1}^{n}\sum_{j=i}^{n} w_i w_j}}\right)^{1/p+q}\right)\right\rangle$$

which completes the proof of theorem 12.

The NULNIGWHM operator has the properties, such as idempotency, monotonicity and boundedness.

Theorem 13. (Idempotency) .

Let all $\tilde{a}_i = \tilde{a}$ for all i $\left(i = 1,2,\cdots,n\right)$, then

$$NULNIGWHM^{p,q}\left(\tilde{a}_1, \tilde{a}_2, \cdots, \tilde{a}_n\right) = \tilde{a}. \tag{48}$$

Proof

Since all $\tilde{a}_i = \tilde{a} = \left\langle [s_\theta, s_\tau], \left(\tilde{t}, \tilde{i}, \tilde{f}\right)\right\rangle$ for all i $(i = 1,2,\cdots,n)$, then we have

$$NULNIGWHM^{p,q}(\tilde{a}_1, \tilde{a}_2, \cdots, \tilde{a}_n) = \left\langle \left[s_{\left(\frac{\sum_{i=1}^{n}\sum_{j=i}^{n} w_i w_j (\theta^p + \theta^q)}{\sum_{i=1}^{n}\sum_{j=i}^{n} w_i w_j}\right)^{\frac{1}{p+q}}}, s_{\left(\frac{\sum_{i=1}^{n}\sum_{j=i}^{n} w_i w_j (\tau^p + \tau^q)}{\sum_{i=1}^{n}\sum_{j=i}^{n} w_i w_j}\right)^{\frac{1}{p+q}}} \right], \right.$$

$$\left(\left(1 - \left(\prod_{i=1}^{n}\prod_{j=i}^{n}\left(1 - \tilde{t}^{\,p+q}\right)^{w_i w_j}\right)^{\frac{1}{\sum_{i=1}^{n}\sum_{j=i}^{n} w_i w_j}}\right)^{1/p+q}, 1 - \left(1 - \left(\prod_{i=1}^{n}\prod_{j=i}^{n}\left(1 - \left(1 - \tilde{i}\right)^{p+q}\right)^{w_i w_j}\right)^{\frac{1}{\sum_{i=1}^{n}\sum_{j=i}^{n} w_i w_j}}\right)^{1/p+q}, \right.$$

$$\left.\left. 1 - \left(1 - \left(\prod_{i=1}^{n}\prod_{j=i}^{n}\left(1 - \left(1 - \tilde{f}\right)^{p+q}\right)^{w_i w_j}\right)^{\frac{1}{\sum_{i=1}^{n}\sum_{j=i}^{n} w_i w_j}}\right)^{1/p+q}\right)\right\rangle = \left\langle [s_\theta, s_\tau], \left(\tilde{t}, \tilde{i}, \tilde{f}\right)\right\rangle$$

Theorem 14.(monotonicity)

Let $\tilde{a}_i = \left\langle [s_{\theta_i}, s_{\tau_i}], \left(\tilde{t}_i, \tilde{i}_i, \tilde{f}_i\right)\right\rangle$ and $\tilde{a}_i^* = \left\langle [s^*{}_{\theta_i}, s^*{}_{\tau_i}], \left(\tilde{t}^*{}_i, \tilde{i}^*{}_i, \tilde{f}^*{}_i\right)\right\rangle$ $(i = 1,2,\cdots,n)$ be two collection of neutrosophic uncertain linguistic fuzzy numbers, and if $\tilde{a}_i \le \tilde{a}_i^*$,i.e. $s_{\theta_i} \le s^*{}_{\theta_i}, s_{\tau_i} \le s^*_{\tau_i}$, $\tilde{t}_i \le \tilde{t}_i^*, \tilde{i}_i \ge \tilde{i}_i^*$ and $\tilde{f}_i \ge \tilde{f}_i^*$, for all i $(i = 1,2,\cdots,n)$, then

$$NULNIGWHM^{p,q}(\tilde{a}_1, \tilde{a}_2, \cdots, \tilde{a}_n) \le NULNIGWHM(\tilde{a}^*{}_1, \tilde{a}^*{}_2, \cdots, \tilde{a}^*{}_n)\,. \tag{49}$$

Proof

(1) Since $s_{\theta_i} \le s^*{}_{\theta_i}, s_{\tau_i} \le s^*_{\tau_i}$, then, $s_{\theta_i^p} \le s^*{}_{\theta_i^p}, s_{\tau_i^q} \le s^*_{\tau_i^q}$, $s_{\theta_i^p \tau_i^q} \le s^*_{\theta_i^p \tau_i^q}$, $s_{w_i w_j \theta_i^p \tau_i^q} \le s^*_{w_i w_j \theta_i^p \tau_i^q}$,

$$s_{\sum_{i=1}^{n}\sum_{j=i}^{n} w_i w_j \theta_i^p \tau_i^q} \le s^*_{\sum_{i=1}^{n}\sum_{j=i}^{n} w_i w_j \theta_i^p \tau_i^q},\ s_{\frac{\sum_{i=1}^{n}\sum_{j=i}^{n} w_i w_j \theta_i^p \tau_i^q}{\sum_{i=1}^{n}\sum_{j=i}^{n} w_i w_j}} \le s^*_{\frac{\sum_{i=1}^{n}\sum_{j=i}^{n} w_i w_j \theta_i^p \tau_i^q}{\sum_{i=1}^{n}\sum_{j=i}^{n} w_i w_j}}.$$

(2) Firstly, since $\tilde{t}_i \le \tilde{t}_i^*$ for all i ,and $p, q > 0$, then we can get

$$\tilde{t}_i{}^p \tilde{t}_j^q \le \tilde{t}_i^{*p} \tilde{t}_j^{*q},\ 1 - \tilde{t}_i{}^p \tilde{t}_j^q \ge 1 - \tilde{t}_i^{*p} \tilde{t}_j^{*q},$$

$$\prod_{i=1}^{n}\prod_{j=i}^{n}\left(1 - \tilde{t}_i{}^p \tilde{t}_j^q\right)^{w_i w_j} \ge \prod_{i=1}^{n}\prod_{j=i}^{n}\left(1 - \tilde{t}_i^{*p} \tilde{t}_j^{*q}\right)^{w_i w_j}, \text{ and}$$

$$\left(\prod_{i=1}^{n}\prod_{j=i}^{n}\left(1-\tilde{t}_i^{\,p}\tilde{t}_j^{\,q}\right)^{w_iw_j}\right)^{\frac{1}{\sum_{i=1}^{n}\sum_{j=i}^{n}w_iw_j}} \ge \left(\prod_{i=1}^{n}\prod_{j=i}^{n}\left(1-\tilde{t}_i^{*p}\tilde{t}_j^{*q}\right)^{w_iw_j}\right)^{\frac{1}{\sum_{i=1}^{n}\sum_{j=i}^{n}w_iw_j}}, \text{ then,}$$

