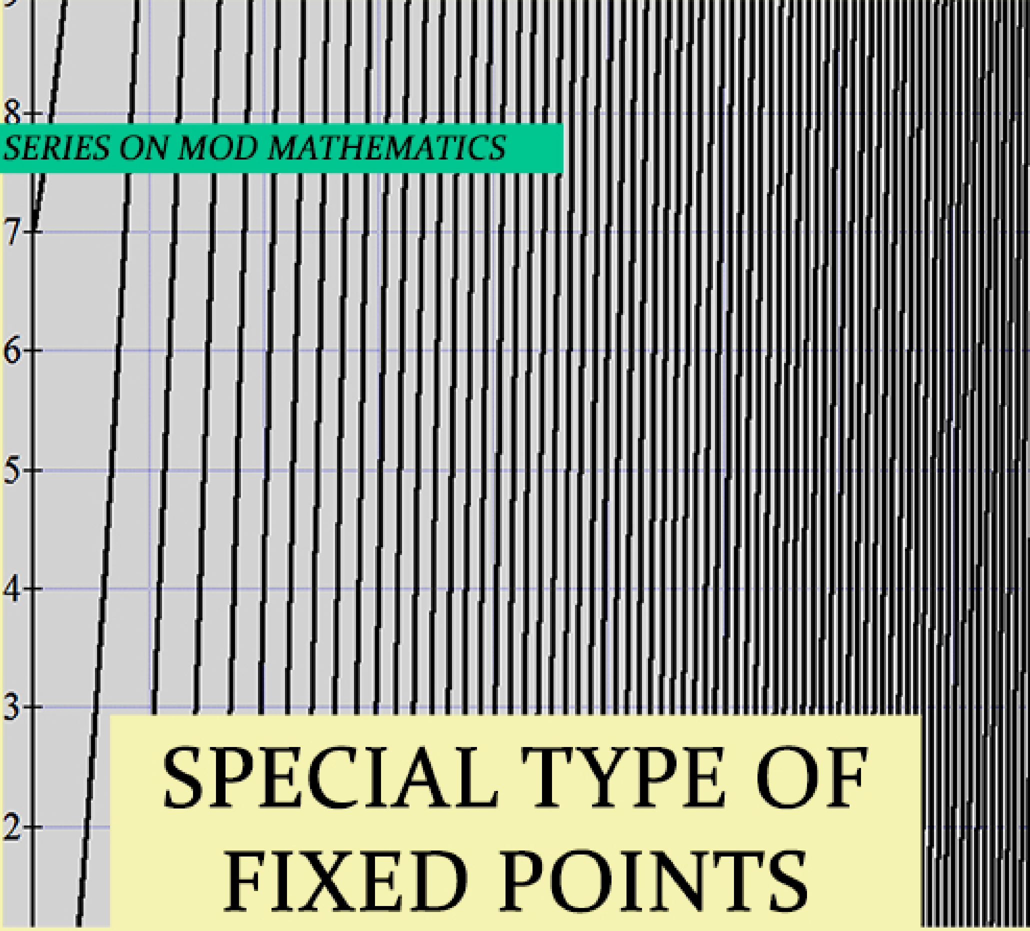

SERIES ON MOD MATHEMATICS

SPECIAL TYPE OF FIXED POINTS OF MOD MATRIX OPERATORS

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PREFACE

In this book authors for the first time introduce a special type of fixed points using MOD square matrix operators. These special type of fixed points are different from the usual classical fixed points.

A study of this is carried out in this book. Several interesting properties are developed in this regard. The notion of these fixed points find many applications in the mathematical models which are dealt systematically by the authors in the forth coming books.

These special type of fixed points or special realized limit cycles are always guaranteed as we use only MOD matrices as operators with its entries from modulo integers. However this sort of results are NP hard problems if we use reals or complex numbers.

These new notions are systemically developed in this book.

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Chapter One

INTRODUCTION

In this book authors for the first time define a special type of fixed point different from the classical fixed points using MOD matrix operators. When the MOD matrices are square matrices they yield a fixed point which is defined as the realized fixed point. The MOD matrices themselves serve as the operators from a collection of row vectors of same order to itself.

Such study is new and innovative leading to several openings both in fixed point theory and in mathematical modeling. Here authors mainly use the modulo integer. Z_n or $\langle Z_n \cup I \rangle$, the neutrosophic integer or $C(Z_n)$ or $\langle Z_n \cup g \rangle$ and so on.

For MOD functions and their properties refer [21].

Clearly the map $\eta_r : R \to [0, n)$ has finite number of classical fixed points [21].

Likewise $\eta : Z \to Z_n$ also has finite number of classical fixed points [1]. However the study of realized fixed points arising from MOD matrix operators are entirely different from the usual or classical fixed points.

We call a square matrix with entries from Z_n as the MOD real matrix operator. This study is carried out in chapter two.

Similarly $\langle Z_n \cup I\rangle$ can be used in the place of Z_n. Likewise $\langle Z_n \cup g\rangle$ or $C(Z_n)$ or $\langle Z_n \cup h\rangle$ or $\langle Z_n \cup k\rangle$ can be used in the place Z_n and realized fixed point and realized limit cycle are found using the MOD matrix as a operator from row matrix collection to itself.

Let M be a $n \times n$ matrix with entries from Z_n (or $\langle Z_n \cup I\rangle$ or $\langle Z_n \cup g\rangle$ or $\langle Z_n \cup h\rangle$ or $\langle Z_n \cup k\rangle$ or $C(Z_n)$). M is called MOD matrix operator and it acts from $B = \{(a_1, \ldots, a_n) \mid a_i \in Z_n; 1 \leq i \leq n\}$ to itself.

M can fix elements of B leading to classical fixed points.

If xM after several iterations takes value y and yM = y then $y \in B$ will be defined as the realized fixed point. It may so happen xM gives y_i and then y_{i+1} and so on once again the y_i after acting on M at each stage.

Then this y_i will be defined as the realized limit cycle. The applications of the operators to mathematical modeling will be given in the forthcoming books.

For the notions of neutrosophic modulo integer $\langle Z_n \cup I\rangle$; $I^2 = I$ refer [3, 4]. For the dual numbers and modulo dual numbers $\langle Z_n \cup g\rangle = \{a + bg \mid g^2 = 0, a, b \in Z_n\}$ refer [12]. For finite complex modulo integers and their properties refer [11].

For special dual like modulo numbers $\langle Z_n \cup h\rangle = \{a, bh \mid a, b \in Z_n; h^2 = h\}$ refer [13].

Finally for the concept of special quasi dual modulo integers $\langle Z_n \cup k\rangle = \{a + bk / a, b \in Z_n, k^2 = (n - 1) k\}$ refer [14].

For MOD structures and their properties refer [21-7].

For thresholding and updating of state vector refer [5].

Chapter Two

MOD-FIXED POINT THEORY

In this chapter we for the first time introduce the notion of MOD-fixed points of MOD-functions [21]. There are several such MOD-functions and the fixed points in those cases are periodically fixed.

This situation will be first represented by examples first and then will be defined.

Example 2.1: Let Z be the integers (both positive and negative) and Z_5 modulo integers.

Define a MOD-function $f : Z \to Z_5$ is as follows:

$f(0) = 0, \quad f(1) = 1 = f(-4)$
$f(2) = 2 = f(-3), f(3) = 3 = f(-2)$
$f(4) = 4 = f(-1), f(\pm 5) = 0$
$f(n5) = 0; \; n = \pm 1, \ldots, \infty$
$f(5n+1) = 1; n = \pm 1, \ldots, \infty$
$f(5n - 1) = 4, f(5n + 2) = 2$
$f(5 n - 2) = 3, \; f(5n + 3) = 3$
$f(5n - 3) = 2; n = \pm 1, \ldots, \infty.$

f is MOD-fixed point function for f fixes 0, 1, 2, 3 and 4.

Example 2.2: Let f: $Z \rightarrow Z_{192}$, f is a MOD-function the MOD-fixed points of f are 0, 1, 2, …, 191.

In view of all these we give the formal definition.

DEFINITION 2.1: *Let Z be the set of positive and negative integers with zero.*

Z_n the integers modulo n.

Define $f : Z \rightarrow Z_n$ by $f(x) = x; \quad 0 \leq x \leq n - 1;$
$f(nt + x) = x;\ 1 \leq t < \infty$
$f(nt - x) = n - x;\ 1 \leq x \leq n - 1.$

Then f is the MOD-function and all elements {0, 1, 2, …, n – 1} of Z are fixed points of f.

This MOD-function behaves is the classical way and the fixed points are also defined in the same way as that of classical one.

Thus we have MOD-functions contributing to finite number of fixed points.

Example 2.3: Let Z_{18} be the modulo integers mod 18 and $f : Z \rightarrow Z_{18}$ be the MOD-function defined by
$f(x) = x;\ 0 \leq x \leq 17.$

$$f(18) = 0$$
$$f(18n + x) = x; \quad 0 \leq x \leq 17$$
$$f(18n - x) = 18 - x \text{ for } \quad 0 \leq x \leq 17;\ n \in Z.$$

Clearly this mod function f fixes the elements 0, 1, 2, …, 17.

Thus the elements of Z_{18} are fixed points of the MOD-function f.

In view of this we prove the following theorem.

THEOREM 2.1: *Let Z be the integers. Z_n be the modulo integers. $f : Z \to Z_n$ be the MOD-function from Z to Z_n. f has {0, 1, 2, .., n – 1} to be the fixed points.*

Proof: Follows from the fact f(x) = x for all x ∈ {0, 1, 2, …, n – 1}. Hence the theorem.

In view of this we can say the MOD function $f : Z \to Z_n$ has n and only n fixed points including 0.

Next our natural questions would be can we have MOD-functions which can have finite number of fixed points or more than n fixed points. The answer is yes.

To this effect some examples are provided.

Example 2.4: Let

$$M = \left\{\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} \text{ where } a, b, c, d, e, f, g, h, i \in Z\right\}$$

be the collection of 3 × 3 matrices.

$$N = \left\{\begin{pmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \end{pmatrix} \mid a_i \in Z_{12}; 1 \leq i \leq 9\right\}$$

be the collection of 3 × 3 matrices with entries from Z_{12}.

Define a function

$$f : M \to N$$

$f(A = (a_{ij})) = (a_{ij})$ if $a_{ij} \in Z_{12}$

$f((a_{ij})) = 12 – a_{ij}$ if a_{ij} is negative and $–12 \leq a_{ij} \leq 0$

$$f((a_{ij})) = f(12\,n + t) \;= t$$
$$f((a_{ij})) = f((12n - t)) = 12 - t.$$

Then f is defined as the MOD-matrix function

f(A) = A if entries of A takes values from $\{0, 1, 2, \ldots, 11\}$.

Thus all elements of $N \subseteq M$ are fixed points are fixed matrices of this MOD-matrix function.

We will illustrate this by some more examples.

Example 2.5: Let

$$M = \left\{ \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \end{bmatrix} \;\middle|\; a_i \in Z; 1 \leq i \leq 5 \right\}$$

be a column matrix with entries from Z.

$$N = \left\{ \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \end{bmatrix} \;\middle|\; a_i \in Z_{10}; 1 \leq i \leq 5 \right\}$$

be the column matrices with entries from Z_{10}.

Define $f : M \to N$ the MOD-matrix function

$$f\left(\begin{bmatrix} 9 \\ 12 \\ 19 \\ -7 \\ 5 \end{bmatrix}\right) = \begin{bmatrix} 9 \\ 2 \\ 9 \\ 3 \\ 5 \end{bmatrix} \in N,\ f\left(\begin{bmatrix} -3 \\ 2 \\ 4 \\ -5 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 7 \\ 2 \\ 4 \\ 5 \\ 0 \end{bmatrix} \in N.$$

$$f\left(\begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 6 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 6 \end{bmatrix},\ f\left(\begin{bmatrix} 3 \\ 3 \\ 3 \\ 3 \\ 3 \end{bmatrix}\right) = \begin{bmatrix} 3 \\ 3 \\ 3 \\ 3 \\ 3 \end{bmatrix} \text{ and so on.}$$

Thus there are several matrices which are kept fixed by the MOD-matrix function.

This is the way MOD-matrix functions are defined and they have certainly a finite number of fixed points but the number of such matrices are greater than 10 in this case.

Example 2.6: Let

$$M = \left\{\begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \end{bmatrix} \mid a_i \in Z;\ 1 \leq i \leq 6\right\}$$

be the collection of 2×3 matrices with entries from Z.

$$N = \left\{\begin{pmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \end{pmatrix} \mid a_i \in Z_{23};\ 1 \leq i \leq 6\right\}$$

be the collection of all 2×3 matrices with entries from Z_{23}.

Define $f : M \to N$ the MOD-matrix function f has several fixed points (matrices).

For instance take

$$A = \begin{pmatrix} 5 & 6 & 8 \\ 9 & 10 & 12 \end{pmatrix} \in M$$

f(A) = A $\in$ N is a fixed point matrix of M.

$$\text{Take } B = \begin{pmatrix} 7 & 8 & 0 \\ 1 & 2 & 22 \end{pmatrix} \in M$$

f(B) = B $\in$ N is again a fixed point (matrix) of M.

Thus M has several fixed points.

Infact all matrices of N which is a subset of M happens to be fixed under the MOD-matrix function f.

$$f\left(\begin{bmatrix} 27 & -3 & 4 \\ -8 & 40 & 12 \\ 0 & -7 & -10 \end{bmatrix}\right) = \begin{bmatrix} 4 & 20 & 4 \\ 15 & 17 & 12 \\ 0 & 16 & 13 \end{bmatrix} \in N.$$

Thus there are matrix in M which are not fixed by N.

In view of all these we prove the follow theorem.

THEOREM 2.2: *Let M = {m $\times$ n matrices with entries from Z} and N = {m $\times$ m matrices with entries from Z_s}.*

Let f : M $\to$ N be the MOD function defined from M to N.

The fixed points (matrices) of the MOD-function f are A = {$(a_{ij})_{m\times n}$ | 0 $\leq a_{ij} \leq$ s – 1}.

Proof: Follows from the fact all elements in $x \in A$ are such that $f(x) = x$. Hence the claim, the MOD matrix function has fixed points.

Next we consider the polynomials in $Z[x]$ and $Z_n[x]$; $2 \leq n < \infty$.

$$Z[x] = \left\{\sum_{i=0}^{\infty} a_o x^i \mid a_i \in Z\right\}$$

be the collection of all polynomials in the variable x with coefficients from Z.

$$Z_n[x] = \left\{\sum_{i=0}^{\infty} a_i x^i \mid a_i \in Z_n\right\}$$

be the collection of all polynomials in x with coefficients from Z_n.

Define $f : Z[x] \to Z_n[x]$;

$f(x) = x$, $f(p(x) = p(x)$;
if $p(x) \in Z_n [x]$, that is all coefficients of p(x) lie in Z_n.

$f(p(x) = \Sigma a_i x^i) = \Sigma f(a_i)\, x^i$; $f(a_i)$ is defined as in case of MOD functions.

$f : Z[x] \to Z_n[x]$ is defined as the MOD-polynomial function. This MOD-polynomial function has infinite number of fixed points.

Let n = 15, $Z_{15}[x]$ be the polynomials with coefficients from Z_{15}.

Let $p(x)= 45x^{10} + 25\, x^8 + 8x^3 + 62x^2 + 75x + 20 \in Z[x]$
$f(p(x)) = 10x^8 + 8x^3 + 2x^2 + 5 \in Z_n[x]$.

Let $q(x) = 3x^3 + 7x^2 + 14x + 12 \in Z_{15}[x]$
$f(q(x)) = 3x^3 + 7x^2 + 14x + 12$;

Thus q(x) is a fixed point.

We have infinitely many fixed points for this MOD-polynomial functions.

f is called the MOD-polynomial function these functions has infinitely many fixed points.

Example 2.7: Let $f : Z[x] \to Z_9[x]$ be the MOD-polynomial function.

Let $p(x) = 9x^{21} + 21x^{17} + 14x^{15} + 29x^7 + 40x^5 + 10x^3 + 16x + 21 \in Z[x]$.

$f(p(x)) = 0 + 3x^{17} + 5x^{15} + 2x^7 + 4x^5 + x^3 + 7x + 0 \in Z_9[x]$.

Thus this MOD-polynomial function f has infinitely many fixed points (polynomials).

Thus examples of these are given.

Let $f : Z[x] \to Z_3[x]$ be the MOD-polynomial function.