$$1-\left(\prod_{i=1}^{n}\prod_{j=i}^{n}\left(1-\tilde{t}_i^{\,p}\tilde{t}_j^{\,q}\right)^{w_iw_j}\right)^{\frac{1}{\sum_{i=1}^{n}\sum_{j=i}^{n}w_iw_j}} \le 1-\left(\prod_{i=1}^{n}\prod_{j=i}^{n}\left(1-\tilde{t}_i^{*p}\tilde{t}_j^{*q}\right)^{w_iw_j}\right)^{\frac{1}{\sum_{i=1}^{n}\sum_{j=i}^{n}w_iw_j}},$$

so, $$\left(1-\left(\prod_{i=1}^{n}\prod_{j=i}^{n}\left(1-\tilde{t}_i^{\,p}\tilde{t}_j^{\,q}\right)^{w_iw_j}\right)^{\frac{1}{\sum_{i=1}^{n}\sum_{j=i}^{n}w_iw_j}}\right)^{1/p+q} \le \left(1-\left(\prod_{i=1}^{n}\prod_{j=i}^{n}\left(1-\tilde{t}_i^{*p}\tilde{t}_j^{*q}\right)^{w_iw_j}\right)^{\frac{1}{\sum_{i=1}^{n}\sum_{j=i}^{n}w_iw_j}}\right)^{1/p+q}$$

Secondly, since $\tilde{i}_i \ge \tilde{i}_i^{*}$ for all i and $p,q>0$, $,\left(1-\tilde{i}_i\right)^p \le \left(1-\tilde{i}_i^{*}\right)^q$ and $\left(1-\tilde{i}_j\right)^p \le \left(1-\tilde{i}_j^{*}\right)^q$,

$$\left(1-\tilde{i}_i\right)^p\left(1-\tilde{i}_j\right)^p \le \left(1-\tilde{i}_i^{*}\right)^q\left(1-\tilde{i}_j^{*}\right)^q,\ \prod_{i=1}^{n}\prod_{j=i}^{n}\left(1-\left(1-\tilde{i}_i\right)^p\left(1-\tilde{i}_j\right)^p\right)^{w_iw_j} \le \prod_{i=1}^{n}\prod_{j=i}^{n}\left(1-\left(1-\tilde{i}_i^{*}\right)^q\left(1-\tilde{i}_j^{*}\right)^q\right)^{w_iw_j},$$

$$\left(\prod_{i=1}^{n}\prod_{j=i}^{n}\left(1-\left(1-\tilde{i}_i\right)^p\left(1-\tilde{i}_j\right)^q\right)^{w_iw_j}\right)^{\frac{1}{\sum_{i=1}^{n}\sum_{j=i}^{n}w_iw_j}} \le \left(\prod_{i=1}^{n}\prod_{j=i}^{n}\left(1-\left(1-\tilde{i}_i^{*}\right)^p\left(1-\tilde{i}_j^{*}\right)^q\right)^{w_iw_j}\right)^{\frac{1}{\sum_{i=1}^{n}\sum_{j=i}^{n}w_iw_j}},$$

$$1-\left(\prod_{i=1}^{n}\prod_{j=i}^{n}\left(1-\left(1-\tilde{i}_i\right)^p\left(1-\tilde{i}_j\right)^q\right)^{w_iw_j}\right)^{\frac{1}{\sum_{i=1}^{n}\sum_{j=i}^{n}w_iw_j}} \ge 1-\left(\prod_{i=1}^{n}\prod_{j=i}^{n}\left(1-\left(1-\tilde{i}_i^{*}\right)^p\left(1-\tilde{i}_j^{*}\right)^q\right)^{w_iw_j}\right)^{\frac{1}{\sum_{i=1}^{n}\sum_{j=i}^{n}w_iw_j}},$$

$$\left(1-\left(\prod_{i=1}^{n}\prod_{j=i}^{n}\left(1-\left(1-\tilde{i}_i\right)^p\left(1-\tilde{i}_j\right)^q\right)^{w_iw_j}\right)^{\frac{1}{\sum_{i=1}^{n}\sum_{j=i}^{n}w_iw_j}}\right)^{\frac{1}{p+q}} \ge \left(1-\left(\prod_{i=1}^{n}\prod_{j=i}^{n}\left(1-\left(1-\tilde{i}_i^{*}\right)^p\left(1-\tilde{i}_j^{*}\right)^q\right)^{w_iw_j}\right)^{\frac{1}{\sum_{i=1}^{n}\sum_{j=i}^{n}w_iw_j}}\right)^{\frac{1}{p+q}}$$

so,

$$1-\left(1-\left(\prod_{i=1}^{n}\prod_{j=i}^{n}\left(1-\left(1-\tilde{i}_i\right)^p\left(1-\tilde{i}_j\right)^q\right)^{w_iw_j}\right)^{\frac{1}{\sum_{i=1}^{n}\sum_{j=i}^{n}w_iw_j}}\right)^{\frac{1}{p+q}} \le 1-\left(1-\left(\prod_{i=1}^{n}\prod_{j=i}^{n}\left(1-\left(1-\tilde{i}_i^{*}\right)^p\left(1-\tilde{i}_j^{*}\right)^q\right)^{w_iw_j}\right)^{\frac{1}{\sum_{i=1}^{n}\sum_{j=i}^{n}w_iw_j}}\right)^{\frac{1}{p+q}}.$$

Thirdly, similar with the previous step, we can prove that

$$1-\left(1-\prod_{i=1}^{n}\prod_{j=i}^{n}\left(1-\left(1-\tilde{f}_i\right)^p\left(1-\tilde{f}_j\right)^p\right)^{w_iw_j}\right)^{\frac{1}{p+q}} \le 1-\left(1-\prod_{i=1}^{n}\prod_{j=i}^{n}\left(1-\left(1-\tilde{f}_i^{*}\right)^q\left(1-\tilde{f}_j^{*}\right)^q\right)^{w_iw_j}\right)^{\frac{1}{p+q}}$$

According to (1)-(2), we can get

$$\left\langle\left[s_{\left(\frac{\sum_{i=1}^{n}\sum_{j=i}^{n}w_iw_j\left(\theta_i^p+\theta_j^q\right)}{\sum_{i=1}^{n}\sum_{j=i}^{n}w_iw_j}\right)^{\frac{1}{p+q}}}, s_{\left(\frac{\sum_{i=1}^{n}\sum_{j=i}^{n}w_iw_j\left(\tau_i^p+\tau_j^q\right)}{\sum_{i=1}^{n}\sum_{j=i}^{n}w_iw_j}\right)^{\frac{1}{p+q}}}\right], \left(\left(1-\left(\prod_{i=1}^{n}\prod_{j=i}^{n}\left(1-\tilde{t}_i^{\,p}\tilde{t}_j^{\,q}\right)^{w_iw_j}\right)^{\frac{1}{\sum_{i=1}^{n}\sum_{j=i}^{n}w_iw_j}}\right)^{1/p+q},\right.\right.$$

$$1-\left(1-\left(\prod_{i=1}^{n}\prod_{j=i}^{n}\left(1-\left(1-\tilde{i}_i\right)^p\left(1-\tilde{i}_j\right)^p\right)^{w_iw_j}\right)^{\frac{1}{\sum_{i=1}^{n}\sum_{j=i}^{n}w_iw_j}}\right)^{\frac{1}{p+q}}, 1-\left(1-\left(\prod_{i=1}^{n}\prod_{j=i}^{n}\left(1-\left(1-\tilde{f}_i\right)^p\left(1-\tilde{f}_j\right)^p\right)^{w_iw_j}\right)^{\frac{1}{\sum_{i=1}^{n}\sum_{j=i}^{n}w_iw_j}}\right)^{\frac{1}{p+q}}\right\rangle \geq$$

$$\left\langle\left[s^*_{\left(\frac{\sum_{i=1}^{n}\sum_{j=i}^{n}w_iw_j\left(\theta_i^p+\theta_j^q\right)}{\sum_{i=1}^{n}\sum_{j=i}^{n}w_iw_j}\right)^{\frac{1}{p+q}}}, s^*_{\left(\frac{\sum_{i=1}^{n}\sum_{j=i}^{n}w_iw_j\left(\tau_i^p+\tau_j^q\right)}{\sum_{i=1}^{n}\sum_{j=i}^{n}w_iw_j}\right)^{\frac{1}{p+q}}}\right], \left(\left(1-\left(\prod_{i=1}^{n}\prod_{j=i}^{n}\left(1-\tilde{t}_i^{*p}\tilde{t}_j^{*q}\right)^{w_iw_j}\right)^{\frac{1}{\sum_{i=1}^{n}\sum_{j=i}^{n}w_iw_j}}\right)^{1/p+q},\right.\right.$$

$$1-\left(1-\left(\prod_{i=1}^{n}\prod_{j=i}^{n}\left(1-\left(1-\tilde{i}_i^*\right)^p\left(1-\tilde{i}_j^*\right)^p\right)^{w_iw_j}\right)^{\frac{1}{\sum_{i=1}^{n}\sum_{j=i}^{n}w_iw_j}}\right)^{\frac{1}{p+q}},$$

$$\left.\left.1-\left(1-\left(\prod_{i=1}^{n}\prod_{j=i}^{n}\left(1-\left(1-\tilde{f}_i^*\right)^p\left(1-\tilde{f}_j^*\right)^p\right)^{w_iw_j}\right)^{\frac{1}{\sum_{i=1}^{n}\sum_{j=1}^{n}w_iw_j}}\right)^{\frac{1}{p+q}}\right)\right\rangle$$

i.e. $NULNIGWHM(\tilde{a}_1,\tilde{a}_2,\cdots,\tilde{a}_n) \leq NULNIGWHM(\tilde{a}^*{}_1,\tilde{a}^*{}_2,\cdots,\tilde{a}^*{}_n)$, which complete the proof of theorem 14.

Theorem 15. (Boundary).

The $NULNIGWHM^{p,q}$ operator lies between the max and min operators: $\tilde{a}_{\min}=\min(\tilde{a}_1,\tilde{a}_2,\cdots,\tilde{a}_n), \tilde{a}_{\max}=\max(\tilde{a}_1,\tilde{a}_2,\cdots,\tilde{a}_n)$, then

$$\tilde{a}_{\min} \leq NULNIGWHM(\tilde{a}_1,\tilde{a}_2,\cdots,\tilde{a}_n) \leq \tilde{a}_{\max}\ . \tag{50}$$

Proof

Since $\tilde{a}_{\min} \leq \tilde{a}$, based on theorems 13 and 14, we can get

$$\tilde{a}_{\min} \leq NULNIGWHM^{p,q}(\tilde{a}^*{}_1,\tilde{a}^*{}_2,\cdots,\tilde{a}^*{}_n)$$

and then $\tilde{a}_{\max} \geq \tilde{a}$, $NULNIGWHM(\tilde{a}_1,\tilde{a}_2,\cdots,\tilde{a}_n) \leq \tilde{a}_{\max}$.

so, $\tilde{a}_{\min} \leq NULNIGWHM(\tilde{a}_1,\tilde{a}_2,\cdots,\tilde{a}_n) \leq \tilde{a}_{\max}$, which complete the proof of theorem 15.

In the following, we will discuss some special cases of the NULNIGGWHM operator in regard to the parameters p and q .

(1) when $p=0$, then

$$NULNIGWHM^{0,q}\left(\tilde{a}_1,\tilde{a}_2,\cdots,\tilde{a}_n\right)=\left\langle\left[s_{\left(\frac{\sum_{i=1}^{n}\sum_{j=i}^{n}w_iw_j\left(1+\theta_j^q\right)}{\sum_{i=1}^{n}\sum_{j=i}^{n}w_iw_j}\right)^{\frac{1}{q}}}, s_{\left(\frac{\sum_{i=1}^{n}\sum_{j=i}^{n}w_iw_j\left(1+\tau_j^q\right)}{\sum_{i=1}^{n}\sum_{j=i}^{n}w_iw_j}\right)^{\frac{1}{q}}}\right], \left(\left(1-\left(\prod_{i=1}^{n}\prod_{j=i}^{n}\left(1-\tilde{t}_j^q\right)^{w_iw_j}\right)^{\frac{1}{\sum_{i=1}^{n}\sum_{j=i}^{n}w_iw_j}}\right)^{1/q},\right.\right.$$

$$1-\left(1-\left(\prod_{i=1}^{n}\prod_{j=i}^{n}\left(1-\left(1-\tilde{i}_j\right)^{q}\right)^{w_i w_j}\right)^{\frac{1}{\sum_{i=1}^{n}\sum_{j=i}^{n} w_i w_j}}\right)^{1/q},\ 1-\left(1-\left(\prod_{i=1}^{n}\prod_{j=i}^{n}\left(1-\left(1-\tilde{f}_j\right)^{q}\right)^{w_i w_j}\right)^{\frac{1}{\sum_{i=1}^{n}\sum_{j=i}^{n} w_i w_j}}\right)^{1/q}\Bigg)\Bigg\rangle. \tag{51}$$

(2) when $q=0$, then

$$NULNIGWHM^{p,0}\left(\tilde{a}_1,\tilde{a}_2,\cdots,\tilde{a}_n\right)=\left\langle\left[s_{\left(\frac{\sum_{i=1}^{n}\sum_{j=i}^{n} w_i w_j\left(\theta_i^{p}+1\right)}{\sum_{i=1}^{n}\sum_{j=i}^{n} w_i w_j}\right)^{\frac{1}{p}}}, s_{\left(\frac{\sum_{i=1}^{n}\sum_{j=i}^{n} w_i w_j\left(\tau_i^{p}+1\right)}{\sum_{i=1}^{n}\sum_{j=i}^{n} w_i w_j}\right)^{\frac{1}{p}}}\right],\right.$$

$$\left(\left(1-\left(\prod_{i=1}^{n}\prod_{j=i}^{n}\left(1-\tilde{t}_i^{\,p}\right)^{w_i w_j}\right)^{\frac{1}{\sum_{i=1}^{n}\sum_{j=i}^{n} w_i w_j}}\right)^{1/p},\ 1-\left(1-\left(\prod_{i=1}^{n}\prod_{j=i}^{n}\left(1-\left(1-\tilde{i}_i\right)^{p}\right)^{w_i w_j}\right)^{\frac{1}{\sum_{i=1}^{n}\sum_{j=i}^{n} w_i w_j}}\right)^{1/p},\right.$$

$$\left.\left.1-\left(1-\left(\prod_{i=1}^{n}\prod_{j=i}^{n}\left(1-\left(1-\tilde{f}_i\right)^{p}\right)^{w_i w_j}\right)^{\frac{1}{\sum_{i=1}^{n}\sum_{j=i}^{n} w_i w_j}}\right)^{1/p}\right)\right\rangle \tag{52}$$