For $p(x) = 7x^5 + 10x^3 - 15x^2 + 5x - 10 \in Z[x]$.

$f(p(x)) = x^5 + x^3 + 2x + 2 \in Z_3[x]$.

Let $p_1(x) = 2x^3 + x^2 + 2x + 2 \in Z[x]$;

$f(p_1(x)) = 2x^3 + x^2 + 2x + 2$.

This is the way MOD-polynomial functions. This $p_1(x)$ is a fixed polynomial of Z[x] so Z[x] has infinitely many fixed (polynomials) points.

Next we proceed onto study MOD interval function from reals R to [0, m); $2 \leq m < \infty$.

We will give first examples of them.

Example 2.8: Let f : R → [0,12) be a function defined as follows.

$f(x) = x$ if $0 \leq x < 11.999$;
$f(\pm 12) = 0$.
$f(12n + x) = x$; $n = \pm 1, \pm 2, \ldots$ $0 \leq x \leq 11.9999$
$f(7.3201) = 7.3201$
$f(18.30125) = 6.30125 + (-7.512) = 4.488$ and so on.
$f(-40.003) = 7.997$.

Thus f has infinitely many fixed points. All x such that $0 \leq x < 11.999\ldots9$ are such that $f(x) = x$.

These are known as MOD function fixed points of intervals.

Example 2.9: Let f : R → [0,11) be a function defined as follows $f(x) = x$ if $0 \leq x \leq 10.999\ldots$ $f(12.0013) = 1.0013$.

$f(-12.0013) = 9.9987$
$f(-2.092) = f(8.908)$ and so on.

This is the way the MOD interval function is defined and this has infinitely many fixed points.

Let us give one more example before we proceed onto derive some properties associated with A.

Example 2.10: Let f : R → [0,118) be the MOD-interval function defined by $f(x) = x$ if $0 \leq x < 117.9999$.

$f(-106.007) = 11.993$ and so on.

Infact there are infinitely many fixed points.

In view of this we have the following theorem the proof of which is left as an exercise to the reader.

THEOREM 2.3: *Let $f : R \to [0,m)$ be the MOD-interval function f has infinitely many fixed points. Infact the interval $[0,m) \subseteq (-\infty, \infty) = R$ are fixed points of f.*

Next we proceed onto define the notion of infinite number of MOD-interval matrix fixed points.

DEFINITION 2.2: *Let M = {$p \times q$ matrices with entries from R} and N = {$p \times q$ matrices with entries from [0,m)}; $2 \leq m < \infty$. Define $f : M \to N$ by $f((a_{ij})) = (a_{ij})$ if $a_{ij} \in [0,m)$ otherwise define function f for each entries in the matrix as that of MOD interval functions.*

Then $f : M \to N$ is defined as the MOD-matrix interval functions.

First we will illustrate this situation by some examples.

Example 2.11: Let

$$M = \left\{ \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \\ a_5 & a_6 \\ a_7 & a_8 \end{bmatrix} \middle| a_i \in R; 1 \leq i \leq 8 \right\}$$

and

$$N = \left\{ \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \\ a_5 & a_6 \\ a_7 & a_8 \end{bmatrix} \middle| a_i \in [0,17); 1 \leq i \leq 8 \right\}$$

be the collection of real and interval 4×2 matrices with entries from R and [0,17) respectively.

Define f: M → N be the MOD-interval matrix.

$$f\left(\begin{bmatrix} 3.1 & 2.7 \\ 1.5 & 0.6 \\ 0.17 & 1.2 \\ 1.7 & 5.1 \end{bmatrix}\right) = \begin{bmatrix} 3.1 & 2.7 \\ 1.5 & 0.6 \\ 0.17 & 1.2 \\ 1.7 & 5.1 \end{bmatrix},$$

$$f\left(\begin{bmatrix} -7 & -8.31 \\ 3.11 & -4.25 \\ 0.33 & -0.67 \\ 1.32 & 6.3 \end{bmatrix}\right) = \begin{bmatrix} 10 & 8.69 \\ 3.11 & 12.75 \\ 0.33 & 16.33 \\ 1.32 & 6.3 \end{bmatrix} \text{ and so on.}$$

f has fixed points which are infinite in number.

Infact N ⊆ N and f(N) = N.

Example 2.12: Let

$$M = \left\{\begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix} \mid a_i \in R;\ 1 \leq i \leq 4\right\}$$

and

$$N = \left\{\begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix} \mid a_i \in [0,44)\ 1 \leq i \leq 4\right\}$$

be the 2 × 2 matrices.

Define f : M → N the MOD-interval matrix function. Clearly f(N) = N.

N considered as a subset of M is an infinite collection of MOD-interval matrix function fixed points.

Next the study of MOD-interval polynomial functions will be described by examples.

Example 2.13: Let

$$M = \left\{ \sum_{i=0}^{\infty} a_i x^i \mid a_i \in R \right\}$$

and

$$N = \left\{ \sum_{i=0}^{\infty} a_i x^i \mid a_i \in [0, 24) \right\}$$

be two real polynomial and MOD-interval polynomials respectively.

A map $f : M \to N$ defined by $f(\Sigma a_i x^i) = \Sigma a_i x^i$ if we have $a_i \in [0,24)$ and $f(\Sigma a_i x^i) = \Sigma f(a_i) x^i$ where $f(a_i)$ is defined as that of MOD interval functions from $R \to [0,24)$.

Let $p(x) = 3.8\, x^8 + 24x^6 + 2.42\, x^4 + 0.762\, x^2 + 27.31 \in R[x]$;

$f(p(x)) = 3.8\, x^8 + 2.42\, x^4 + 0.762\, x^2 + 3.31$.

Let $g(x) = 64\, x^{10} + 48x^9 + 24.007\, x^5 + 3.74\, x^4 - 6.31\, x^3 + 10.31\, x^2 + 4x - 27.3 \in R[x]$.

$f(g(x)) = 16x^{10} + 0.007x^5 + 3.74x^4 + 17.69x^3 + 10.31x^2 + 4x + 21.7 \in N$.

Thus f the MOD interval polynomial function has infinite number of fixed points (polynomials).

Further it is to be noted as a set $N \subseteq M$; N is a proper subset of M and N is of infinite cardinality and $f(N) = N$; so f is a MOD-interval polynomial function which has infinitely many polynomials which are fixed points.

In view of this we have the following theorem.

THEOREM 2.4: *Let*

$$M = \left\{\sum_{i=0}^{\infty} a_i x^i \mid a_i \in R\right\}$$

be the collection of all polynomials with real coefficients.

$$N = \left\{\sum_{i=0}^{\infty} a_i x^i \mid a_i \in [0,m);\ 2 \leq m < \infty\right\}$$

be the MOD-interval polynomials with coefficients from [0,m).

The MOD-interval polynomial function f: M → N fixes infinitely many points.

The fixed points of f are N that is f(N) = N as N ⊆ M.

Proof is direct and hence left as an exercise to the reader.

Next fixed point MOD function on $\underbrace{Z \times Z \times \ldots \times Z}_{\text{n times}}$ to $\underbrace{Z_m \times Z_m \times \cdots \times Z_m}_{\text{m−times}}$ will be discussed by examples.

Example 2.14: Let

$$V = \{Z \times Z \times Z = \{(a, b, c) \mid a, b, c \in Z\}\}$$

be the triple product of integers.

Let $W = \{Z_7 \times Z_7 \times Z_7 = \{(a, b, c) \mid a, b, c, \in Z_7\}\}$ be the triple product of modulo integers.

f : V → W be the MOD-function defined by f(a, b, c) = (x_1, x_2, x_3) where $x_1 = a$, $x_2 = b$ and $x_3 = c$ if $0 \leq a, b, c \leq 6$.

f(a, b, c) = (x_1, x_2, x_3) where if $a \geq 7$ then

$$\begin{aligned} a &= 7t + x_1 \text{ where } 0 \leq x_1 \leq 6; \\ \text{if } a \leq 7 \text{ then } a &= 7t + y \\ &= 7 - y \\ &= x_1. \end{aligned}$$

Similar working for b and c.

We see if x = (8.3, –7.5, 5.31) ∈ V then

f(x) = f((8.3, –7.5, 5.31)) = (1.3, 6.5, 5.31) ∈ W.

Let y = (3.331, 4.44, 6.302) ∈ V;

$$\begin{aligned} \text{we see } f(y) &= f((3.331, 4.44, 6.302)) \\ &= (3.331, 4.44, 6.302) \\ &= y \in W. \end{aligned}$$

Clearly as $W \not\subseteq V$ are see f(W) = W is the collection of all fixed points of V, by the MOD function f which is only a finite collection.

Example 2.15: Let

$$V = \{Z \times Z \times Z \times Z \times Z) = \{(a_1, a_2, a_3, a_4, a_5) \mid a_i \in Z; \; 1 \leq i \leq 5\}\}$$

be the 5-tuple product of integers.

$W = \{(Z_9 \times Z_9 \times Z_9 \times Z_9 \times Z_9) = \{(x_1, x_2, x_3, x_4, x_5) \mid x_i \in Z_9; 1 \leq i \leq 9\}\}$ be the 5-tuples of Z_9 the modulo integers.

Define f : V → W to the MOD function.

Then associated with x are only 8^3 fixed points and nothing more.

Let x = (9, 8.3, 10.3) be in V.

$$\begin{aligned} f(x) &= f((9, 8.3, 10.3) \\ &= (0, 8.3, 1.3) \in W. \end{aligned}$$

So x is not fixed by f.

Let $x_1 = (2.3, 0, 5.2) \in V$

$$\begin{aligned} f(x_1) &= f((2.3, 0, 5.2)) \\ &= (2.3, 0, 5.2) = x_1 \in W. \end{aligned}$$

Thus this x_1 is fixed by f.

Let $x_2 = (-3.7, -22.5, -17.2) \in V$.

$f(x_2) = f((-3.7, -22.5, -17.2)) = (5.3, 4.5, 0.8) \in W$.

Thus x_2 is not a fixed point of f.

In view of all these we have the following theorem.

THEOREM 2.5: *Let $V = (Z \times Z \times \ldots \times Z) = \{(a_1, a_2, \ldots, a_n) \mid a_i \in Z;\ 1 \leq i \leq n\}$ and $W = (Z_m \times Z_m \times \ldots \times Z_m) = \{(x_1, x_2, \ldots, x_n)$ where $x_i \in Z_m;\ 1 \leq i \leq n\}$ be the n tuples of real and modulo integers respectively.*

Let $f : V \to W$ be a MOD function defined from V to W.

f fixes only m^n points in V and no more.

Proof is direct and hence left as an exercise to the reader.

Now we give some more examples to this effect.

Example 2.16: Let V = Z × Z × Z × Z × Z = $\{(a_1, a_2, a_3, a_4, a_5) \mid a_i \in Z; 1 \leq i \leq 5\}$ and W = $Z_3 \times Z_8 \times Z_5 \times Z_{12} \times Z_9 = \{(d_1, d_2, d_3, d_4, d_5) \mid d_1 \in Z_3, d_2 \in Z_8, d_3 \in Z_5, d_4 \in Z_{12}$ and $d_5 \in Z_9\}$ be the 5 tuple reals and 5-tuple mixed modulo integers.

Let f : V → W be the MOD function.

Consider x = (5.3, 47.2, 9.89, 12.83, 14.67) ∈ V

$$\begin{aligned} f(x) &= f((5.3, 47.2, 9.89, 12.83, 14.67) \\ &= (2.3 \pmod 3), 1.2 \pmod 8, \\ &\quad 4.89 \pmod 5, 0.83 \pmod{12}, 5.67 \pmod 9)) \\ &= (2.3, 1.2, 4.89, 0.83, 5.67) \in W. \end{aligned}$$

This is the very special way by which the MOD function is defined.

Let y = (–7.3, –10.52, –4.8, –15.72, –10.8) ∈ V;

$$\begin{aligned} \text{now } f(y) &= f((-7.3, -10.52, -4.8, -15.72, -10.8)) \\ &= (-1.7 \pmod 3)\ 5.48 \pmod 8, 0.2 \pmod 5, \\ &\quad 8.38 \pmod{12}, 7.2 \pmod 9)) \\ &= (1.7, 5.48, 0.2, 8.38, 7.2) \in V. \end{aligned}$$

Thus if entries are negative the MOD function f : V → W is defined.

Next consider the element s = (1.2, 7.2, 4.5, 10.35, 6.331) ∈ V.

f(s) = f((1.2, 7.2, 4.5, 10.35, 6.331)) = (1.2, 7.2, 4.5, 10.35, 6.331) = s ∈ W.

Thus s is a fixed point of V fixed by the MOD function f.

Infact f fixes exactly 3.8.5.12.9 = 12, 960 number of elements in V.

This is the way MOD functions on mixed modulo product is defined.

We will illustrate this situation by one more example.

Example 2.17: Let $V = Z \times Z \times Z \times Z \times Z \times Z \times Z = \{(a_1, a_2, a_3, a_4, a_5, a_6, a_7) \mid a_i \in Z; 1 \leq i \leq 7\}$ be the 7-tuple of integers.

$W = (Z_{10} \times Z_6 \times Z_{13} \times Z_2 \times Z_{16} \times Z_6 \times Z_2) = \{d_1, d_2, d_3, d_4, d_5, d_6, d_7) \mid d_1 \in Z_{10}, d_7, d_2 \in Z_6, d_3 \in Z_{13}, d_4, d_7 \in Z_2, d_5 \in Z_{16}\}$ be the 7-tuple of mod integers.

$f : V \to W$; be the MOD function defined on V.

Let $(12.3, 9.6, 16.1, 6.332, 19.31, 8.312, 5.1102) \in V$.

$f((12.3, 9.6, 16.1, 6.332, 19.31, 8.312, 5.1102))$
$= (2.3 \text{ (mod 10)}, 3.6 \text{ (mod 6)}, 3.1 \text{ (mod 13)},$
$0.332 \text{ (mod 2)}, 3.31 \text{ (mod 16)}, 2.312 \text{ (mod 6)},$
$1.1102 \text{ (mod 2)} \in W.$
$= (2.3, 36, 3.1, 0.332, 3.31, 2.312, 1.1102) \in W$

Consider $x = (-10.3, -4.2, -7.5, -5.3, -0.3, -6.3, -7.6) \in V$

$f(x) = f((-10.3, -4.2, -7.5, -5.3, -0.3, -6.3, -7.6))$

$= (9.7 \text{ (mod 10)}, 1.8 \text{ (mod 6)}, 5.5 \text{ (mod 13)}, 0.7 \text{ (mod 2)},$
$15.7 \text{ (mod 16)}, 5 \text{ (mod 6)}, 0.4 \text{ (mod 2)})$

$= (9.7, 1.8, 5.5, 0.7, 15.7, 5.7, 0.4) \in W.$

This is the way MOD function f is defined.

Clearly the MOD function of fixes
$10 \times 6 \times 13 \times 2 \times 16 \times 6 \times 2 = 299520$ as fixed points of W.

This type of MOD function fixes only finite number of point or fixed points associated with f are 299520.

In view of all these we have the following theorem.

THEOREM 2.6: *Let $V = Z \times Z \times \ldots \times Z = \{(a_1, a_2, \ldots, a_n) \mid a_i \in Z\}$ $1 \leq i \leq n\}$ and*

$W = Z_{m_1} \times Z_{m_2} \times \ldots \times Z_{m_n} = \{(x_1 \ldots x_n) ; x_i \in Z_{m_i}$ $1 \leq i \leq n$ *and m_i's are finite positive integers\} be n-tuple of integers and modulo integers respectively.*

$f : V \to W$ be the MOD function from V to W.

f fixes exactly $m_1 \times m_2 \times \ldots \times m_n$ number of points in V.

The proof is direct and hence left as an exercise to the reader.

Next we study MOD-functions from p × q matrix collection from reals to p × q matrices with entries from modulo integers.

We will first describe this by an example or two.

Example 2.18: Let

$$M = \left\{\begin{bmatrix} a & b \\ c & d \end{bmatrix} \mid a,\ b, c, d \in Z\right\}$$

and

$$N = \left\{\begin{bmatrix} x & y \\ z & w \end{bmatrix} \mid x \in Z_{15}, y \in Z_6, z \in Z_3, w \in Z_7\right\}$$

be the collection of 2 × 2 matrices with entries from Z and MOD integers respectively.