(3) when $p=q=1$, then

$$NULNIGWHM^{1,1}\left(\tilde{a}_1,\tilde{a}_2,\cdots,\tilde{a}_n\right)=\left\langle\left[s_{\left(\frac{\sum_{i=1}^{n}\sum_{j=i}^{n} w_i w_j\left(\theta_i+\theta_j\right)}{\sum_{i=1}^{n}\sum_{j=i}^{n} w_i w_j}\right)^{\frac{1}{2}}}, s_{\left(\frac{\sum_{i=1}^{n}\sum_{j=i}^{n} w_i w_j\left(\tau_i+\tau_j\right)}{\sum_{i=1}^{n}\sum_{j=i}^{n} w_i w_j}\right)^{\frac{1}{2}}}\right],\right.$$

$$\left(\left(1-\left(\prod_{i=1}^{n}\prod_{j=i}^{n}\left(1-\tilde{t}_i\tilde{t}_j\right)^{w_i w_j}\right)^{\frac{1}{\sum_{i=1}^{n}\sum_{j=i}^{n} w_i w_j}}\right)^{1/2},\ 1-\left(1-\left(\prod_{i=1}^{n}\prod_{j=i}^{n}\left(1-\left(1-\tilde{i}_i\right)\left(1-\tilde{i}_j\right)\right)^{w_i w_j}\right)^{\frac{1}{\sum_{i=1}^{n}\sum_{j=i}^{n} w_i w_j}}\right)^{1/2},\right.$$

$$\left.\left.1-\left(1-\left(\prod_{i=1}^{n}\prod_{j=i}^{n}\left(1-\left(1-\tilde{f}_i\right)\left(1-\tilde{f}_j\right)\right)^{w_i w_j}\right)^{\frac{1}{\sum_{i=1}^{n}\sum_{j=i}^{n} w_i w_j}}\right)^{1/2}\right)\right\rangle \tag{53}$$

4.2 The NULNIGGWHM operator

Definition 16. Let $p,q\geq 0$, and $\tilde{a}_i=\left\langle[s_{\theta_i},s_{\tau_i}],\left(\tilde{t}_i,\tilde{i}_i,\tilde{f}_i\right)\right\rangle$ $(i=1,2,\cdots,n)$ be a collection of single neutrosophic uncertain linguistic numbers. $W=\left(w_1,w_2,\cdots,w_n\right)^T$ is the weight vector of $\tilde{a}_i\left(i=1,2,\cdots,n\right)$, and satisfies $w_i \geq 0$, $\sum_{i=1}^{n} w_i = 1$.If

$$NULNIGGWHM^{p,q}\left(a_1,a_2,\cdots,a_n\right)=\frac{1}{p+q}\prod_{i=1}^{n}\prod_{j=i}^{n}\left(pa_i+qa_j\right)^{\frac{2(n+1-i)}{n(n+1)}\frac{w_j}{\sum_{k=i}^{n}w_k}} \tag{54}$$

then $\text{NULNIGGWHM}^{p,q}$ is called the neutrosophic uncertain linguistic number improved generalized geometric weighted Heronian mean (NULNIGGWHM) operator.

Theorem 16. Let $p,q \geq 0$, and $\tilde{a}_i=\left\langle [s_{\theta_i},s_{\tau_i}],\left(\tilde{t}_i,\tilde{i}_i,\tilde{f}_i\right)\right\rangle$ $\left(i=1,2,\cdots,n\right)$be a collection of the single valued neutrosophic uncertain linguistic numbers. $W=\left(w_1,w_2,\cdots,w_n\right)^T$ is the weight vector of $\tilde{a}_i\left(i=1,2,\cdots,n\right)$, and satisfies $w_i \geq 0$, $\sum_{i=1}^{n} w_i = 1$, then the result aggregated from Definition 16 is still a NLUN, and

$$NULNIGGWHM^{p,q}\left(a_1,a_2,\cdots,a_n\right)=\frac{1}{p+q}\prod_{i=1}^{n}\prod_{j=i}^{n}\left(pa_i+qa_j\right)^{\frac{2(n+1-i)}{n(n+1)}\frac{w_j}{\sum_{k=i}^{n}w_k}}=$$

$$\left\langle\left[s_{\frac{1}{p+q}\prod_{i=1}^{n}\prod_{j=i}^{n}\left(p\theta_i+q\theta_j\right)^{\frac{2(n+1-i)}{n(n+1)}\frac{w_j}{\sum_{k=i}^{n}w_k}}},s_{\frac{1}{p+q}\prod_{i=1}^{n}\prod_{j=i}^{n}\left(p\tau_i+q\tau_j\right)^{\frac{2(n+1-i)}{n(n+1)}\frac{w_j}{\sum_{k=i}^{n}w_k}}}\right],\left(1-\left(1-\prod_{i=1}^{n}\prod_{j=i}^{n}\left(1-\left(1-\tilde{t}_i\right)^p\left(1-\tilde{t}_j\right)^q\right)^{\frac{2(n+1-i)}{n(n+1)}\frac{w_j}{\sum_{k=i}^{n}w_k}}\right)^{1/p+q},\right.\right. \tag{55}$$

$$\left.\left.\left(1-\prod_{i=1}^{n}\prod_{j=i}^{n}\left(1-\tilde{i}_i^{\,p}\,\tilde{i}_j^{\,q}\right)^{\frac{2(n+1-i)}{n(n+1)}\frac{w_j}{\sum_{k=i}^{n}w_k}}\right)^{1/p+q},\left(1-\prod_{i=1}^{n}\prod_{j=i}^{n}\left(1-\tilde{f}_i^{\,p}\,\tilde{f}_j^{\,q}\right)^{\frac{2(n+1-i)}{n(n+1)}\frac{w_j}{\sum_{k=i}^{n}w_k}}\right)^{1/p+q}\right)\right\rangle$$

The proof is similar with the theorem 12, and it is omitted here.

Similar to Theorems 13-15, it is easy to prove the NULNIGGWHM operator has the following properties.

Theorem 17. (Idempotency) .

Let all $\tilde{a}_i=\tilde{a}$ for all i $\left(i=1,2,\cdots,n\right)$, then

$$NULNIGGWHM^{p,q}\left(\tilde{a}_1,\tilde{a}_2,\cdots,\tilde{a}_n\right)=\tilde{a}. \tag{56}$$

Theorem 18. (Boundary).