Let f : M → N this new type of MOD function is defined as follows, which is only described by the example.

$$f\left(\begin{bmatrix} 17.3 & -0.7 \\ 4.5 & -10.3 \end{bmatrix}\right) = \begin{bmatrix} 2.3(\text{mod}\,15) & 5.3(\text{mod}\,6) \\ 1.5(\text{mod}\,3) & 4.7(\text{mod}\,7) \end{bmatrix}$$

$$= \begin{bmatrix} 2.3 & 5.3 \\ 1.5 & 4.7 \end{bmatrix} \in N$$

This is the way the MOD function acts on M.

Now we give some fixed points of M.

$$f\left(\begin{bmatrix} 0.38 & 0.46 \\ 1.12 & 1.07 \end{bmatrix}\right) = \begin{bmatrix} 0.38 & 0.46 \\ 1.12 & 1.07 \end{bmatrix}$$

is a fixed point of f; the MOD function.

Clearly there are $15 \times 6 \times 3 \times 7 = 1890$ number of fixed elements in M.

So we have different types of MOD functions from integer matrices to different MOD integer matrices.

Example 2.19: Let

$$M = \left\{\begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \\ a_{10} & a_{11} & a_{12} \\ a_{13} & a_{14} & a_{15} \end{bmatrix} \middle| a_i \in Z; 1 \leq i \leq 15\right\}$$

and

$$N = \left\{\begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \\ a_{10} & a_{11} & a_{12} \\ a_{13} & a_{14} & a_{15} \end{bmatrix} \middle| \; a_1 \in Z_7,\, a_2 \in Z_9,\, a_3 \in Z_{12},\right.$$

$$\left. a_4 \in Z_{10},\, a_5,\, a_6 \in Z_4,\, a_7 \in Z_3,\, a_9,\, a_8 \in Z_5,\, a_{10},\, a_{11} \in Z_2,\, a_{12},\, a_{13} \in Z_{11},\, a_{14},\, a_{15} \in Z_{15}\right\}$$

be the collection of integer matrices and mod integer matrices.

Let $f : M \to N$ is defined as follows

$$f\left(\begin{bmatrix} -3.7 & 10.3 & -3.4 \\ -4.3 & 6.5 & 7.3 \\ 4.2 & 9.7 & -1.3 \\ 4.3 & 3.1 & 13.7 \\ -0.3 & 16.3 & 18.3 \end{bmatrix}\right)$$

$$= \begin{bmatrix} 3.3(\text{mod}\,7) & 1.3(\text{mod}\,9) & 8.6(\text{mod}\,12) \\ 5.7(\text{mod}\,10) & 2.5(\text{mod}\,4) & 3.3(\text{mod}\,4) \\ 1.2(\text{mod}\,3) & 4.7(\text{mod}\,5) & 3.7(\text{mod}\,5) \\ 0.3(\text{mod}\,2) & 1.1(\text{mod}\,2) & 2.7(\text{mod}\,11) \\ 10.7(\text{mod}\,11) & 1.3(\text{mod}\,15) & 3.3(\text{mod}\,15) \end{bmatrix}.$$

This is the way mod function is performed.

Clearly this has several fixed points.

However the number of fixed points are only finite given by 7.9.12.10.4.4.3 5.5 2.2 11.11. 15.15 = 987940800000.

Thus there are many fixed points, but are only finite in number.

This is expressed by the following theorem.

THEOREM 2.7: *Let M = {(collection of all p × q matrices with entries from Z} and N = {(collection of all p × q matrices with entries from* $\{Z_{m_1}, Z_{m_2}, Z_{m_3}, \ldots, Z_{m_{p\times q}}\}$ *be the collection of all p × q matrices with entries from integer Z and from mod integers from* $Z_{m_1}, \ldots, Z_{m_{p\times q}}$.

Let f : M → N defined by $f((a_{ij})) = (b_{ij})$

$f(a_{ij}) = (b_{ij}\ (mod\ Z_{m_{i\times j}}))$.

f fixes $m_1 \times m_2 \times \ldots \times m_{p\times q}$ *number of elements.*

Proof is direct and hence, left as an exercise to the reader.

Now we give examples of function with infinite MOD function.

Example 2.20: Let M = {(R × R × R) = (a, b, c); a, b c ∈ R} and N = {[0, 9) × [0, 9) × [0, 90) = {(x_1, x_2, x_3) / x_1 ∈ [0,19), x_2 ∈ [0,9) x_3 ∈ [0,90); 1 ≤ i ≤ 3} be the real 3-tuple and 3-tuple MOD intervals.

Let f : M → N be defined

f((23.001, 7.02, 110.314)) = (4.001, 7.02, 20.314) ∈ N where (23.001, 7.02, 110.314) ∈ M.

Let x = (–0.72, –14.004, 16.003) ∈ M;

f(x) = f((–0.72, –14.004, 16.003))

= (18.28, 3.996, 16.003) ∈ N.

Let y = (2.003, 4.556, 7.006) ∈ M;

$f(y) = f((2.003, 4.556, 7.006)) = (2.003, 4.556, 7.006) = y$ is a fixed point.

Since $N \subseteq M$ we see every element of N is a fixed point and the MOD function f has infinitely many fixed points.

However as algebraic structure both N and M are very distinct, one can see N becomes a proper subset of M.

Example 2.21: Let $M = R \times R \times R \times R \times R \times R = \{(a_1, a_2, a_3, a_4, a_5, a_6) / a_i \in R; 1 \leq i \leq 6\}$ be the 6-tuple of reals.

$N = \{([0,15) \times [0,51) \times [0, 25) \times [0,15) \times [0,5) \times [0,50)) = \{(x_1, x_2, x_3, x_4, x_5, x_6) / x_1, x_4 \in [0,15), x_2 \in [0,51), x_3 \in [0,25)$ $x_4 \in [0, 15), x_5 \in [0,5), x_6 \in [0,50)\}$; be the 6-tuple of MOD interval.

Clearly N is a subset of M.

$$\text{Let } f : M \to N$$

$f((7.02, 0.52, -3.26, 9.87, -4.27, 10.34))$

$$= (7.02, 0.52, 21.74, 9.87, 0.73, 10.34) \in N.$$

This is the way MOD function f is defined

$f(3.111, 2.555, 0.748, 1.041, 4.033, 0.142)$

$$= (3.111, 2.555, 0.748, 1.041, 4.033, 0.142) \in N.$$

Thus this point is a fixed point.

Thus the MOD function f has infinitely many fixed points.

In view of all this we have proved the following theorem.

THEOREM 2.8: *Let $M = \{R \times R \times \ldots \times R = \{(a_1, a_2, \ldots, a_n) / a_i \in R; 1 \leq i \leq n\}\}$ be the n-tuple of reals $N = \{[0,m_1) \times [0,m_2) \times$*

… $\times$ [0,m_n) = {(x_1, x_2, …, x_n) / $x_i \in$ [0,m_i); $1 \leq i \leq n$} be the n-tuple of mod intervals.

Let f : M $\rightarrow$ N is the MOD function. Mod f has infinite number of fixed elements (or fixed n-tuples).

Proof follows from the very definition of MOD functions.

Thus infinite number of points are fixed by the MOD function.

Next the infinite number of fixed points given by the MOD function using matrices is given by some example.

Example 2.22: Let

$$M = \left\{ \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \end{bmatrix} \mid a_i \in R;\ 1 \leq i \leq 5 \right\}$$

and

$$N = \left\{ \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \end{bmatrix} \mid a_2 \in [0,3),\ a_4 \in [0,2),\ a_5 \in [0,10); \right.$$

$$\left. a_3, a_1 \in [0,12);\ 1 \leq i \leq 5 \right\}$$

are 5 × 1 matrices with reals and the MOD interval [0,3) [0,2) [0,10) and [0,12) respectively.

Define f : M $\rightarrow$ N the MOD function as follows.

$$f\left(\begin{bmatrix} -3.71 \\ 13.05 \\ 4.25 \\ -0.75 \\ -14.08 \end{bmatrix}\right) = \begin{bmatrix} 8.29 \\ 1.05 \\ 4.25 \\ 1.25 \\ 5.92 \end{bmatrix}.$$

Infact we have infinite collection of matrices which are kept fixed by the MOD function f.

$$\text{Let } x = \begin{bmatrix} 0.3 \\ 0.2 \\ 1.2 \\ 0.9 \\ 5.7 \end{bmatrix} \in M;\ f(x) = f\left(\begin{bmatrix} 0.3 \\ 0.2 \\ 1.2 \\ 0.9 \\ 5.7 \end{bmatrix}\right) = \begin{bmatrix} 0.3 \\ 0.2 \\ 1.2 \\ 0.9 \\ 5.7 \end{bmatrix} \in N.$$

This x is a fixed point. Infact $N \subseteq M$, and N a subset of M and every element in N are fixed by f.

That is f(N) = N.

Hence the MOD function f fixed infinitely many points of M.

Example 2.23: Let

$$M = \left\{\begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \end{bmatrix} \mid a_i \in R; 1 \le i \le 9\right\}$$

and

$$N = \left\{\begin{bmatrix} x_1 & x_2 & x_3 \\ x_4 & x_5 & x_6 \\ x_7 & x_8 & x_9 \end{bmatrix} \middle| \; x_1, x_4 \in [0,7)\right.$$

$$\left. x_2, x_3, x_5 \in [0,17) \; x_6, x_7, x_8, x_9 \in [0,13); 1 \leq i \leq 9\right\}$$

be 3×3 real matrices and MOD interval matrices.

We have a MOD function

$f : M \to N$

$$f\left(\begin{bmatrix} 19.3 & 3.3 & 1.7 \\ 1.2 & -1.1 & 9.2 \\ 10.8 & 10.7 & 6.9 \end{bmatrix}\right) = \begin{bmatrix} 5.3 & 3.3 & 1.7 \\ 1.2 & 15.9 & 9.2 \\ 10.8 & 10.9 & 6.9 \end{bmatrix}$$

$$f\left(\begin{bmatrix} 0.3 & 6.3 & 1.1 \\ 6.9 & 3.9 & 4.8 \\ 7.2 & 8.9 & 9.1 \end{bmatrix}\right) = \begin{bmatrix} 0.3 & 6.3 & 1.1 \\ 6.9 & 3.9 & 4.8 \\ 7.2 & 8.9 & 9.1 \end{bmatrix}$$

is a fixed point of M.

Infact MOD function f has infinite number of fixed points.

Example 2.24: Let

$$M = \left\{\begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \\ \vdots & \vdots \\ a_{11} & a_{12} \end{bmatrix} \text{where } a_i \in R; 1 \leq i \leq 12\right\}$$

$$N = \left\{ \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \\ \vdots & \vdots \\ a_{11} & a_{12} \end{bmatrix} \right| a_1, a_2, a_4, a_3 \in [0,3), a_5 \in [0,6);$$

$$a_6 \text{ to } a_{10} \in [0,9), a_{11}, a_{12} \in [0,8); 1 \leq i \leq 12\}$$

be 6×2 matrices with entries from reals R and MOD-interval [0, 6).

We see $N \subseteq M$ is a proper subset of M.

Clearly if $f : M \to N$ is defined as that of a MOD function then every $A \in N \subseteq M$ is such that $f(A) = A$.

Thus f has infinite number of fixed points.

In view of all these we have the following theorem.

THEOREM 2.9: *Let M = {(collection of all $p \times q$ matrices with entries from the reals} and N = {collection of all $p \times q$ matrices with entries from $[0, m_1)$, $[0, m_2)$, …, $[0, m_{p\times q})$}; f: $M \to N$ is a MOD interval matrix function which fixes infinitely many points. That is f has infinitely many fixed points.*

Proof is direct and hence left as an exercise to the reader.

Next we proceed define a special type of function which does not follow the laws of function.

According to the definition if f is a map from a non empty set X to another nonempty set Y then we have any $x \in X$ has only one y associated with it in Y many x in X may be mapped onto the same y in Y.

However x in X cannot be mapped on to y_1 and y_2 in Y where $y_1 \neq y_2$.

However we have infinitely many functions f from X to Y such that one element x in X is mapped onto infinitely many elements in Y.

Such function we call as multivalued MOD function.

Clearly this is also misnomer. But as multivalued function is a misnomer so is multivalued MOD function.

We first illustrate the multivalued MOD-integer function.

Example 2.25: Let Z_{20} be the set of modulo integers. Z be the collection of integers.

Define a map $f_m : Z_{20} \to Z$ as follows
$f_m(0) = 20n$ ($n = 0, \pm1, \pm2, \ldots$)

$f_m(1) = (20\,n + 1)$
$f_m(2) = (20\,n + 2)$
$f_m(3) = (20\,n + 3)$ and so on
$f_m(19) = (20n + 19)$, $n = 0, \pm1, \pm2, \pm3, \ldots$,.

This f_m is defined as the multivalued MOD-function from a finite set is mapped onto an infinite collection.

This MOD-multivalued function f_m fixed every element of Z_{20}.

Example 2.26: Let $f_n : Z_{53} \to Z$ be the MOD-multivalued function. f_n fixes all the elements of Z_{53}.

For if $x \in Z_{53}$ then $f(x) = x$ is not possible as
$f_m(x) = 53n + x$ $n = 0 \pm1, \pm2, \ldots$

Thus this element x is mapped to infinite number of points.

For take x = 2 then
$f_m(x) = 2, 55, –51, 108, –104, 161, –157$ and so on.

If x = 5 then $f_m(5)$ = 5, 58, 48,101, 111 and so on.

$f_m(50)$ = 50, 103, –3, and so on.

Thus all elements of Z are exhausted by the multivalued MOD function f_m.

Infact we have sequence of points for every single point in Z_t.

The definition of this situation is as follows:

DEFINITION 2.3: *Let Z_n be the modulo integers. Z be the set of integers. f_m: $Z_n \to Z$ be the multivalued MOD function defined is as $f_m(0) = nt$; $t = 0, \pm 1, \pm 2, \ldots$*

$$f_m(1) = nt + 1, \ t = 0, \pm 1, \ldots$$
$$f_m(2) = nt + 2, t = 0, \pm 1, \pm 2, \ldots$$
$$f_n(n-1) = nt + (n-1); \ t = 0, \pm 1, \pm 2.$$

Thus $f(0) = 0$, $f(1) = 1$,
$f(2) = 2$ and so on
$f(n) = n$ all this happens for $t = 0$.

Now interested author can find what is f o f_m and f_m o f.

where f : $Z \to Z_n$ and $f_m : Z_n \to Z$ defined as earlier.

Next we find the multivalued MOD function from the t-tuple $\underbrace{Z_n \times \ldots \times Z_n}_{t-\text{times}}$ to the t-tuple of $\underbrace{Z \times Z \times \ldots \times Z}_{t-\text{times}}$.

First we will illustrate this situation by some examples.

Example 2.27: Let

$$S = (Z_{12} \times Z_{12} \times Z_{12} \times Z_{12}) = \{(x_1, x_2, x_3, x_4) \mid x_i \in Z_{12}; \ 1 \leq i \leq 4\}$$

and

$$T = (Z \times Z \times Z \times Z) = \{(a_1, a_2, a_3, a_4) \mid a_i \in Z; 1 \leq i \leq 4\}$$

be the 4-tuples of MOD integers Z_{12} and Z respectively.

Define f_m: S → T as the MOD multivalued function.

If x = (3, 8, 4, 9) ∈ S then

$f_m(x) = \{(12n + 3, 12n +8, 12n + 4, 12n + 9)\} = \{(3, 8, 4, 9), (15, 20, 16, 21), (9, 4, 8, 3), (27, 32, 28, 33), (21, 16, 20, 15)$ and so on$\}$ = P has infinite number of elements associated with it.

Similarly for any x in S. Thus given any y ∈ T we have a unique element associated with it in S.

For if y = (–78, 105, –3, –7) ∈ T
then $f_m^{-1}(y) = (6, 9, 9, 5) \in S$,

$f_m(6, 9, 9, 5) = (12n_0 + 6, 12n_1 + 9, 12n_2 + 9, 12n_3 + 5)$ for we see we can take $n_0 = 0$, $n_1 = 1$, $n_2 = –3$ and $n_3 = –10$.

So we see the set P has other different elements for the n can take mixed values and so on.

Thus when we put same n still it is to be kept in mind we permute it for varying values of n.

So P has lot more elements for one n can be m_1 another n is m_2, another n is m_3 and the forth n is m_4.

This sort of values alone can cater for all the elements of $Z \times Z \times Z \times Z$.