The $NULNIGGWHM^{p,q}$ operator lies between the max and min operators: $\tilde{a}_{\min}=\min(\tilde{a}_1,\tilde{a}_2,\cdots,\tilde{a}_n),\tilde{a}_{\max}=\max(\tilde{a}_1,\tilde{a}_2,\cdots,\tilde{a}_n)$, then

$$\tilde{a}_{\min} \leq NULNIGGWHM(\tilde{a}_1,\tilde{a}_2,\cdots,\tilde{a}_n) \leq \tilde{a}_{\max}. \tag{57}$$

Theorem 19.(monotonicity)

Let $\tilde{a}_i=\left\langle [s_{\theta_i},s_{\tau_i}],\left(\tilde{t}_i,\tilde{i}_i,\tilde{f}_i\right)\right\rangle$ and $\tilde{a}^*{}_i=\left\langle [s^*{}_{\theta_i},s^*{}_{\tau_i}],\left(\tilde{t}^*{}_i,\tilde{i}^*{}_i,\tilde{f}^*{}_i\right)\right\rangle$ $\left(i=1,2,\cdots,n\right)$ be two collection of neutrosophic uncertain linguistic fuzzy numbers, and if $s_{\theta(a_i)} \leq s^*{}_{\theta(a_i)}, s_{\tau(a_i)} \leq s^*{}_{\tau(a_i)}$, $\tilde{t}_i \leq \tilde{t}_i^*, \tilde{\delta}_i \geq \tilde{\delta}_i^*$ and $\tilde{f}_i \geq \tilde{f}_i^*$, for all i $\left(i=1,2,\cdots,n\right)$,then

$$NULNIGGWHM(\tilde{a}_1,\tilde{a}_2,\cdots,\tilde{a}_n)\le NULNIGGWHM(\tilde{a}^*{}_1,\tilde{a}^*{}_2,\cdots,\tilde{a}^*{}_n)\,. \tag{58}$$

In the following, we will discuss some special cases of the NULNIGGWHM operator in regard to the parameters p and q.

(1) when $p=0$, then

$$NULNIGGWHM^{0,q}(a_1,a_2,\cdots,a_n)=\left\langle\left[s_{\frac{1}{q}\prod_{i=1}^{n}\prod_{j=i}^{n}(q\theta_j)^{\frac{2(n+1-i)}{n(n+1)}\frac{w_j}{\sum_{k=i}^{n}w_k}}},s_{\frac{1}{q}\prod_{i=1}^{n}\prod_{j=i}^{n}(q\tau_j)^{\frac{2(n+1-i)}{n(n+1)}\frac{w_j}{\sum_{k=i}^{n}w_k}}}\right],\right.$$

$$\left(1-\left(1-\prod_{i=1}^{n}\prod_{j=i}^{n}\left(1-(1-\tilde{t}_j)^q\right)^{\frac{2(n+1-i)}{n(n+1)}\frac{w_j}{\sum_{k=i}^{n}w_k}}\right)^{1/q},\left(1-\prod_{i=1}^{n}\prod_{j=i}^{n}\left(1-\tilde{i}_j^{\,q}\right)^{\frac{2(n+1-i)}{n(n+1)}\frac{w_j}{\sum_{k=i}^{n}w_k}}\right)^{1/q},\right.$$

$$\left.\left.\left(1-\prod_{i=1}^{n}\prod_{j=i}^{n}\left(1-\tilde{f}_j^{\,q}\right)^{\frac{2(n+1-i)}{n(n+1)}\frac{w_j}{\sum_{k=i}^{n}w_k}}\right)^{1/q}\right)\right\rangle \tag{59}$$

(2) when $q=0$, then

$$NULNIGGWHM^{p,0}(a_1,a_2,\cdots,a_n)=\left\langle\left[s_{\frac{1}{p}\prod_{i=1}^{n}\prod_{j=i}^{n}(p\theta_i)^{\frac{2(n+1-i)}{n(n+1)}\frac{w_j}{\sum_{k=i}^{n}w_k}}},s_{\frac{1}{p}\prod_{i=1}^{n}\prod_{j=i}^{n}(p\tau_i)^{\frac{2(n+1-i)}{n(n+1)}\frac{w_j}{\sum_{k=i}^{n}w_k}}}\right],\right.$$

$$\left(1-\left(1-\prod_{i=1}^{n}\prod_{j=i}^{n}\left(1-(1-\tilde{t}_i)^p\right)^{\frac{2(n+1-i)}{n(n+1)}\frac{w_j}{\sum_{k=i}^{n}w_k}}\right)^{1/p},\left(1-\prod_{i=1}^{n}\prod_{j=i}^{n}\left(1-\tilde{i}_i^{\,p}\right)^{\frac{2(n+1-i)}{n(n+1)}\frac{w_j}{\sum_{k=i}^{n}w_k}}\right)^{1/p},\right.$$

$$\left.\left.\left(1-\prod_{i=1}^{n}\prod_{j=i}^{n}\left(1-\tilde{f}_i^{\,p}\right)^{\frac{2(n+1-i)}{n(n+1)}\frac{w_j}{\sum_{k=i}^{n}w_k}}\right)^{1/p}\right)\right\rangle \tag{60}$$

(2) when $q=p=1$, then

$$NULNIGGWHM^{1,1}(a_1,a_2,\cdots,a_n)=\left\langle\left[s_{\frac{1}{2}\prod_{i=1}^{n}\prod_{j=i}^{n}(\theta_i+\theta_j)^{\frac{2(n+1-i)}{n(n+1)}\frac{w_j}{\sum_{k=i}^{n}w_k}}},s_{\frac{1}{2}\prod_{i=1}^{n}\prod_{j=i}^{n}(\tau_i+\tau_j)^{\frac{2(n+1-i)}{n(n+1)}\frac{w_j}{\sum_{k=i}^{n}w_k}}}\right],\right.$$

$$\left(1-\left(1-\prod_{i=1}^{n}\prod_{j=i}^{n}\left(1-(1-\tilde{t}_i)(1-\tilde{t}_j)\right)^{\frac{2(n+1-i)}{n(n+1)}\frac{w_j}{\sum_{k=i}^{n}w_k}}\right)^{1/2},\left(1-\prod_{i=1}^{n}\prod_{j=i}^{n}\left(1-\tilde{i}_i\tilde{i}_j\right)^{\frac{2(n+1-i)}{n(n+1)}\frac{w_j}{\sum_{k=i}^{n}w_k}}\right)^{1/2},\right.$$

$$\left.\left.\left(1-\prod_{i=1}^{n}\prod_{j=i}^{n}\left(1-\tilde{f}_i\tilde{f}_j\right)^{\frac{2(n+1-i)}{n(n+1)}\frac{w_j}{\sum_{k=i}^{n}w_k}}\right)^{1/2}\right)\right\rangle \tag{61}$$

5. The decision-making methods based on the NULIGWHM operator and NULIGGWHM operator

In order to strengthen the efficiency of this decision-making, we can make several experts participate in the decision-making under neutrosophic uncertain linguistic fuzzy environment.