Of course the all elements of $Z_{12} \times Z_{12} \times Z_{12} \times Z_{12}$ is fixed for n = 0 when n ≠ 0 they generate the totality of $Z\times Z \times Z\times Z$.

We will work another example for a better understanding of this concept.

Example 2.28: Let

$S = \{Z_3 \times Z_3 \times Z_3 = \{(x_1, x_2, x_3) \mid x_i \in Z_3; 1 \leq i \leq 3\}\}$ and

$T = \{Z \times Z \times Z = \{(a_1, a_2, a_3) \mid a_i \in Z; 1 \leq i \leq 3\}\}$

be two 3-tuples.

$$f_m : S \to T$$

$$\begin{aligned} f_m(0, 0, 0) &= (0, 0, 0) \\ f_m(1, 2, 0) &= (1, 2, 0) \\ &= (3n+1, 3n_1 + 2, 3n_3) \\ &= \{(1, 2, 0), (1, 2, 3), (1, 2, 6), (1, 2, 9), \\ &\quad (1, 2, -3), (1, 2, -6), (1, 2, -9), (1, 2, -12), \\ &\quad (1, 2, 12) \end{aligned}$$

and so on.

(4, 2, 0), (7, 2, 0), (–2, 2, 0) (–2, 2, 3), (–2, 2, 6) (–2, 2, –3) and so on and so forth (1, –1, 3), (1, –4, 3), (1, –7, 3), (1, –10, 3) …, (1, 5, 3), (1, 8, 3), (1, 11, 3) and so on}.

In actuality one has to work like this so f_m is a very special type of multivalued MOD function.

Example 2.29: Let $S = \{Z_{18} \times Z_8 = \{(a, b) \mid a, b \in Z_{18}\}\}$ and

$T = \{Z \times Z = \{(x, y) \mid x, y \in Z\}$.

f_m: $S \to T$ is the multivalued MOD function.

This fixes the set S for f(S) = S but more for every $x \in S$ is mapped onto a infinite periodically placed pairs.

Just we represent this situation by some illustrations.

Let $y = (3, 9)$

$f_m(y) = (3, 9)$ and

$f_m(y) = (m18 + 3, n18 + 9)$ $m, n \in Z \setminus \{0\}$.

$f_m(y) = \{(3, 9)$ (when both $m = 0 = n$); $(21, 27), (-15, -9), (3, 27), (21, -9), (-15, 9)$ $(-15, +27)$ and so on$\}$.

Thus infinitely periodic pairs are being mapped by (3, 9) of $Z_{18} \times z_{18} = S$.

$f : Z \times Z \to Z_n \times Z_n$ the MOD function has only a finite number of fixed points.

Infact infinite many points in $Z \times Z$ is mapped onto a finite set.

Likewise we can extend the study of multivalued MOD function to intervals.

$f_m: [0, n) \to R$ this map is as follows.

$f_n(x) = x$ for all $x \in [0, n)$ and $f_n(x) = nt + x$ $t \in Z \setminus \{0\}$.

By this method this MOD multivalued interval function does not leave even a single element in R left without being mapped.

We can say the interval $[0, n)$ which has infinite number of points is being fixed by f_m.

We will illustrate this situation by an example or two.

Example 2.30: Let $f_m: [0,6) \to R$ defined by

$$f(3.001) = 3.001$$
$$= 6t + 3.001 \ (t \in Z \setminus \{0\})$$
$$= \{9.001, -2.999, 3.001 \text{ (when } t = 0),$$
$$15.001, -9.999, 21.001, -14.999 \text{ and so on}\}.$$

This is mapped onto an infinite collection.

Consider $0.1 \in [0,6)$

$$f(0.1) = \{0.1, 7.1, -5.9, 12.1, -11.9, \text{and so on}\}$$

Thus f_m the multivalued MOD interval function behave in an odd way by mapping a single element of [0,6) on an infinite collection which is made periodically using both positive and negative integers.

The authors leave it as an open conjecture to study about the properties the MOD interval function f and the MOD multivalued function f_m.

We supply one more example to this effect.

Example 2.31: Let $f_m : [0, 13) \to R$ defined by $f_m(x) = \{x, 13t + x, t \in Z \setminus \{0\}; x \in [0, 13)\}$; this is an infinite collection which periodically fills the real line R.

Next we give some examples of the MOD multivalued function

$$f_m: [0, n) \times [0, n) \times [0, n) \to R \times R \times R.$$

This is also defined in a similar way as that of

$$f_m: [0,n) \to R.$$

Here we see if $x = (x_1, x_2, x_3) \in [0, n) \times [0,n) \times [0, n)$ then

$$f_m(x) = \{nt_1 + x_1, nt_2 + x_2, nt_3 + x_3); t_1, t_2, t_3 \in Z.$$

So this $f_m(x)$ a single point x is mapped by the multivalued MOD interval function into infinitely many triple points or dense triple intervals covering the entire region.

This sort of study is very different as MOD multivalued interval functions are not functions as they do not obey the classical properties of functions.

Next we can have MOD multivalued matrix (transformation) function.

Example 2.32: Let

$$S = \left\{\begin{bmatrix} a_1 & a_2 & a_3 & a_4 & a_5 \\ a_6 & a_7 & a_8 & a_9 & a_{10} \end{bmatrix} \mid a_i \in Z_{10};\ 1 \leq i \leq 10\right\}$$

and

$$T = \left\{\begin{bmatrix} a_1 & a_2 & a_3 & a_4 & a_5 \\ a_6 & a_7 & a_8 & a_9 & a_{10} \end{bmatrix} \mid a_i \in Z;\ 1 \leq i \leq 10\right\}$$

be two sets of 2×5 matrices built using Z_{10} and Z respectively.

f_n: S → T is defined by $f_n\left(\begin{bmatrix} a_1 & \ldots & a_5 \\ a_6 & \ldots & a_{10} \end{bmatrix}\right)$

$$= \begin{bmatrix} 10n_1 + a_1 & 10n_2 + a_2 & \ldots & 10n_5 + a_5 \\ 10n_6 + a_6 & 10n_7 + a_7 & \ldots & 10n_{10} + a_{10} \end{bmatrix}$$

where $n_1, n_2, \ldots, n_{10}$ takes all values from Z.

For instance if $A = \begin{bmatrix} 3 & 1 & 7 & 0 & 5 \\ 2 & 0 & 1 & 8 & 9 \end{bmatrix}$

$$f_m(A) = \begin{bmatrix} 10n_1 + 3 & 10n_2 + 1 & 10n_3 + 7 & 10n_4 & 10n_5 + 5 \\ 10n_6 + 2 & 10n_7 & 10n_8 + 1 & 10n_9 + 8 & 10n_{10} + 9 \end{bmatrix}$$

$$= \left\{ \begin{bmatrix} 3 & 1 & 7 & 0 & 5 \\ 2 & 0 & 1 & 8 & 9 \end{bmatrix}, \begin{bmatrix} -7 & -9 & -3 & 0 & -5 \\ 12 & 10 & -9 & 42 & -11 \end{bmatrix}, \right.$$

$$\left. \begin{bmatrix} 13 & 21 & 27 & 20 & 25 \\ 22 & 20 & 21 & 28 & 29 \end{bmatrix} \text{ and so on} \right\}.$$

Thus $f_m(A)$ is an infinite collection which contains A.

This sort of study using multivalued MOD matrix functions is an interesting problem.

Example 2.33: Let

$$W = \left\{ \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix} \mid a_i \in Z_{29}, 1 \le i \le 4 \right\}$$

and

$$V = \left\{ \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix} \mid a_i \in Z; 1 \le i \le 4 \right\}$$

be the square matrix collection with entries from Z_{29} and Z respectively.

The map f_m: W → V defined by for

$$A = \begin{bmatrix} 10 & 12 \\ 18 & 0 \end{bmatrix} \in W;$$

$$f_m(A) = \begin{bmatrix} 29t_2 + 10 & 29t_1 + 12 \\ 29t_3 + 18 & 29t_4 \end{bmatrix}$$

t_1, t_2, t_3 and t_4 takes values from Z.

$f_m(A) = A$ if $t_1 = t_2 = t_3 = t_4 = 0$.

$$f_m(A) = \{A, \begin{bmatrix} 39 & 41 \\ 47 & 29 \end{bmatrix} \begin{bmatrix} 29 & 41 \\ -11 & 0 \end{bmatrix}, \begin{bmatrix} 39 & 41 \\ -11 & 0 \end{bmatrix}, \begin{bmatrix} 10 & 41 \\ 18 & 29 \end{bmatrix}$$

and so on}.

This is an infinite collection.

This the study of multivalued MOD matrix function is an interesting one.

We can also have mixed multivalued matrix MOD functions which will be described by examples.

Example 2.34: Let

$$M = \left\{ \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{bmatrix} \middle|\ a_i \in Z_{40},\ a_2 \in Z_4,\ a_3 \in Z_5 \text{ and } a_4 \in Z_{12} \right\} \text{ and}$$

$$N = \left\{ \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{bmatrix} \middle|\ a_i \in Z;\ 1 \leq i \leq 4 \right\}$$

be the collection of 4 × 1 column matrices with entries from different modulo integers and the ring of integers respectively.

f_m: M → N the MOD multivalued matrix function is defined as follows.

$$\text{Let } A = \begin{bmatrix} 28 \\ 3 \\ 4 \\ 8 \end{bmatrix} \in M,$$

$$f_m(A) = f\left(\begin{bmatrix} 28 \\ 3 \\ 4 \\ 8 \end{bmatrix}\right) = \begin{bmatrix} 40n_1 + 28 \\ 4n_2 + 3 \\ 5n_3 + 4 \\ 12n_4 + 8 \end{bmatrix} = \left\{\begin{bmatrix} 28 \\ 3 \\ 4 \\ 8 \end{bmatrix}, \begin{bmatrix} 12 \\ -1 \\ -1 \\ -4 \end{bmatrix}, \text{ and so on}\right\}.$$

Thus MOD multivalued matrix function f_m fixes matrix collection M.

This is the way MOD-multivalued matrix function is defined and developed.

Next we proceed onto describe MOD multivalued interval matrix function f_m by these following examples.

Example 2.35: Let

$$M = \left\{\begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \end{bmatrix} \mid a_1 \in Z_{10}, a_2, a_3 \in Z_{15}, a_4, a_5 \in Z_3,\right.$$

$$\left. a_6, a_7 \in Z_4, a_8, a_9 \in Z_6\right\}$$

$$N = \left\{\begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \end{bmatrix} \mid a_i \in Z; 1 \le i \le 9\right\}$$

be the collection of 3×3 matrices with entries from M the set of mod integers and N be the collection of 3×3 matrices with entries from Z.

Let f_m: M $\to$ N defined by the following way.

$$f_m\left(\begin{bmatrix} 9 & 10 & 12 \\ 1 & 2 & 3 \\ 0 & 5 & 2 \end{bmatrix}\right) = \left\{\begin{bmatrix} 10n_1+9 & 15n_2+10 & 15n_3+12 \\ 3n_4+1 & 3n_5+2 & 4n_6+3 \\ 4n_7 & 6n_8+5 & 6n_9+2 \end{bmatrix}\right.$$

where $n_i \in$ Z; $1 \leq i \leq 9$; it is to be noted each n_i can take any value from Z and so all possible combinations are exhausted as all possible values from Z are taken by all the n_i's : i = 1, 2, …, 9}.

We find $f_m(A)$ where A = $\begin{bmatrix} 9 & 10 & 12 \\ 1 & 2 & 3 \\ 0 & 5 & 2 \end{bmatrix}$.

Thus this set is an infinite collection but still not the totality of N.

Hence

$$f_m(A) = \{A, \begin{pmatrix} 19 & 25 & 27 \\ 4 & 5 & 7 \\ 4 & 11 & 8 \end{pmatrix}, \begin{pmatrix} -1 & -5 & -3 \\ -2 & -1 & -1 \\ -4 & -1 & -4 \end{pmatrix} \text{ and so on}\}$$

is an infinite collection.

Consider B = $\begin{bmatrix} 7 & 6 & 5 \\ 0 & 1 & 2 \\ 3 & 4 & 5 \end{bmatrix} \in$ M

$$f_m(B) = \left\{\begin{bmatrix} 10n_1+7 & 15n_2+6 & 15n_3+12 \\ 3n_4 & 3n_5+1 & 4n_6+2 \\ 4n_7+3 & 6n_8+4 & 6n_9+5 \end{bmatrix} n_i \in Z; \right.$$

$$\left. i = 1, 2, \ldots, 9\right\}$$

is again an infinite collection.

However $f_m(A) \cap f_m(B) = \phi$.

$$\text{Consider } \theta = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \in M;$$

$$f_m(\theta) = \left\{\begin{bmatrix} 10n_1 & 15n_2 & 15n_3 \\ 3n_4 & 3n_5 & 4n_6 \\ 4n_7 & 6n_8 & 6n_9 \end{bmatrix} \mid n_i \in Z; 1 \le i \le 9\right\}$$

is again an infinite collection.

$$f_m(\theta) = \left\{\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 10 & 15 & 15 \\ 3 & 3 & 4 \\ 4 & 6 & 6 \end{bmatrix}, \right.$$

$$\left.\begin{bmatrix} -10 & 15 & -15 \\ 3 & -3 & 4 \\ -4 & 6 & -6 \end{bmatrix}, \begin{bmatrix} -10 & -15 & -15 \\ -3 & -3 & -4 \\ -4 & 6 & -6 \end{bmatrix} \text{ and so on}\right\}.$$

We see $f_m(\theta) \cap f_m(A) = \phi$ $f_m(\theta) \cap f_m(B) = \phi$ and the fact is for each $A \in M$ is such that $f_m(A)$ is a special collection of elements such that $f_m(A)$ is disjoint with every element of N and $\sum_i f_m(A_i) = N$.

Infact one can realize the MOD-multivalued matrix function f_m as a partition on N.

The notion of equivalence classes and equivalence relation can be defined in a routine way.

Thus these MOD multivalued function behaves in a unique way.

However in case of MOD functions f: $Z \to Z_n$ reverse way of work is carried out as $f([a]) = f(a)$; $a \in Z_n$.

That is an infinite collection of elements is mapped onto a single element.

Such study is new and innovative for we are not in a position to fully analyse the behavior of these MOD functions and MOD-multivalued functions.

Study of the special properties associated with these functions happens to be a open problem.

Next we proceed onto describe MOD multivalued interval functions f_m: $[0, n) \to R$ by the following examples.

Example 2.36: Let $f_m : [0, 10) \to R$ be the MOD multivalued interval function defined by

$$f_n(x) = \{x \text{ or } 10n + x;\ n \in Z\}.$$

Several interesting properties can be derived using the MOD multivalued interval function f_m.

$f_m(0.3) = \{0.3, 10.3, 9.7, 20.3, 29.7$ and so on$\}$; this is an infinite collection.

Let $3 \in [0, 10)$; $f_m(3) = \{3, 13, 7, 23, 17, \ldots\}$.

It is easily verified $f_m(0.3) \cap f_m(3) = \phi$.

Infact this is true for every element of [0, 10).

Thus each $x \in [0, 10)$ has $f_m(x)$ and $\bigcup_{x \in [0,10)} f_m(x) = R$ and

$f_m(x) \cap f_m(y) = \phi$ if $x \neq y$ and $x, y \in [0,10)$.

Example 2.37: Let $f_m : [0,41) \to R$ be the MOD-multivalued interval function $f_m(x) = \{41n + x/ n \in Z\} = \{x, 41 + n, -41 + x$ and so on$\}$.

Let $4.3 \in [0, 41)$, then

$f_m(4.3) = \{4.3, 45.3, -36.7, -77.7, -118.7, 85.3, 127.3$ and so on$\}$.

For $10.3 \in [0,41)$;

$f_m (10.3) = \{41n + 10.3 \mid n \in Z\} = \{10.3, 51.3, +92.3, 133.3, -30.7, -71.7$ and so on$\}$

Clearly $f_m(4.3) \cap f_m(10.3) = \phi$.

Infact the elements of R are partitioned by the MOD multivalued interval function f_m and the interval [0, n).

In view of this the following theorems are left as an exercise.

THEOREM 2.9: *Let f_m: $Z_n \to Z$ be the MOD multivalued function.*

i) f_m partitions Z into equivalence classes
ii) $f_m(x) \cap f_m(y) = \phi$; $\bigcup_{x \in Z_n} f_m(x) = Z$.