Considering the multiple attribute group decision making problems with neutrosophic uncertain linguistic fuzzy information described as follows. Let $A=\{A_1,A_2,\cdots,A_m\}$ be a set of alternatives, and $C=\{C_1,C_2,\cdots,C_n\}$ be the set of attributes, and $W=\{w_1,w_2,\cdots,w_n\}$ be the weight vector of the attribute C_j $(j=1,2,\cdots,n)$, where $w_j \geq 0$, $j=1,2,\cdots,n$, $\sum_{j=1}^{n} w_j = 1$ .Let $D=\{D_1,D_2,\cdots,D_t\}$ be the set of decision makers, and $\lambda=(\lambda_1,\lambda_2,\cdots\lambda_t)^T$ be the weight vector of decision makers $D_e(e=1,2,\cdots,t)$, where $\lambda_e \geq 0$, $\sum_{e=1}^{t}\lambda_e=1$. Suppose $H^{(e)}=[h_{ij}^{(e)}]_{m\times n}$ are the decision matrices where $\tilde{h}_{ij}^{(e)}=\left\langle [s_{\theta_{ij}}^{(e)},s_{\tau_{ij}}^{(e)}],\left(\tilde{t}_{ij}^{(e)},\tilde{i}_{ij}^{(e)},\tilde{f}_{ij}^{(e)}\right)\right\rangle$ takes the form of the single valued neutrosophic uncertain linguistic variables given by the decision maker D_e for alternative A_i with respect to attribute C_j. Then, the ranking of alternatives is finally acquired.

The methods involve the following steps:

Step 1. Utilize the NULIGWHM operator

$$\tilde{h}_i^{(e)} = NULIGWHM\left(\tilde{h}_{i1}^{(e)},\tilde{h}_{i2}^{(e)},\cdots,\tilde{h}_{in}^{(e)}\right) \tag{62}$$

or NULIGGWHM operator

$$\tilde{h}_i^{(e)} = NULIGGWHM\left(\tilde{h}_{i1}^{(e)},\tilde{h}_{i2}^{(e)},\cdots,\tilde{h}_{in}^{(e)}\right) \tag{63}$$

to get the comprehensive attribute values of each alternative for decision maker D_e .

Step 2. Utilize the NULIGWHM operator

$$\tilde{h}_i = NULIGWHM\left(\tilde{h}_i^{(1)},\tilde{h}_i^{(2)},\cdots,\tilde{h}_i^{(t)}\right) \tag{64}$$

or NULIGGWHM operator

$$\tilde{h}_i = NULIGGWHM\left(\tilde{h}_i^{(1)},\tilde{h}_i^{(2)},\cdots,\tilde{h}_i^{(t)}\right) \tag{65}$$

to aggregate the evaluation values of the single decision maker to the collective comprehensive values for each alternative.

Step 3. Calculate the value $E(h_i)$ of h_i .

Step 4. Rank $h_i(i=1,2,\cdots,m)$ in descending order according to the comparison method of INULNs described in Definition 14.

Step 5. End.

6. A numerical example

In this section, we will provide an example to illustrate the application of IULFPEWA and IULFPEWG operators. Suppose that an investment company wants to invest an amount of money to a company. There are four candidate companies $A_i \quad (i = 1,2,3,4)$ evaluated by three decision makers $\{D_1, D_2, D_3\}$. The weight vector of the decision makers is $\lambda = (0.314, 0.355, 0.331)^T$, and the considered attributes include: C_1 (the risk index) , C_2 (the growth index), C_3 (the social-political impact index), and C_4 (the environmental impact index). Suppose the attribute weight vector is $w = (0.4, 0.20, 0.40)^T$ .The three decision makers $\{D_1, D_2, D_3\}$ evaluate the four companies $A_i \quad (i = 1,2,3,4)$ with respect to the attributes $C_j\,(j = 1,2,3)$ by using the neutrosophic uncertain linguistic variables (suppose that the decision makers use linguistic term set $S = (s_0, s_1, s_2, s_3, s_4, s_5, s_6)$ to express their evaluation results) and construct three following decision matrices $H^{(e)} = [h_{ij}^{(e)}]_{4\times 4} \quad (e = 1,2,3)$ as listed in Tables 1-3.

Table 1. Decision matrix $H^{(1)}$.

	C1	C2	C3
A1	$\langle[S_5,S_5],(0.265,0.350,0.385)\rangle$	$\langle[S_2,S_3],(0.330,0.390,0.280)\rangle$	$\langle[S_5,S_6],(0.245,0.275,0.480)\rangle$
A2	$\langle[S_4,S_5],(0.345,0.245,0.410)\rangle$	$\langle[S_5,S_5],(0.430,0.290,0.280)\rangle$	$\langle[S_3,S_4],(0.245,0.375,0.380)\rangle$
A3	$\langle[S_3,S_4],(0.365,0.300,0.335)\rangle$	$\langle[S_4,S_4],(0.480,0.315,0.205)\rangle$	$\langle[S_4,S_5],(0.340,0.370,0.290)\rangle$
A4	$\langle[S_6,S_6],(0.430,0.300,0.270)\rangle$	$\langle[S_2,S_3],(0.460,0.245,0.295)\rangle$	$\langle[S_3,S_4],(0.310,0.520,0.170)\rangle$

Table 2. Decision matrix $H^{(2)}$.

	C1	C2	C3
A1	$\langle[S_3,S_4],(0.125,0.470,0.405)\rangle$	$\langle[S_3,S_4],(0.220,0.420,0.360)\rangle$	$\langle[S_3,S_4],(0.345,0.490,0.165)\rangle$
A2	$\langle[S_5,S_6],(0.355,0.315,0.330)\rangle$	$\langle[S_3,S_4],(0.300,0.370,0.330)\rangle$	$\langle[S_4,S_5],(0.205,0.630,0.165)\rangle$
A3	$\langle[S_4,S_5],(0.315,0.380,0.305)\rangle$	$\langle[S_4,S_4],(0.330,0.565,0.105)\rangle$	$\langle[S_2,S_3],(0.280,0.520,0.200)\rangle$
A4	$\langle[S_5,S_5],(0.365,0.365,0.270)\rangle$	$\langle[S_4,S_5],(0.355,0.320,0.325)\rangle$	$\langle[S_2,S_3],(0.425,0.485,0.090)\rangle$

Table 3. Decision matrix $H^{(3)}$.

	C1	C2	C3
A1	$\langle[S_5,S_5],(0.260,0.425,0.315)\rangle$	$\langle[S_3,S_4],(0.220,0.450,0.330)\rangle$	$\langle[S_4,S_5],(0.255,0.500,0.245)\rangle$
A2	$\langle[S_4,S_5],(0.270,0.370,0.360)\rangle$	$\langle[S_5,S_5],(0.320,0.215,0.465)\rangle$	$\langle[S_2,S_3],(0.135,0.575,0.290)\rangle$
A3	$\langle[S_4,S_4],(0.245,0.465,0.290)\rangle$	$\langle[S_5,S_5],(0.250,0.570,0.180)\rangle$	$\langle[S_1,S_3],(0.175,0.660,0.165)\rangle$

A4	<[S_3,S_4],(0.390,0.340,0.270)>	<[S_3,S_4],(0.305,0.475,0.220)>	<[S_4,S_5],(0.465,0.485,0.050)>

6.1 The decision-making method based on NULIGWHM operator

Step 1. Get the comprehensive attribute values of each alternative for decision maker D_e by the NULIGWHM operator in Eq.(62).(suppose $q = p = 1$)

$\tilde{h}_1^{(1)} =$ <$[s_{2.925}, s_{3.046}]$,(0.269,0.324,0.401)>, $\tilde{h}_2^{(1)} =$ <[$s_{2.320}, s_{2.717}$],(0.322,0.303,0.373)>,

$\tilde{h}_3^{(1)} =$ <[$s_{2.810}, s_{3.146}$],(0.376,0.330,0.291)>, $\tilde{h}_4^{(1)} =$ <[$s_{2.953}, s_{3.274}$],(0.388,0.369,0.229)>,

$\tilde{h}_1^{(2)} =$ <[$s_{2.925}, s_{3.046}$],(0.240,0.469,0.283)>, $\tilde{h}_2^{(2)} =$ <[$s_{2.462}, s_{2.839}$],(0.287,0.438,0.253)>,

$\tilde{h}_3^{(2)} =$ <[$s_{2.717}, s_{2.953}$],(0.303,0.466,0.218)>, $\tilde{h}_4^{(2)} =$ <[$s_{2.943}, s_{2.943}$],(0.389,0.403,0.188)>,

$\tilde{h}_1^{(3)} =$ <[$s_{2.139}, s_{2.690}$],(0.251,0.459,0.287)>, $\tilde{h}_2^{(3)} =$ <[$s_{3.049}, s_{3.156}$],(0.227,0.412,0.347)>,

$\tilde{h}_3^{(3)} =$ <[$s_{2.470}, s_{2.847}$],(0.218,0.558,0.214)>, $\tilde{h}_4^{(3)} =$ <[$s_{2.437}, s_{2.818}$],(0.408,0.420,0.144)>.

Step 2. Get the collective comprehensive values for each alternative by the NULIGWHM operator in Eq.(64).(suppose $q = p = 1$)

$\tilde{h}_1 =$<[$s_{2.354}, s_{2.461}$],(0.250,0.422,0.323)>, $\tilde{h}_2 =$<[$s_{2.343}, s_{2.472}$],(0.276,0.389,0.323)>,

$\tilde{h}_3 =$<[$s_{2.245}, s_{2.385}$],(0.298,0.453,0.243)>, $\tilde{h}_4 =$<[$s_{2.322}, s_{2.434}$],(0.391,0.402,0.189)>

Step 3.Calculate the value $E(h_i)$ of h_i.

$E(h_1) = s_{1.208}$, $E(h_2) = s_{1.255}$, $E(h_3) = s_{1.236}$, $E(h_4) = s_{1.427}$.

Step 4. Rank $h_i(i = 1,2,\cdots,m)$ in descending order according to the comparison method of INULNs described in Definition 16.

So, $E(h_4) > E(h_2) > E(h_3) > E(h_1)$.

i.e., A_4 is the best choice.

Step 5. End.

6.2 The decision-making method based on NULIGWHM operator

Step 1’. Get the comprehensive attribute values of each alternative for decision maker D_e by the NULIGWHM operator in Eq.(63).(suppose $q = p = 1$)

$\tilde{h}_1^{(1)} =$ <[$s_{4.245}, s_{4.463}$],(0.274,0.333,0.394)>, $\tilde{h}_2^{(1)} =$ <[$s_{2.644}, s_{3.652}$],(0.326,0.308,0.365)>,

$\tilde{h}_3^{(1)} =$ <[$s_{3.934}, s_{4.945}$],(0.385,0.331,0.284)>, $\tilde{h}_4^{(1)} =$ <[$s_{4.326}, s_{5.329}$],(0.390,0.373,0.240)>,

$\tilde{h}_1^{(2)} =$ <[$s_{4,245}, s_{4.643}$],(0.221,0.464,0.307)>, $\tilde{h}_2^{(2)} =$ <[$s_{2.938}, s_{3.958}$],(0.279,0.459,0.270)>,

$\tilde{h}_3^{(2)} =$ <[$s_{3.652}, s_{4.326}$],(0.306,0.488,0.217)>, $\tilde{h}_4^{(2)} =$ <[$s_{4.314}, s_{4.314}$],(0.384,0.400,0.220)>,

$\tilde{h}_1^{(3)} =$ <[$s_{2.115}, s_{3.588}$],(0.247,0.461,0.293)>, $\tilde{h}_2^{(3)} =$ <[$s_{4.556}, s_{4.957}$],(0.225,0.413,0.364)>,

$\tilde{h}_3^{(3)} =$ <[$s_{2.942}, s_{3.961}$],(0.218,0.572,0.216)>, $\tilde{h}_4^{(3)} =$ <[$s_{2.915}, s_{3.934}$],(0.393,0.435,0.178)>.

Step2'. Get the collective comprehensive values for each alternative by the NULIGWHM operator in Eq.(65)(suppose $q = p = 1$).

$\tilde{h}_1 =$ <[$s_{3.653}, s_{4.469}$],(0.247,0.422,0.332)>, $\tilde{h}_2 =$ <[$s_{3.747}, s_{4.601}$],(0.275,0.397,0.333)>,

$\tilde{h}_3 =$ <[$s_{3.177}, s_{4.035}$],(0.298,0.470,0.239)>, $\tilde{h}_4 =$ <[$s_{3.442}, s_{4.282}$],(0.389,0.403,0.213)>

Step 3'. Calculate the value $E(h_i)$ of h_i .

$E(h_1) = s_{2.020}$, $E(h_2) = s_{2.150}$, $E(h_3) = s_{1.909}$, $E(h_4) = s_{2.282}$.

Step 4'. Rank $h_i(i = 1,2,\cdots,m)$ in descending order according to the comparison method of INULNs described in Definition 16.

So, $E(h_4) > E(h_2) > E(h_1) > E(h_3)$.

i.e., A_4 is the best choice.