THEOREM 2.10: *Let f_m : {$p \times q$ matrices with entries from Z_n} = $M \to N$ = {collections of all $p \times q$ matrices with entries from Z} be the MOD-multivalued matrix function.*

i) *Every matrix A in M is divided by the f_m into equivalence classes.*
ii) $f_m(A) \cap f_m(B) = \phi$ *if* $A \neq B$.
iii) $\bigcup_{A \in M} f_m(A) = N$.

Next we proceed onto develop the properties associated with MOD multivalued interval functions f_m: $[0, n) \to R$.

THEOREM 2.11: *Let $f_m : [0, n) \to R$ be the MOD multivalued interval function. Then the following are true.*

i) *f_m partitions R into equivalence classes for every $x \in [0, n)$.*
ii) $f_m(x) \cap f_m(y) = \phi$ *if* $x \neq y$; $x, y \in [0, n)$.
iii) $\bigcup_{x \in [0,n)} f_m(x) = R$.

Proof is left as an exercise to the reader.

THEOREM 2.12: *Let f_m: $M \to N$ where*

M = {$p \times q$ matrices collection with entries from $[0,n)$} and

N = {collection of $p \times q$ matrices with entries from R} be a MOD interval multivalued matrix function. Then the following conditions are satisfied by f_m.

i) *Every $A \in M$ has a class of matrices associated with $f_m(A)$; such that N is partitioned into matrices classes.*
ii) $f_m(A) \cap f_m(B) = \phi$ *if* $A \neq B$; $A, B \in M$.
iii) $\bigcup_{A \in M} f_m(A) = N$.

Proof is left as an exercise to the reader.

We suggest the following problems some of which are at research level.

Problems

1. Develop the special properties enjoyed by the MOD function

$$f : Z \to Z_n \ (n < \infty;\ 2 \leq n < \infty).$$

2. Let f: $Z \to Z_m$ be the MOD function from Z to Z_m; $\{\{f(\{x\}) = n \mid x \in Z\} = n \in Z_m\}$ = {collection of all $x \in Z$ such that $f(x) = n$}; n a fixed number.

   i) Prove if $n_1 = f(\{x_1\})$ and $n_2 = f(\{x_2\})$ then $n_1 \neq n_2$.
   ii) $f(\{x_1\}) \cap f(\{x_2\}) = \phi$.

3. Enumerate the special and distinct features enjoyed by

$$f : Z \to Z_n;\ 2 \leq n < \infty.$$

4. Let $f : (Z \times Z \times Z) \to Z_5 \times Z_5 \times Z_5$ be the MOD function.

   Study all properties associated with f.

5. Let f : M = {all p × q matrices with entries from Z} → N = {all p × q matrices with entries from Z_n} be the MOD matrix function.

   Study all the special features enjoyed by f.

6. Let $f : Z \to Z_{15}$ be the MOD function.

   i) Find all special features of this MOD function f.
   ii) Can we say there are only 15 disjoint sets of Z as the pull back of f or t = f(x) = {those element x in Z mapped onto t of Z_{15}}?

7. Let $M = \{5 \times 3$ matrices with entries from $Z\}$ and

   $N = \{5 \times 3$ matrices with entries from $Z_{20}\}$

   Let $f : M \to N$ be the MOD function.

   i) Study all special features enjoyed by f.

   ii) Prove if $A \in N$ then all those $X \in M$ such that $\{f(X)\} = A$ is an infinite collection and if $\{f(X)\} = A$ and $\{f(Y)\} = B$ where $\{f(X)\} = \{$all those elements in Z mapped onto $A\}$
   $\{f(Y)\} = \{$All those elements in Z mapped onto $B\}$
   then $\{f(X)\} \cap \{f(Y)\} = \phi$ if $A \neq B$; $A, B \in N$.

   iii) Can we say the association of every $A \in N$ makes M into disjoint sets such that it is a partition of M?

8. Let $M = \{$All 3×3 matrices with entries from $Z\}$ and $N = \{$collection of all 3×3 matrices with entries from $Z_{45}\}$.

   $f : N \to M$ be the MOD matrix function.

   Study questions (i) to (iii) of problem (7) for this $f : M \to N$.

9. Let $f: R \to [0,20)$ be MOD interval function.

   Study questions (i) to (iii) of problem (7) for this function.

10. Specify all special features associated with MOD interval function $f : R \to [0,m)$; $2 \leq m < \infty$.

11. Let $f : R \to [0,9)$ be the MOD interval function.

Let $\{f(x)\} = \{x \in R \mid f(x) = 3.77\}$ and $\{f(y) = \{y \in R \mid f(y) = 4.8\}$

i) Prove $\{f(x)\} \cap \{f(y)\} = \phi$.

ii) Prove $\bigcup_{f(x) \in [0,9)} \{f(x)\} = R$.

12. Let $f : R \to [0,25)$ be a MOD interval function.

    Study questions (i) to (iii) of problem (7) for this f.

13. Let $f : R \times R \to [0,7) \times [0,7)$ be the MOD interval function.

    Study questions (i) to (iii) of problem (7) for this f.

14. Let $f: R \times R \times R \times R \times R \to [0,23) \times [0,23) \times [0,23) \times [0,23) \times [0,23) \times [0,23)$ be the MOD interval function.

    Study questions (i) to (iii) of problem (7) for this f.

15. Let $f : \left\{\begin{bmatrix} a_1 & a_2 & a_3 & a_4 \\ a_5 & a_6 & a_7 & a_8 \end{bmatrix} \mid a_i \in R; 1 \leq i \leq 8\right\} = M \to$

    $N = \left\{\begin{bmatrix} x_1 & x_2 & x_3 & x_4 \\ x_5 & x_6 & x_7 & x_8 \end{bmatrix} \mid x_i \in [0,43); 1 \leq i \leq 8\right\}$

    be the MOD interval matrix function.

    Study questions (i) to (iii) of problem (7) for this f.

16. When $f : M \to N$ is a MOD-interval matrix function, find all the special features enjoyed by such f.

17. Let f : $\left\{\sum_{i=0}^{\infty} a_i x^i / a_i \in Z\right\} \to \left\{\sum_{i=0}^{\infty} a_i x^i \mid a_i \in Z_{40}\right\}$ be the MOD polynomial function.

    Can questions (i) to (iii) of problem (7) be true.

    Justify your claim.

18. Find all the special and distinct features associated with MOD-polynomial functions.

    i) Are these different from f: $Z \to Z_n$ the MOD function?

    ii) Are they similar or different from the MOD matrix functions?

19. Can we say MOD matrix function to satisfy (i) to (iii) of problem 7?

    Justify your claim.

20. Let f_m: $Z_{15} \to Z$, MOD-multivalued function.

    i) Study all the special features associated with f_m.

    ii) Can f_m partition the range space into a finite number of sets but each of them are of infinite cardinality?

    iii) Can we say f_m is a sort of equivalence relation on Z?

21. Let f_m: $Z_n \to Z$ be the MOD-multivalued function.

    Study questions (i) to (iii) of problem (20) for this f_m.

22. Study questions (i) to (iii) of problem (21) for the function $f_m : Z_{53} \to Z$.

23. Compare f: $Z \to Z_n$ with f_m: $Z_n \to Z$.

24. Can we say f_m: $Z_n \rightarrow Z$ the MOD multivalued functions are not functions in the classical sense?

25. Let f_m: $Z_5 \times Z_5 \times Z_5 \rightarrow Z \times Z \times Z$ be the MOD multivalued function.

    Can we prove the questions (i) to (iii) of problem (20) are true for this f_m?

26. Let f_m: N = {collection of all 5×5 matrices with entries from Z_{12} → M = {collection of all 5×5 matrices with entries from Z} be the MOD-multivalued multifunction.

    Can questions (i) to (iii) of problem (20) be true for this f_m?

27. Let f_m: $M = \left\{ \begin{bmatrix} a_1 & a_2 & a_3 & a_4 \\ a_5 & a_6 & a_7 & a_8 \\ a_9 & a_{10} & a_{11} & a_{12} \end{bmatrix} \mid a_1, a_2 \in Z_{10}, a_3, a_4 \in Z_5, \right.$

    $a_5, a_6 \in Z_7, a_7, a_8 \in Z_{19}, a_{10}, a_9 \in Z_{12}, a_{11}, a_{12} \in Z_{23}\} \rightarrow$

    $N = \left\{ \begin{bmatrix} a_1 & a_2 & a_3 & a_4 \\ a_5 & a_6 & a_7 & a_8 \\ a_9 & a_{10} & a_{11} & a_{12} \end{bmatrix} \mid a_i \in Z; 1 \leq i \leq 12 \right\}$

    be the MOD-multivalued matrix function.

    Can questions (i) to (iii) of problem (20) be true for this f_m.

28. Study questions (i) to (iii) of problem (20) for the $f_n : Z_{10} \times Z_{19} \times Z_{48} \rightarrow Z \times Z \times Z$ the MOD-multivalued function.

29. Let f_m: $[0,23) \to R$ be the MOD-multivalued interval function.

    Study questions (i) to (iii) of problem (20) for this f_m.

30. Let $f_m : [0, n) \to R$, $(2 \leq n < \infty)$ be the MOD-multivalued interval function.

    i) Describe and develop all important features enjoyed by f_m.

    ii) Distinguish this f_m from g_m: $Z_n \to Z$.

31. Let f_m: $[0,43) \to R$ be the MOD-multivalued interval function.

    Study all questions (i) to (iii) of problem (20) for this f_m.

32. Let f_m: M = {collection of all 2×7 matrices with entries from [0,24)} $\to$ N = {collection of all 2×7 matrices with entries from R} be the MOD-multivalued interval matrix function.

    Can questions (i) to (iii) of problem (20) be true for this f_m?

33. Let f_m: M = {collection of all 4×4 matrices with entries from [0,23)} $\to$ N = {collection of all 4×4 matrices with entries from R} be the MOD interval multivalued matrix function.

    i) Study questions (i) to (iii) of problem (20) for this f_m.

    ii) Compare f_m^1 $[0,23) \to R$ with the above f_m where f_m^1 is the MOD-multivalued interval function.

34. Let f_m^2 : [0,24) × [0,43) → R × R be the MOD-multivalued interval function.

    i) Compare f_m^2 with f_m^1 in problem (33).

    ii) Compare f_m^2 with f_m of problem (33).

35. Let $f_m^3 : \left\{\begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix}\right.$ $a_i \in [0,5)$; $a_2 \in [0,42)$, $a_3 \in [0,427)$ and

    $A_4 \in [0,12)\}$ = M → N = $\left\{\begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix}\right.$ | $a_i \in R$ $1 \leq i \leq 4\}$

    be MOD-multivalued interval matrix function

    i) Study questions (i) to (iii) of problem (20) for this f_m^3.

    ii) Compare f_m of problem (32) with this f_m^3.

    iii) Compare f_m^1 of problem 33 with this f_m^3.

    iv) Compare f_m^2 of problem (34) with this f_m^3.

36. Let f_m^p : M = $\{\Sigma a_i x^i \mid a_i \in [0,43)\}$ → N = $\{\Sigma a_i x^i \mid a_i \in R\}$ be the MOD multivalued interval polynomial function.

    i) Study questions (i) to (iii) of problem (20) for this f_m^p.

    ii) Compare f_m^p with f_m^2 of problem (34).

    iii) Compare f_m^p with f_m^3 of problem (35).

37. Let $f_m : [0,144) \to R$ be the MOD-multivalued interval function.

    i) Into how many disjoint set R is partition by f_m and the interval [0,144)?

    ii) Can we say if $f : R \to [0,144)$ then $f \circ f_m = f_m \circ f$ = identity map?

    iii) Find $f \circ f_m$

    iv) Find $f_m \circ f$.

38. Can we say the study of MOD-multivalued interval functions f_m gives infinitely many fixed points?

39. Find all the fixed points of $f : Z \to Z_5$.

40. Find all fixed points of the MOD interval function $f : R \to [0,23)$.

41. Find all the fixed points of $f : Z \times Z \times Z \to Z_7 \times Z_{12} \times Z_{31}$ where f is the MOD function.

42. Find all fixed points of the MOD interval function $f : R \times R \times R \times R \to [0,3) \times [0,20) \times [0,143) \times [0,7)$.

43. What are the special features associated with the fixed points of the MOD function and that of any classical function?

44. What are fixed points of the MOD-multivalued function $f_m : Z_{48} \to Z$?

45. Find all the fixed points of the MOD-multivalued interval function $f_m : [0,43) \to R$.

46. What are fixed points of MOD-multivalued functions

i) $f_m^1\ [0,20) \times [0,48) \to R \times\ R$?

ii) $f_m^2 : ([0,19) \times [0,22) \times [0,19)) \to R \times R \times R$?

iii) $f_m^3 : N\left\{\begin{bmatrix} [0,40) & [0,3) & [0,7) \\ [0,12) & [0,72) & [0,5) \end{bmatrix}\right\} \to$ all 2×3 matrices

with entries from R?

Chapter Three

FIXED ELEMENTS OF MOD-MATRIX OPERATORS

Here for the first time the notion of MOD-matrix operators using MOD-integers is defined, described and developed. Further fixed elements which are row vectors or column vectors are obtained in the case of MOD-modulo integer matrix operators.

Throughout this chapter only square matrices will be used and they take entries only from the MOD-integers. So the number of $n \times n$ square matrices with entries from Z_m the ring of modulo integers is finite.

Further the collection of all row or column matrices with entries from Z_m is also finite.

This property is mainly exploited to get a fixed row vector or a fixed column vector depending on the way the operations are performed.

First this situation is represented by an example or two.

Example 3.1: Let M = $\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$ be a matrix with entries from Z_3.

Consider the row vectors

{(0, 0), (1, 0), (0, 1), (2, 0), (0, 2), (1, 2), (2, 1), (1,1), (2,2)}= A.

Let x = (1, 0) we find when will x become a fixed point.

Here if we take x = (1, 0) while updating we continue to keep the second coordinate to be always one.

xM = (1, 0) $\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$ = (2, 1) → (1, 1) = y '→' show the

vector is updated.

$$(1, 1) \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} = (0, 0) \to (1, 0) = x.$$

Thus x = (1, 0) is a fixed point.

Let y = (2, 1) ∈ A

$$yM = (2,1) \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} = (2, 1) = y.$$

So the matrix operator yields y to be a fixed point.

We call these fixed point as classical MOD matrix fixed points.

The fixed point may occur at the first stage or at second stage and so on.

Let $(2, 2) = z \in A$;

$$zM = (2, 2)\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} = (0, 0) = (2, 2)$$ after updating is a fixed point.

However if the notion of fixed point does not exist we call it as zero divisors or zero vectors.

Example 3.2: Let

$$M = \begin{bmatrix} 3 & 2 & 1 & 5 \\ 4 & 0 & 3 & 1 \\ 2 & 1 & 1 & 0 \\ 1 & 0 & 4 & 2 \end{bmatrix}$$

be the matrix with entries from Z_6.

We find the fixed row vectors by the MOD operator matrix M.

Let $A = \{(x_1, x_2, x_3, x_4)$ where $x_i \in Z_6; 1 \leq i \leq 4\}$.

Take $x = (3, 0, 2, 0) \in A$;

$$xM = (3, 0, 2, 0)\begin{bmatrix} 3 & 2 & 1 & 5 \\ 4 & 0 & 3 & 1 \\ 2 & 1 & 1 & 0 \\ 1 & 0 & 4 & 2 \end{bmatrix} = (1, 2, 2, 3)$$

$$\to (3, 2, 2, 3) = y_1 \text{ (say)}$$

($\rightarrow$ denote the vector has been updated)

$y_1M \rightarrow (3, 2, 2, 5) = y_2$ (say).

$y_2M \rightarrow (3, 2, 2, 3) = y_3$ (say)

$y_3M \rightarrow (3, 2, 2, 5) = y_2$.

Thus it is a limit cycle and not a fixed point.

Suppose we do not use the technique of updating we find

$$xM = (3, 0, 2, 0)\begin{bmatrix}3 & 2 & 1 & 5\\4 & 0 & 3 & 1\\2 & 1 & 1 & 0\\1 & 0 & 4 & 2\end{bmatrix} = (1, 2, 5, 3) = y_1.$$

$$y_1M = (1, 2, 5, 3)\begin{bmatrix}3 & 2 & 1 & 5\\4 & 0 & 3 & 1\\2 & 1 & 1 & 0\\1 & 0 & 4 & 2\end{bmatrix} = (0, 1, 0, 1) = y_2$$

$$y_2M = (0, 1, 0, 1)\begin{bmatrix}3 & 2 & 1 & 5\\4 & 0 & 3 & 1\\2 & 1 & 1 & 0\\1 & 0 & 4 & 2\end{bmatrix} = (5, 0, 1, 3) = y_3$$

$$y_3M = (5, 0, 1, 3)\begin{bmatrix}3 & 2 & 1 & 5\\4 & 0 & 3 & 1\\2 & 1 & 1 & 0\\1 & 0 & 4 & 2\end{bmatrix} = (2, 5, 0, 1) = y_4$$

$$y_4M = (2, 5, 0, 1)\begin{bmatrix} 3 & 2 & 1 & 5 \\ 4 & 0 & 3 & 1 \\ 2 & 1 & 1 & 0 \\ 1 & 0 & 4 & 2 \end{bmatrix} = (3, 4, 3, 5) = y_5$$

$$y_5M = (3, 4, 3, 5)\begin{bmatrix} 3 & 2 & 1 & 5 \\ 4 & 0 & 3 & 1 \\ 2 & 1 & 1 & 0 \\ 1 & 0 & 4 & 2 \end{bmatrix} = (0, 3, 2, 5) = y_6$$

$$y_6M = (0, 3, 2, 5)\begin{bmatrix} 3 & 2 & 1 & 5 \\ 4 & 0 & 3 & 1 \\ 2 & 1 & 1 & 0 \\ 1 & 0 & 4 & 2 \end{bmatrix} = (0, 2, 1, 1) = y_7$$

$y_7M = (5, 1, 5, 4) = y_8$; $y_8M = (3, 3, 2, 1) = y_9$;
$y_9M = (2, 2, 0, 2) = y_{10}$; $y_{10}M = (4, 4, 4, 4) = y_{11}$;
$y_{11}M = (4, 0, 0, 4) = y_{12}$; $y_{12}M = (4, 0, 0, 1) = y_{13}$;
$y_{13}M = (1, 0, 0, 1) = y_{14}$; $y_{14}M = (4, 0, 0, 2) = y_{15}$;
$y_{15}M = (2, 0, 0, 4) = y_{16}$; $y_{16}M = (4, 0, 0, 2) = y_{15}$.

Thus this point is a limit cycle getting (4, 0, 0, 2) to (2 0, 0, 4), (4, 0, 0, 2) $\rightarrow$ (2, 0, 0, 4) $\rightarrow$ (4, 0, 0, 2) $\rightarrow$ (2, 0, 0, 4) and so on.

Thus we have three types of fixed points or limit cycles.

This will be defined systematically in the following.

DEFINITION 3.1: *Let $S = (s_{ij})$ be a $n \times n$ matrix with entries from Z_m.*

$P = \{(a_1, \ldots, a_n) / a_i \in Z_m;\ 1 \leq i \leq n\}$ collection of all row vectors. S is called the MOD operator on elements of P.

For any $X \in P$ we have $X \times S \in P$. Now if for $X \in P$; $X \times S = X$ then X is defined as the classical fixed point of the MOD matrix operator S.

If for any $X \in P$;
$XP \to y_1 \to y_2 \to \ldots \to y_t \to y_t \to y_{t+1} \ldots$ then X is defined as a limit cycle.

If $X \in P$ after some p number of iterations;

$XS \to Y_1, \ldots, Y_{p-1}$ and $Y_{p-1}S = Y_{p-1}$ then X is defined as the realized fixed point of the MOD matrix operator S or MOD realized fixed point of S.

If $X \in P$; $XS \to Y_1$ and if the coordinates of Y_1 are updated that is if in X; $a_i, a_j, \ldots, a_k$ points exists then in Y_1 also we replace $a_i, a_j, \ldots, a_k$ and only zero entries of X not updated then we find $Y_1S \to Y_2$.

Y_2 is also updated, by this method after a finite number of steps we may arrive at a Y_n where $Y_nS = Y_n$ then we call Y_n the updated fixed point of X of the MOD-matrix operator S.

If we do not get a fixed point but a limit cycle say Z_n we call Z_n the updated limit cycle of X of the MOD matrix operator S.

Thus we have several types of fixed points (row vectors) associated with the MOD matrix operator.

We will first illustrate this situation by some examples.

Example 3.3: Let

$$S = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{bmatrix}$$

be the 4×4 matrix with entries from $Z_2 = \{0, 1\}$.

S is the MOD matrix operator acts on the set of state row vectors P = {(0, 0, 0, 0), (1, 0, 0, 0), (0, 1, 0, 0), …, (1, 1, 1, 1)}; o(P) = 16.

Let $x = (1, 1, 0, 1) \in P$; $xS = (0, 0, 0, 1) = y_1$ (say)

$y_1S = (1, 1, 0, 1) = y_2 = x$.

So we see $x \to y_1 \to x \to y$ is a limit cycle.

Thus the MOD operator matrix S makes x only a limit cycle of length one.

Let $x_1 = (0, 1, 1, 1) \in P$;
$x_1S = (0, 1, 0, 0) = y_1$ (say)
$y_1S = (0, 1, 1, 1) = x_2$.

Thus $x_1 = (0, 1, 1, 1)$ is again the limit cycle on the MOD operator matrix S.

Let $x_2 = (1, 0, 1, 1) \in P$.
$x_2S = (1, 0, 0, 0) = y_1$,
$y_1S = (1, 0, 1, 1) \in P$.

x_2 is also a limit cycle of the MOD-matrix operator.

$x_3 = (1, 1, 1, 0) \in P$.

The resultant or effect of x_3 on the MOD matrix operator S is as follows.

$x_3S = (0, 0, 1, 0) = y_1$; $y_1S = (1, 1, 1, 0) = x_3$

is again a limit cycle using the MOD matrix operator S.

Consider $a_1 = (1, 1, 0, 0) \in P$.

The effect of a_1 on the MOD-matrix operator S is as follows: $a_1S = (1, 1, 0, 0)$.

So a_1 is a classical fixed point of the MOD matrix operator S.

Let $a_2 = (1, 0, 1, 0) \in P$.

The effect of a_2 on S; $a_2S = (0, 1, 0, 1) = b_1$;
$b_1S = (1, 0, 1, 0) \in P$.

a_2 is a limit cycle for
$(1, 0, 1, 0) \to (0, 1, 0, 1) \to (1, 0, 1, 0) \to (0, 1, 0, 1)$.

Let $a_3 = (1, 0, 0, 1) \in P$.

The resultant of a_3 on the MOD matrix operator S.
$a_3S = (0, 1, 1, 0) = b_1$.

$b_1S = (1, 0, 0, 1) = a_3$.

Thus a_3 is only a limit cycle for the MOD matrix operator S as

$(1, 0, 0, 1) \to (0, 1, 1, 0) \to (1, 0, 0, 1) \to (0, 1, 1, 0)\ldots$

Next consider $a_4 = (0, 1, 1, 0) \in P$.

The effect of a_4 on S is given in the following.

$a_4S = (1, 0, 0, 1) = b_1$; $b_1S = (0, 1, 1, 0) = a_4$.

Thus $a_4 \to b_1 \to a_4 \to b_1$ is a limit cycle.

Let $a_5 = (0, 1, 0, 1) \in P$.

The resultant of a_5 on S is as follows $a_5S = (1, 0, 1, 0) = b_1$.

$b_1S = (0, 1, 0, 1) = a_5$.

Thus a_5 is a limit cycle of the MOD matrix operator S as

$(0, 1, 0, 1) \to (1, 0, 1, 0) \to (0, 1, 0, 1) \to (1, 0, 1, 0)$.

Let $a_6 = (0, 0, 1, 1) \in P$.

The resultant of a_6 on the MOD matrix operator S is as follows.

$a_6S = (0, 0, 1, 1)$ is again a classical fixed point of S.

Consider $d_1 = (1, 0, 0, 0) \in S$.

The resultant of d_1 on S is as follows.

$d_1S = (1, 0, 1, 1) = b_1$, $b_1S = (1, 0, 0, 0) = d_1$.

Thus $(1, 0, 0, 0) \to (1, 0, 1, 1) \to (1, 0, 0, 0) \to (1, 0, 1, 1)$ is only a limit cycle of the MOD matrix operator S.

Let $d_2 = (0, 1, 0, 0) \in P$; $d_2S = (0, 1, 1, 1) = b_1$;
$b_1S = (0, 1, 0, 0)$.

Thus
$(0, 1, 0, 0) \to (0, 1, 1, 1) \to (0, 1, 0, 0) \to (0, 1, 1, 1)$ is again limit cycle of the MOD matrix operator S.

Let $d_3 = (0, 0, 1, 0) \in P$; the resultant of d_3 on the MOD-matrix operator S is $d_3S = (1, 1, 1, 0) = b_1$, $b_1S = (0, 0, 1, 0) = d_3$.

d_3 is a limit cycle of the MOD-matrix operator S as
(0, 0, 1, 0) → (1, 1, 1, 0) → (0, 0, 1, 0) → (1, 1, 1, 0).

Now let d_4 = (0, 0, 0, 1) ∈ P; the effect of d_4 on the MOD-matrix operator S is as follows.

d_4S = (1, 1, 0, 1) = b_1; b_1S = (0, 0, 0, 1) = d_4.

Thus
(0, 0, 0, 1) → (1, 1, 0, 1) → (0, 0, 0, 1) → (1, 1, 0, 1) is a limit cycle of the MOD matrix operator S.

Let c = (1, 1, 1, 1) ∈ P.

The effect of c on S is cS = (1, 1, 1, 1) = c is a fixed point.

Thus all the elements of P are either a fixed point of the MOD matrix operator or a limit cycle of length one.

Now we change the MOD-matrix operator from S to

$$M = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}.$$

Now to find the effect of the elements of P on M.

Let x_1 = (1, 0, 0, 0) ∈ P.

The effect of x_1 on M.

x_1M = (1, 1, 0, 0) = y_1; y_1M = (0, 1, 1, 1) = y_2 ;
y_2M = (1, 1, 0, 0) = y_3; y_3M = (0, 1, 1, 1) and so.

So x_1 is a limit cycle of the MOD matrix operator M

(1, 0, 0, 0) → (1, 1, 0, 0) → (0, 1, 1, 1) → (1, 1, 0, 0) →

$$(0, 1, 1, 1) \to (1, 1, 0, 0)$$

Let $x_2 = (0\ 1\ 0\ 0) \in P$; the effect of x_2 on M.

$x_2M = (1\ 0\ 1\ 1) = y_1$; $\quad y_1M = (1\ 0\ 1\ 1) = y_2 = y_1$

x_2 of P is only a realized fixed point of M as $x_2M \pm x_2$ but
$x_2M = (1\ 0\ 1\ 1) = y_1$; $\quad y_1M = y_1$.

Let $x_3 = (0\ 0\ 1\ 0) \in P$.

The effect of x_3 on M.

$x_3M = (0\ 0\ 0\ 1) = y_1$; $\quad y_1M = (0\ 1\ 1\ 0) = y_2$;
$y_2M = (1\ 0\ 1\ 0) = y_3$; $\quad y_3M = (1\ 1\ 0\ 1) = y_4$;
$y_4M = (0\ 0\ 0\ 1) = y_1$.

Thus
$(0\ 0\ 1\ 0) \to (0\ 0\ 0\ 1) \to (0\ 1\ 1\ 0) \to (1\ 0\ 1\ 0) \to (1\ 1\ 0\ 1) \to$
$(0\ 0\ 0\ 1) \to (0\ 1\ 1\ 0) \to (1\ 0\ 1\ 0) \to (1\ 1\ 0\ 1) \to (0\ 0\ 0\ 1)$.

So x_3 is a realized limit cycle on the MOD matrix operator M.

Consider $x_4 = (0\ 0\ 0\ 1)$; to find the effect of x_4 on M;

$x_4M = (0\ 1\ 1\ 0) = y_1$; $\quad y_1M = (1\ 0\ 1\ 0) = y_2$;
$y_2M = (1\ 1\ 0\ 1) = y_3$; $\quad y_3M = (0\ 0\ 0\ 1) = x_4$.

So x_4 is a realized classical fixed point as after four iterations $x_4M = x_4$.

Thus is a very special type of fixed point $x_5 = (1\ 1\ 0\ 0) \in P$.

The effect of x_5 on the MOD matrix operator M.

$x_5M = (0\ 1\ 1\ 1) = y_1$; $\quad y_1M = (1\ 1\ 0\ 0) = y_2$;
$y_2M = (0\ 1\ 1\ 1)$.

Thus x_5 is a realized fixed point of M after one iteration.

Let $x_6 = (1\ 0\ 1\ 0) \in P$, the effect of x_6 on M is given below

$x_6M = (1\ 1\ 0\ 1) = y_1$; $\quad$ $y_1M = (0\ 0\ 0\ 1) = y_2$;
$y_2M = (0\ 1\ 1\ 0) = y_3$; $\quad$ $y_3M = (1\ 0\ 1\ 0) = x_6$;

Thus x_6 is a realized fixed point of M after three iterations.

Consider $x_7 = (1\ 0\ 0\ 1) \in P$.

$x_7M = (1\ 0\ 1\ 0) = y_1$; $\quad$ $y_1M = (1\ 1\ 0\ 1) = y_2$;
$y_2M = (0\ 0\ 0\ 1) = y_3$; $\quad$ $y_3M = (0\ 1\ 1\ 0) = y_4$;
$y_4M = (1\ 0\ 1\ 0) = y_1$.

Thus the resultant of x_7 on M is a limit cycle given by
$(1\ 0\ 0\ 1) \to (1\ 0\ 1\ 0) \to (1\ 1\ 01) \to (0\ 0\ 0\ 1) \to (0\ 1\ 1\ 0) \to (1\ 0\ 1\ 0)$.

Next let $x_8 = (0\ 1\ 1\ 0) \in P$; to find effect of x_8 on M.

$x_8M = (1\ 0\ 1\ 0) = y_1$; $\quad$ $y_1M = (1\ 1\ 0\ 1) = y_2$;
$y_2M = (0\ 0\ 0\ 1) = y_3$; $\quad$ $y_3M = (0\ 1\ 1\ 0) = y_4 = x_8$.

Thus the row vector x_8 is a realized fixed point after three iterations.

Let $x_9 = (0\ 1\ 0\ 1) \in P$; to find the effect of x_9 on M.

$x_9M = (1\ 1\ 0\ 1) = y_1$; $\quad$ $y_1M = (0\ 0\ 0\ 1) = y_2$;
$y_2M = (0\ 1\ 1\ 0) = y_3$; $\quad$ $y_3M = (1\ 0\ 1\ 0) = y_4$;
$y_4M = (1\ 1\ 0\ 1) = y_5$.

Thus it is a realized fixed point as $(0\ 1\ 0\ 1) \to (1\ 1\ 0\ 1) \to (0\ 0\ 0\ 1) \to (0\ 1\ 1\ 0) \to (1\ 0\ 1\ 0) \to (1\ 1\ 0\ 1)$.

Let $x_{10} = (0\ 0\ 1\ 1) \in P$. To find the effect of x_{10} on M.

$x_{10}M = (0\ 1\ 1\ 1) = y_1$; $\quad$ $y_1M = (1\ 1\ 0\ 0) = y_2$;

$y_2M = (0\ 1\ 1\ 1) = y_3 = y_1$.

Thus $(0\ 0\ 1\ 1) \to (0\ 1\ 1\ 1) \to (1\ 1\ 0\ 0) \to (0\ 1\ 1\ 1)$.

Hence x_{10} is a realized fixed point of M.

$x_{11} = (1\ 1\ 1\ 0) \in P$.

$x_{11}M = (0\ 1\ 1\ 0) = y_1$; $\quad y_1M = (1\ 0\ 1\ 0) = y_2$;
$y_2M = (1\ 1\ 0\ 1) = y_3$; $\quad y_3M = (0\ 0\ 0\ 1) = y_4$;
$y_4M = (0\ 1\ 1\ 0) = y_5\ (= y_1)$.

Thus the state vector is a realized fixed point of M.

$x_{12} = (1\ 1\ 0\ 1) \in P$. To find the effect of x_{12} on M.

$x_{12}M = (0\ 0\ 0\ 1) = y_1$; $\quad y_1M = (0\ 1\ 1\ 0) = y_2$;
$y_2M = (1\ 0\ 1\ 0) = y_3$; $\quad y_3M = (1\ 1\ 0\ 1) = y_4 = x_{12}$.