Step 5'. End

6.3 The influences of the parameters p, q on the decision-making problem

Table 4 Ordering of the alternatives by the different parameters p and q in NULIGWHM operator

p, q	$E(h_i)$	ranking
$p = 0, q = 1$	$E(h_1) = s_{3.135}$, $E(h_2) = s_{2.948}$, $E(h_3) = s_{2.687}$, $E(h_4) = s_{3.564}$.	$A_4 \succ A_1 \succ A_2 \succ A_3$
$p = 0, q = 2.1$	$E(h_1) = s_{2.285}$, $E(h_2) = s_{2.179}$, $E(h_3) = s_{1.940}$, $E(h_4) = s_{2.572}$.	$A_4 \succ A_1 \succ A_2 \succ A_3$
$p = 0, q = 2.2$	$E(h_1) = s_{2.274}$, $E(h_2) = s_{2.173}$, $E(h_3) = s_{1.933}$, $E(h_4) = s_{2.559}$.	$A_4 \succ A_1 \succ A_2 \succ A_3$
$p = 0, q = 10$	$E(h_1) = s_{2.589}$, $E(h_2) = s_{2.675}$, $E(h_3) = s_{2.429}$, $E(h_4) = s_{3.000}$.	$A_4 \succ A_2 \succ A_1 \succ A_3$
$p = 1, q = 0$	$E(h_1) = s_{3.083}$, $E(h_2) = s_{3.548}$, $E(h_3) = s_{3.247}$, $E(h_4) = s_{3.863}$.	$A_4 \succ A_2 \succ A_3 \succ A_1$
$p = 2, q = 0$	$E(h_1) = s_{2.291}$, $E(h_2) = s_{2.645}$, $E(h_3) = s_{2.342}$, $E(h_4) = s_{2.926}$.	$A_4 \succ A_2 \succ A_3 \succ A_1$
$p = 10, q = 0$	$E(h_1) = s_{2.619}$, $E(h_2) = s_{2.889}$, $E(h_3) = s_{2.548}$, $E(h_4) = s_{3.378}$.	$A_4 \succ A_2 \succ A_1 \succ A_3$
$p = 2, q = 1$	$E(h_1) = s_{1.120}$, $E(h_2) = s_{1.202}$, $E(h_3) = s_{1.166}$, $E(h_4) = s_{1.348}$.	$A_4 \succ A_2 \succ A_3 \succ A_1$
$p = 10, q = 1$	$E(h_1) = s_{1.988}$, $E(h_2) = s_{2.167}$, $E(h_3) = s_{1.962}$, $E(h_4) = s_{2.509}$.	$A_4 \succ A_2 \succ A_1 \succ A_3$
$p = 1, q = 2$	$E(h_1) = s_{1.120}$, $E(h_2) = s_{1.137}$, $E(h_3) = s_{1.100}$, $E(h_4) = s_{1.305}$.	$A_4 \succ A_2 \succ A_1 \succ A_3$
$p = 1, q = 10$	$E(h_1) = s_{1.959}$, $E(h_2) = s_{2.038}$, $E(h_3) = s_{1.889}$, $E(h_4) = s_{2.274}$.	$A_4 \succ A_2 \succ A_1 \succ A_3$

$p=1, q=1$	$E(h_1)=s_{1.208}$, $E(h_2)=s_{1.255}$, $E(h_3)=s_{1.236}$, $E(h_4)=s_{1.427}$.	$A_4 \succ A_2 \succ A_3 \succ A_1$

Table 5 Ordering of the alternatives by the different parameters p and q in NULIGGWHM operator

p, q	$E(h_i)$	Ranking
$p=0, q=1$	$E(h_1)=s_{2.036}$, $E(h_2)=s_{1.783}$, $E(h_3)=s_{1.510}$, $E(h_4)=s_{2.196}$.	$A_4 \succ A_1 \succ A_2 \succ A_3$
$p=0, q=2$	$E(h_1)=s_{2.014}$, $E(h_2)=s_{1.749}$, $E(h_3)=s_{1.494}$, $E(h_4)=s_{2.159}$.	$A_4 \succ A_1 \succ A_2 \succ A_3$
$p=0, q=10$	$E(h_1)=s_{1.865}$, $E(h_2)=s_{1.561}$, $E(h_3)=s_{1.380}$, $E(h_4)=s_{2.014}$.	$A_4 \succ A_1 \succ A_2 \succ A_3$
$p=0.01, q=0$	$E(h_1)=s_{1.924}$, $E(h_2)=s_{2.445}$, $E(h_3)=s_{2.151}$, $E(h_4)=s_{2.435}$.	$A_2 \succ A_4 \succ A_3 \succ A_1$
$p=1, q=0$	$E(h_1)=s_{1.912}$, $E(h_2)=s_{2.423}$, $E(h_3)=s_{2.127}$, $E(h_4)=s_{2.415}$.	$A_2 \succ A_4 \succ A_3 \succ A_1$
$p=2, q=0$	$E(h_1)=s_{1.899}$, $E(h_2)=s_{2.3940}$, $E(h_3)=s_{2.099}$, $E(h_4)=s_{2.3941}$.	$A_4 \succ A_2 \succ A_3 \succ A_1$
$p=2, q=1$	$E(h_1)=s_{1.982}$, $E(h_2)=s_{2.219}$, $E(h_3)=s_{1.972}$, $E(h_4)=s_{2.306}$.	$A_4 \succ A_2 \succ A_3 \succ A_1$
$p=1, q=1$	$E(h_1)=s_{2.020}$, $E(h_2)=s_{2.150}$, $E(h_3)=s_{1.909}$, $E(h_4)=s_{2.282}$.	$A_4 \succ A_2 \succ A_1 \succ A_3$
$p=10, q=1$	$E(h_1)=s_{1.822}$, $E(h_2)=s_{2.073}$, $E(h_3)=s_{1.892}$, $E(h_4)=s_{2.240}$.	$A_4 \succ A_2 \succ A_3 \succ A_1$
$p=1, q=2$	$E(h_1)=s_{2.092}$, $E(h_2)=s_{2.085}$, $E(h_3)=s_{1.834}$, $E(h_4)=s_{2.243}$.	$A_4 \succ A_1 \succ A_2 \succ A_3$
$p=1, q=10$	$E(h_1)=s_{1.878}$, $E(h_2)=s_{1.649}$, $E(h_3)=s_{1.471}$, $E(h_4)=s_{2.035}$.	$A_4 \succ A_1 \succ A_2 \succ A_3$

According to the Tables 4 and 5, the ranking results may be different for the different parameter values p, q in NULIGWHM and NULIGGWHM operators. In general, the best alternative is always A_4 in two tables. We can take the values of the two parameters as $p=q=1$, which is not merely intuitive and simple but also takes the correlations of the aggregated arguments into consideration completely.

7. Conclusions

The MAGDM problems widely exist in real decision making, and the aggregation operators are the important tools these problems. Especially, Heronian mean (HM) can catch the interrelations of the aggregated arguments. In addition, the neutrosophic uncertain linguistic set can be better to express the incomplete, indeterminate and inconsistent information. In this paper, we proposed the neutrosophic uncertain linguistic numbers by combining neutrosophic set and uncertain linguistic variables, and developed some Heronian mean operators on the basis of neutrosophic uncertain linguistic numbers, included the neutrosophic uncertain linguistic number improved generalized weighted Heronian mean (NULNIGWHM) operator and the neutrosophic uncertain linguistic number improved generalized geometric weighted Heronian mean (NLUNIGGWHM) operator and discussed the properties of them in detail. In the meantime, we studied the some special cases in consideration of the values of p and q. Moreover, we developed two methods which have the advantages that they can take the correlations of the attributes into account fully to deal with the multi-attribute group decision making (MAGDM) problems under neutrosophic uncertain linguistic number environment. We also gave a numerical example to show the steps of the proposed methods and to discuss the influences of different values of p and q on the ranking results. In the future research, we can extend the application scopes of the proposed operators to other fields such as option of sponsors, science-technology assessment, the performance evaluation, and so on.

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