The state vector $x_{12} = (1\ 1\ 0\ 1)$ is a realized fixed point after three iterations $x_{13} = (1\ 0\ 1\ 1) \in P$.

The effect of x_{13} on M is as follows.

$x_{13}M = (1\ 0\ 1\ 1) = x_{13}$ is a fixed classical point of M.

Let $x_{14} = (0\ 1\ 1\ 1) \in P$.

The effect of x_{14} on M is as follows.

$x_{14}M = (1\ 1\ 0\ 0) = y_1$; $\quad y_1M = (0\ 1\ 1\ 1) = y_2 = x_{14}$.

The point x_{14} is a realized fixed point after one iteration.

$x_{15} = (1\ 1\ 1\ 1) \in P$.

The effect of x_{15} on M is as follows.

$x_{15}M = (0\ 0\ 0\ 0) = y_1$; $\quad y_1M = (0\ 0\ 0\ 0)$.

Thus the effect of x_{15} is a fixed point or is zero.

So if S and M are two 4 × 4 matrices with entries from $Z_2 = \{0, 1\}$ the effect of each element varies as is clearly seen.

Let us consider the 4 × 4 matrix

$$N = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix} \text{ with entries from } Z_2.$$

To find the effect of
$P = \{(a_1, a_2, a_3, a_4)$ where $a_i \in Z_2 = \{0, 1\}; 1 \leq i \leq 4\}$ on N.

Let $x_1 = (1\ 0\ 0\ 0) \in P$

$x_1N = (1\ 0\ 0\ 0) = x_1$ is a classical fixed point of N.

Let $x_2 = (0\ 1\ 0\ 0) \in P$, to find effect of x_2 on N.

$x_2N = (0\ 1\ 1\ 0) = y_1$; $y_1N = (0\ 0\ 0\ 1) = y_2$;
$y_1N = (1\ 1\ 1\ 1) = y_3$; $y_3N = (0\ 1\ 1\ 0) = y_4$;
$y_4N = (0\ 0\ 0\ 1) = y_5 = (y_2)$.

We see

$x_2 = (0\ 1\ 0\ 0) \to (0\ 1\ 1\ 0) \to (0\ 0\ 0\ 1) \to (1\ 1\ 1\ 1) \to$
$(0\ 1\ 1\ 0) \to (0\ 0\ 0\ 1)$.

Thus x_2 is not the classical fixed point but x_2 is a realized fixed point.

Let $x_3 = (0\ 0\ 1\ 0) \in P$, to find effect of on z_3 on N.

$x_3N = (0\ 1\ 1\ 1) = y_1$; $y_1N = (1\ 1\ 1\ 0) = y_2$;

$y_2N = (1\ 0\ 0\ 1) = y_3$; $\quad y_3N = (0\ 1\ 1\ 1) = y_4\ (= y_1)$.

Thus x_3 is the realized fixed point of N.

Let $x_4 = (0\ 0\ 0\ 1) \in P$, to find effect of x_4 on N.

$x_4N = (1\ 1\ 1\ 1) = y_1$; $\quad y_1N = (0\ 1\ 1\ 0) = y_2$
$y_2N = (0\ 0\ 0\ 1) = x_4$ is a fixed point after two iterations.

$x_5 = (1\ 1\ 0\ 0) \in P$.

The effect of x_5 on N is as follows.

$x_5N = (1\ 1\ 1\ 0) = y_1$; $\quad y_1N = (1\ 0\ 0\ 1) = y_2$;
$y_2N = (0\ 1\ 1\ 1) = y_3$; $\quad y_3N = (1\ 1\ 1\ 0) = y_4 = y_1$.

Thus effect of x_5 is a realized fixed point
$(1\ 1\ 0\ 0) \to (1\ 1\ 1\ 0) \to (1\ 0\ 0\ 1) \to (0\ 1\ 1\ 1) \to (1\ 1\ 1\ 0)$.

To find the effect of $x_6 = (1\ 0\ 1\ 0) \in P$ on N.

$x_6N = (1\ 1\ 1\ 1) = y_1$; $\quad y_1N = (0\ 1\ 1\ 0) = y_2$;
$y_2N = (0\ 0\ 0\ 1) = y_3$; $\quad y_3N = (1\ 1\ 1\ 1) = y_4\ (= y_1)$.

Thus x_6 is a realized fixed point using N.

$x_7 = (1\ 0\ 0\ 1) \in P$.

To find the effect of x_7 on N.

$x_7N = (0\ 1\ 1\ 1) = y_1$; $\quad y_1N = (1\ 1\ 1\ 0) = y_2$;

$y_2N = (1\ 0\ 0\ 1) = x_7$; thus x_7 is a realized fixed point of N.

Consider $x_8 = (0\ 1\ 1\ 0) \in P$, to find the effect of x_8 on N.

$x_8N = (0\ 0\ 0\ 1) = y_1$; $\quad y_1N = (1\ 1\ 1\ 1) = y_2$;
$y_2N = (0\ 1\ 1\ 0) = y_3\ (= x_8)$.

Thus x_8 is a realized fixed point after two iterations.

Let $x_9 = (0\ 1\ 0\ 1) \in P$; to find the effect of x_9 on N.

$x_9N = (1\ 0\ 0\ 1) = y_1$; $\quad$ $y_1N = (0\ 1\ 1\ 1) = y_2$;
$y_2N = (1\ 1\ 1\ 0) = y_3$; $\quad$ $y_3N = (1\ 0\ 0\ 1) = y_4\ (=y_1)$.

Hence the effect is a realized fixed point given in the following.

$(0\ 1\ 0\ 1) \rightarrow (1\ 0\ 0\ 1) \rightarrow (0\ 1\ 1\ 1) \rightarrow (1\ 1\ 1\ 0) \rightarrow (1\ 0\ 0\ 1)$.

Consider $x_{10} = (0\ 0\ 1\ 1) \in P$.

To find the effect of x_{10} on N.

$x_{10}N = (1\ 0\ 0\ 0) = y_1$; $\quad$ $y_1N = (1\ 0\ 0\ 0) = y_2 = y_1$.

Thus x_{10} give the MOD realized fixed point after one iteration.

Let $x_{11} = (1\ 1\ 1\ 0) \in P$.

To find the effect of x_{11} on N.

$x_{11}N = (1\ 0\ 0\ 1) = y_1$; $\quad$ $y_1N = (0\ 1\ 1\ 1) = y_2$;
$y_2N = (1\ 1\ 1\ 0) = y_3$.

$(1\ 1\ 1\ 0) \rightarrow (1\ 0\ 0\ 1) \rightarrow (0\ 1\ 1\ 1) \rightarrow (1\ 1\ 1\ 0)$.

Thus x_{11} is a MOD realized fixed point of N.

$x_{12} = (1\ 1\ 0\ 1) \in P$.

To find the effect of x_{12} on N.

$x_{12}N = (0\ 0\ 0\ 1) = y_1$; $\quad$ $y_1N = (1\ 1\ 1\ 1) = y_2$;
$y_2N = (0\ 1\ 1\ 0) = y_3$; $\quad$ $y_3N = (0\ 0\ 0\ 1) = y_4\ (= y_1)$.

Hence

$(1\ 1\ 0\ 1) \rightarrow (0\ 0\ 0\ 1) \rightarrow (1\ 1\ 1\ 1) \rightarrow (0\ 1\ 1\ 0) \rightarrow (0\ 0\ 0\ 1)$ is a realized fixed point of x_{12}.

$x_{13} = (1\ 0\ 1\ 1) \in P$. To find the effect of x_{13} on N.

$x_{13}N = (0\ 0\ 0\ 0)$ is a realized fixed point.

Let $x_{14} = (0\ 1\ 1\ 1) \in P$.

To find the effect of x_{14} on N.

$x_{14}N = (1\ 1\ 1\ 0) = y_1$; $\quad y_1N = (1\ 0\ 0\ 1) = y_2$;
$y_2N = (0\ 1\ 1\ 1) = y_3\ (= x_{14})$.

Clearly x_{14} after some iteration is a fixed point of N.

$(0\ 1\ 1\ 1) \rightarrow (1\ 1\ 1\ 0) \rightarrow (1\ 0\ 0\ 1) \rightarrow (0\ 1\ 1\ 1)$.

Let $x_{15} = (1\ 1\ 1\ 1) \in P$.

To find the effect of x_{15} on N.

$x_{15}N = (0\ 1\ 1\ 0) = y_1$; $\quad y_1N = (0\ 0\ 0\ 1) = y_2$;
$y_2N = (1\ 1\ 1\ 1) = y_3$.

Thus x_{15} is a MOD realized fixed point as x_{15} only.

Hence use of three different MOD matrix operators give different effect on the elements of P.

Let us consider yet another MOD matrix operator with entries from $Z_2 = \{0, 1\}$.

Let $W = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix}$ be the MOD matrix operator.

To find effect of
$P = \{(a_1, a_2, a_3, a_4) \mid a_i \in \{0, 1\} = Z_2; 1 \leq i \leq 4\}$.

Let $x_1 = (1\ 0\ 0\ 0) \in P$. To find effect of x_1 on W.

$$x_1W = (1\ 0\ 0\ 0) = x_1.$$

Thus x_1 is a fixed point of the MOD matrix operator.

$$x_2 = (0\ 1\ 0\ 0) \in P.$$

To find the effect of x_2 on W.

$x_2W = (1\ 1\ 0\ 0) = y_1$; $\quad y_1W = (0\ 1\ 0\ 0) = y_2 = x_2$.

Thus x_2 is a realized fixed point.

Let $x_3 = (0\ 0\ 1\ 0) \in P$.

To find the effect of x_3 on W.

$x_3W = (1\ 1\ 1\ 0) = y_1$; $\quad y_1W = (1\ 0\ 1\ 0) = y_2$;
$y_2W = (0\ 1\ 1\ 0) = y_3$; $\quad y_3W = (0\ 0\ 1\ 0) = x_3$.

Thus x_3 is a realized fixed point of W.

$x_4 = (0\ 0\ 0\ 1) \in P$.

To find the effect of x_4 on W.

$x_4W = (1\ 1\ 1\ 1) = y_1$; $\quad y_1W = (0\ 1\ 0\ 1) = y_2$;
$y_2W = (0\ 0\ 1\ 1) = y_3$; $\quad y_3W = (0\ 0\ 0\ 1) = y_4 = x_4$.

Thus x_4 is a realized fixed point on W and leads to a same fixed point after four iterations.

Let $x_5 = (1\ 1\ 0\ 0) \in P$.

To find the effect of x_5 on W.

$x_5W = (0\ 1\ 0\ \ 0) = y_1$; $y_1W = (1\ 1\ 0\ 0) = y_2$ (= x_5).
$y_2W = (0\ 1\ 0\ 0) = y_3$.

Thus the realized fixed point of x_5.

$$\text{Hence } (1\ 1\ 0\ 0) \rightarrow (0\ 1\ 0\ 0) \rightarrow (1\ 1\ 0\ 0).$$

Let $x_6 = (1\ 0\ 1\ 0) \in P$.

To find the effect of x_6 on W.

$x_6W = (0\ 1\ 1\ 0) = y_1$; $y_1W = (0\ 0\ 1\ 0) = y_2$;
$y_2W = (1\ 1\ 1\ 0) = y_3$; $y_3W = (1\ 0\ 1\ 0) = x_6$.

$$(1\ 0\ 1\ 0) \rightarrow (0\ 1\ 1\ 0) \rightarrow (0\ 0\ 1\ 0) \rightarrow (1\ 0\ 1\ 0).$$

Thus after iteration this is a fixed point by the MOD operator W.

$x_7 = (1\ 0\ 0\ 1) \in P$.

To find the effect of x_7 on W.

$x_7W = (0\ 1\ 1\ 1) = y_1$; $y_1W = (1\ 1\ 0\ 1) = y_2$;
$y_2W = (1\ 0\ 1\ 1) = y_3$; $y_3W = (1\ 0\ 0\ 1) = x_7$.

Thus we have
$(1\ 0\ 0\ 1) \rightarrow (0\ 1\ 1\ 1) \rightarrow (1\ 1\ 0\ 1) \rightarrow (1\ 0\ 1\ 1) \rightarrow (1\ 0\ 0\ 1)$ which gives a realized fixed point.

Let $x_8 = (0\ 1\ 1\ 0) \in P$.

To find the effect of x_8 on W.

$x_8W = (0\ 0\ 1\ 0) = y_1$; $y_1W = (1\ 1\ 1\ 0) = y_2$;
$y_2W = (1\ 0\ 1\ 0) = y_3$; $y_3W = (0\ 1\ 1\ 0) = x_8$.

Hence
(0 1 1 0) → (0 0 1 0) → (1 1 1 0) → (1 0 1 0) → (0 1 1 0).

Thus x_8 is a realized fixed point fixing the same point.

Let $x_9 = (0\ 1\ 0\ 1) \in P$.

To find the effect of x_9 on W.

$x_9W = (0\ 0\ 1\ 1) = y_1$; $\quad y_1W = (0\ 0\ 0\ 1) = y_2$;
$y_2W = (1\ 1\ 1\ 1) = y_3$; $\quad y_3W = (0\ 1\ 0\ 1) = x_9$.

(0 1 0 1) → (0 0 1 1) → (0 0 01) → (1 1 1 1) → (0 1 0 1).

Hence after three iterations we get the same point so x_9 is the realized fixed point of W.

$x_{10} = (0\ 0\ 1\ 1) \in P$.

To find the effect of x_{10} on W.

$x_{10}W = (0\ 0\ 0\ 1) = y_1$; $\quad y_1W = (1\ 1\ 1\ 1) = y_2$;
$y_2W = (0\ 1\ 0\ 1) = y_3$; $\quad y_3W = (0\ 0\ 1\ 1) = x_{10}$.

Hence
(0 0 1 1) → (0 0 01) → (1 1 1 1) → (0 1 0 1) → (0 0 1 1).

Thus x_{10} is a realized fixed point fixing x_{10} after three to four iterations.

$x_{11} = (1\ 1\ 1\ 0) \in P$.

To find the effect of x_{11} on W.

$x_{11}W = (1\ 0\ 1\ 0) = y_1$; $\quad y_1W = (0\ 1\ 1\ 0) = y_2$;
$y_2W = (0\ 0\ 1\ 0) = y_3$; $\quad y_3W = (1\ 1\ 1\ 0) = x_{11}$.

(1 1 1 0) → (1 0 1 0) → (0 1 1 0) → (0 0 1 0) → (1 1 1 0).

Thus x_{11} is a realized fixed point as x_{11} after three iterations.

Let $x_{12} = (1\ 1\ 0\ 1) \in P$.

To find the effect of x_{12} on W is as follows.

$x_{12}W = (1\ 0\ 1\ 1) = y_1$; $y_1W = (1\ 0\ 0\ 1) = y_2$;
$y_2W = (0\ 1\ 1\ 1) = y_3$; $y_3W = (1\ 1\ 0\ 1) = x_{12}$.

$(1\ 1\ 0\ 1) \to (1\ 0\ 1\ 1) \to (1\ 0\ 0\ 1) \to (0\ 1\ 1\ 1) \to (1\ 1\ 0\ 1)$.

Thus x_{12} is a realized fixed point after three iterations x_{12} is got $x_{13} = (0\ 1\ 1\ 1) \in P$.

To find the effect of x_{13} on W.

$x_{13}W = (1\ 1\ 0\ 1) = y_1$; $y_1W = (1\ 0\ 1\ 1) = y_2$;
$y_2W = (1\ 0\ 0\ 1) = y_3$; $y_3W = (0\ 1\ 1\ 1) = x_{13}$.

$(0\ 1\ 1\ 1) \to (1\ 1\ 0\ 1) \to (1\ 0\ 1\ 1) \to (1\ 0\ 0\ 1) \to (0\ 1\ 1\ 1)$.

Thus x_{13} is a realized fixed point after three iterations.

$x_{14} = (1\ 0\ 1\ 1) \in P$.

To find the effect of x_{14} on W.

$x_{14}W = (1\ 0\ 0\ 1) = y_1$; $y_1W = (0\ 1\ 1\ 1) = y_2$;
$y_2W = (1\ 1\ 0\ 1) = y_3$; $y_3W = (1\ 0\ 1\ 1) = x_{14}$.

$(1\ 0\ 1\ 1) \to (1\ 0\ 0\ 1) \to (0\ 1\ 1\ 1) \to (1\ 1\ 0\ 1) \to (1\ 0\ 1\ 1)$.

Thus x_{14} is a realized fixed point by the MOD matrix W and is a fixed point x_{14}.

Let $x_{15} = (1\ 1\ 1\ 1) \in P$.

To find effect of x_{15} on W.

$x_{15}W = (0\ 1\ 0\ 1) = y_1$; $\quad y_1W = (0\ 0\ 1\ 1) = y_2$;
$y_2W = (0\ 0\ 0\ 1) = y_3$; $\quad y_3W = (1\ 1\ 1\ 1) = x_{15}$.

$(1\ 1\ 1\ 1) \to (0\ 1\ 0\ 1) \to (0\ 0\ 1\ 1) \to (0\ 0\ 0\ 1) \to (1\ 1\ 1\ 1)$.

Thus x_{15} is a realized fixed point after 3 iterations.

We will give one example using MOD matrix operation with elements from Z_4.

Example 3.4: Let

$$S = \begin{bmatrix} 1 & 2 & 0 & 1 \\ 0 & 1 & 3 & 2 \\ 2 & 0 & 1 & 3 \\ 1 & 3 & 2 & 0 \end{bmatrix}$$

be a 4×4 MOD matrix operator with entries from Z_4.

$P = \{(a_1, a_2, a_3, a_4) \mid a_i \in Z_4; 1 \leq i \leq 4\}$ be the collection of 4^4 number of state vectors.

Let $x_1 = (1\ 0\ 0\ 0) \in P$.

To find the effect of x on S.

$x_1S = (1\ 2\ 0\ 1) = y_1$; $\quad y_1S = (2\ 3\ 0\ 1) = y_2$;
$y_2S = (3\ 2\ 3\ 0) = y_3$; $\quad y_3S = (1\ 0\ 3\ 0) = y_4$;
$y_4S = (3\ 2\ 3\ 2) = y_5$; $\quad y_5S = (3\ 2\ 1\ 0) = y_6$;
$y_6S = (1\ 0\ 3\ 2) = y_7$; $\quad y_7S = (1\ 0\ 3\ 2) = y_8\ (= y_7)$.

Thus x_1 gives a realized fixed point after 8 iterations.

$x_2 = (0\ 1\ 0\ 0) \in P$. To find the effect of x_2 on S.

$x_2S = (0\ 1\ 3\ 2) = y_1$; $\quad y_1S = (0\ 3\ 2\ 3) = y_2$;
$y_2S = (3\ 0\ 1\ 0) = y_3$; $\quad y_3S = (1\ 2\ 1\ 2) = y_4$;
$y_4S = (0\ 2\ 3\ 0) = y_5$; $\quad y_5S = (2\ 2\ 1\ 1) = y_6$;

$y_6S = (1\ 3\ 1\ 1) = y_7$; $\quad y_7S = (0\ 0\ 0\ 2) = y_8$;
$y_8S = (2\ 2\ 0\ 0) = y_9$; $\quad y_9S = (2\ 2\ 2\ 2) = y_{10}$;
$y_{10}S = (0\ 0\ 0\ 0) = y_{11}$; $\quad y_1S = (0\ 0\ 0\ 0)$.

Thus the resultant is a realized fixed point given by (0 0 0 0).

$x_3 = (0\ 0\ 1\ 0) \in P$.

To find the effect of x_3 on S.

$x_3S = (2\ 0\ 1\ 3) = y_1$; $\quad y_1S = (3\ 1\ 3\ 1) = y_2$;
$y_2S = (2\ 2\ 0\ 2) = y_3$; $\quad y_3S = (0\ 0\ 2\ 2) = y_4$;
$y_4S = (2\ 2\ 2\ 2) = y_5$; $\quad y_5S = (0\ 0\ 0\ 0) = y_6$;
$y_6S = (0\ 0\ 0\ 0) = y_7$.

Thus this is also a fixed point only; a realized fixed point and is (0 0 0 0).

Let $x_4 = (0\ 0\ 0\ 1) \in P$.

$x_4S = (1\ 3\ 2\ 0) = y_1$; $\quad y_1S = (0\ 1\ 3\ 1) = y_2$;
$y_2S = (0\ 0\ 0\ 3) = y_3$; $\quad y_3S = (3\ 1\ 2\ 0) = y_5$;
$y_5S = (3\ 3\ 1\ 1) = y_6$; $\quad y_6S = (1\ 0\ 0\ 0) = y_7$;
$y_7S = (1\ 2\ 0\ 1) = y_8$; $\quad y_8S = (2\ 3\ 0\ 1) = y_9$;
$y_9S = (3\ 2\ 3\ 0) = y_{10}$; $\quad y_{10}S = (1\ 0\ 1\ 0) = y_{11}$;
$y_{11}S = (3\ 2\ 1\ 0) = y_{12}$; $\quad y_{12}S = (1\ 0\ 3\ 3) = y_{13}$;
$y_{13}S = (2\ 3\ 1\ 2) = y_{14}$; $\quad y_{14}S = (2\ 3\ 2\ 3) = y_{15}$;
$y_{15}S = (1\ 0\ 1\ 2) = y_{16}$; $\quad y_{16}S = (1\ 0\ 1\ 0) = y_{17}$;
$y_{17}S = (3\ 2\ 1\ 0) = y_{18} = y_{12}$.

This is a realized limit cycle.

$x_5 = (2\ 0\ 0\ 1) \in P$.

To find the resultant of x_5 on S.

$x_5S = (3\ 3\ 2\ 2) = y_1$; $\quad y_1S = (1\ 3\ 3\ 3) = y_2$;
$y_2S = (2\ 1\ 2\ 0) = y_3$; $\quad y_3S = (2\ 1\ 1\ 2) = y_4$;

$y_4S = (2\ 3\ 0\ 3) = y_5$; $y_5S = (1\ 2\ 3\ 0) = y_6$;
$y_6S = (3\ 0\ 1\ 2) = y_7$; $y_7S = (3\ 0\ 1\ 2) = y_8$ (= y_7).

Thus the resultant of $x_5 = (2\ 0\ 0\ 1)$ is a realized fixed point (3 0 1 2).

For after seven iteration the effect of x_5 on the MOD operator matrix S results in the realized fixed point (3 0 1 2).

Let $x_6 = (2\ 0\ 2\ 0) \in P$.

To find the effect of x_6 on the MOD operator matrix S.

$x_6S = (2\ 0\ 2\ 0)$. Thus $x_6S = x_6$ is the classical fixed point by the MOD matrix operator S.

Consider $x_7 = (0\ 2\ 0\ 2) \in P$.

The effect of x_7 on the MOD matrix operator S.

$x_7S = (2\ 0\ 2\ 0) = y_1$; $y_1S = (2\ 0\ 2\ 0) = y_1$.

Thus x_7 is a realized fixed point after first iteration.

We get the fixed point (2 0 2 0).

Let $x_8 = (2\ 2\ 2\ 2) \in P$.

To find the effect of x_8 on the MOD matrix operator S.

$x_8S = (0\ 0\ 0\ 0) = y_1$; $y_1S = (0\ 0\ 0\ 0) = y_1$.

Thus this is a not a classical fixed point of S only, but realized fixed point of S.

Now the following observation is very important.

We see $x_6 = (2\ 0\ 2\ 0)$ is a classical fixed point so $x_6S = x_6$, $x_7 = (0\ 2\ 0\ 2)$ is a realized fixed point after one iteration given by (2 0 2 0).

We see $x_8 = (2\ 2\ 2\ 2) = x_6 + x_7$ sum of these two state vectors.

$x_8S = (0\ 0\ 0\ 0)$ is the realized fixed point.

So $x_8 = (2\ 2\ 2\ 2) = (2\ 0\ 2\ 0) + (0\ 2\ 0\ 2) = x_6 + x_7$ is such that $x_8S = x_6S + x_7S = (2\ 0\ 2\ 0) + (2\ 0\ 2\ 0) = (0\ 0\ 0\ 0)$

(As $x_7S = x_6$ and $x_6S = x_6$).

But will this property be true for all state vectors in P.

We see more illustrations about the behavior of the effect of these state vector before we arrive at any conclusion.

Let $x_9 = (1\ 2\ 1\ 2) \in P$.

The effect of x_9 on S is given in the following.

$x_9S = (1, 2, 3, 0) = y_1$; $y_1S = (3, 0, 3, 2) = y_2$;
$y_2S = (3, 0\ 3\ 0) = y_3$; $y_3S = (1\ 2\ 3\ 0) = y_4\ (= y_1)$.

Thus x_9 is a realized fixed cycle given by (1, 2, 3, 0).

Let $x_{10} = (2\ 1\ 2\ 1) \in P$.

The effect of x_{10} on S is given in the following.

$x_{10}S = (3\ 0\ 3\ 2) = g_1$; $g_1S = (3\ 0\ 3\ 0) = g_2$;
$g_2S = (1\ 2\ 3\ 0) = g_3$; $g_3S = (3\ 0\ 3\ 2) = g_4\ (=g_1)$.

Thus the resultant of x_{10} is a realized limit cycle.

Consider $x_{11} = x_9 + x_{10}$
$= (1\ 2\ 1\ 2) + (2\ 1\ 2\ 1)$

$= (3\ 3\ 3\ 3) \in P.$

To find the effect of x_{11} on S

$x_{11}S = (0\ 2\ 2\ 2) = y_1$; $y_1S = (2\ 0\ 2\ 2) = y_2$;
$y_2S = (0\ 2\ 2\ 0) = y_3$; $y_3S = (0\ 2\ 0\ 2) = y_4$;
$y_4S = (2\ 0\ 2\ 0) = y_5$; $y_5S = (2\ 0\ 2\ 0) = y_6$ $(=y_5)$

is a realized fixed point.

$x_9 = (1, 2, 1, 2)$ the resultant associated with it is (1 2 3 0).

For $x_{10} = (2, 1, 2, 1)$ the resultant associated with it is (3 0 3 2). $x_{11} = x_9 + x_{10}$ but resultant of x_{11} is (0 2 2 2).

Thus there is no relation with this sum on S.

Let $x_{12} = (1\ 0\ 3\ 0)$ and $x_{13} = (3\ 0\ 1\ 0) \in P$.

We will find the effect of x_{12} and x_{13} on S.

$x_{12}S = (0\ 2\ 3\ 2) = y_1$; $y_1S = (3\ 2\ 1\ 0) = y_2$;
$y_2S = (1\ 0\ 3\ 2) = y_3$; $y_3S = (1\ 0\ 3\ 2) = y_4 (= y_3)$.

Thus the resultant of x_{12} is a realized fixed point (1 0 3 2) -- I

The resultant of x_{13} on S.

$x_{13}S = (1\ 2\ 1\ 2) = y_1$; $y_1S = (0\ 2\ 3\ 0) = y_2$;
$y_2S = (2\ 2\ 1\ 1) = y_3$; $y_3S = (1\ 1\ 1\ 1) = y_4$;
$y_4S = (0\ 2\ 2\ 2) = y_5$; $y_5S = (2\ 0\ 0\ 2) = y_6$;
$y_6S = (0\ 2\ 0\ 2) = y_7$; $y_7S = (2\ 0\ 2\ 0) = y_8$;
$y_8S = (2\ 0\ 2\ 0) = y_7 (= y_8)$.

Thus the resultant of x_{13} is a realized fixed point given by

(2 0 2 0) – II

Now $x_{12} + x_{13} = (1\ 0\ 3\ 0) + (3\ 0\ 1\ 0) = (0\ 0\ 0\ 0)$ and effect of $(0\ 0\ 0\ 0)$ on S is $(0\ 0\ 0\ 0)$ it is a trivial classical fixed point.

$$\begin{aligned} \text{But } x_{13}S + x_{12}S &\rightarrow \text{I} + \text{II} \\ &= (10, 3, 2) + (2\ 0\ 2\ 0) \\ &= (3\ 0\ 1\ 2). \end{aligned}$$

Thus the resultant behaves in a chaotic way.

Hence in view of this example the following theorem is evident.

THEOREM 3.1: *Let $S = (a_{ij})$ be a $n \times n$ matrix with entries from Z_m be the MOD matrix operator on $P = \{(a_1, \ldots, a_n) / a_i \in Z_m; 1 \leq i \leq n\}$ the set of state vectors. If x and y $\in$ P and if resultant of x on the MOD matrix operator is t and that of y is s then the resultant of x + y on S need not in general be t + s.*

Proof. Follows from the above example.

In view of all these the following conjecture is left open.

Conjecture 3.1. If S and P be as in theorem 3.1. Characterize all those x and y $\in$ P such that the sum of the resultants of x and y is the resultant sum of x and y.

Conjecture 3.2. Let S and P be given as theorem 3.1.

(i) Characterize all those classical fixed points of S.
(ii) Can we say the fixed points are related to entries of S?

Conjecture 3.3. Let S and P be as in theorem 3.1.

Characterize all those realized fixed points of S.

Conjecture 3.4. Can we say for some MOD matrix operator S all elements of P are classical fixed points?

Conjecture 3.5. Does for some MOD operator say a $n \times n$ matrix S there is an element

$x \in \{(a_1, \ldots, a_n) \mid a_i \in Z_m;\ 1 \leq i \leq n\} = P$ which is fixed point after $n^n - 2$ iterations?

Conjecture 3.6. Can there be a MOD operator matrix S for which every element is a realized fixed point after 5 iterations each?

Example 3.5: Let

$$S = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 3 & 0 \\ 2 & 1 & 3 \end{bmatrix}$$

be a 3×3 MOD matrix operator with elements from Z_6.

$P = \{(a_1, a_2, a_3) \mid a_i \in Z_6;\ 1 \leq i \leq 3\}$ be the collection of all state vectors.

Let $x_1 = (1\ 1\ 1) \in P$.

To find the effect of x on S.

$x_1S = (0\ 0\ 0)$.

Thus x_1 is a realized fixed point not a classical fixed point.

Let $x_2 = (2\ 2\ 2) \in P$.

To find the effect of x_2 on S is as follows.

$x_2S = (0\ 0\ 0)$ is a MOD realized fixed point and a classical fixed point of S.

Let $x_3 = (3\ 3\ 3) \in P$.

To find the effect of x_3 on S.

$x_3S = (0\ 0\ 0)$ is a realized fixed not and is not a classical fixed point.

Let $x_4 = (4, 4, 4) \in P$. The effect of x_4 on S is as follows.

$x_4S = (0, 0, 0)$ is a realized fixed point not a classical fixed point.

Let $x_5 = (5, 5, 5) \in P$; the effect of x_5 on S is as follows.

$x_5S = (0, 0, 0)$ is a realized fixed point and not a classical fixed point.

Let $x_6 = (2, 0, 2) \in P$.

$x_6S = (0, 0, 0)$ is a realized fixed point.

$x_7 = (0, 2, 0) \in P$, the effect of S on x_7 is as follows:

$x_7S = (0, 0, 0)$, is a realized fixed point

$x_8 = (2, 0, 0) \in P$; the effect on x_8 on S is as follows.

$x_8S = (2, 4, 0) = y_1$; $y_1S = (2, 4, 0) = y_2$ $(=y_1)$.

The resultant is only a realized fixed point of S.

Let $x_9 = (0\ 0\ 2) \in P_1$ to find the effect of x_9 on S

$x_9S = (4\ 2\ 0) = y_1$; $\qquad$ $y_1S = (4\ 2\ 0) = y_2$ $(= y_1)$.

This is the realized fixed point.

$x_8 = (2\ 0\ 0)$ and $x_9 = (0\ 0\ 2)$; $\qquad$ $x_8 + x_9 = (2\ 0\ 2) = x_6$

$x_8S \rightarrow (2, 4, 0)$; $\qquad$ $x_9S \rightarrow (4, 2, 0)$

$(x_8 + x_9)S = x_6S$ as realized fixed point of P.

Let $x_{10} = (1\ 2\ 3) \in$ P; to find the effect of x_{10} on S is as follows.

$x_{10}S = (1\ 5\ 0) = y_1$; $y_1S = (4\ 5\ 3) = y_2$;
$y_2S = (1\ 5\ 3) = y_3$; $y_3S = (4\ 2\ 0) = y_4$;
$y_4S = (4\ 2\ 0) = y_5\ (= y_4)$.

Thus $x_{10} = (1, 2, 3)$ is only a realized fixed point.

$x_1 = (1\ 0\ 0) \in$ P.

The effect of x_1 on S.

$x_1S = (1\ 2\ 3) = y_1$; $y_1S = (1\ 5\ 0) = y_2$;
$y_2S = (4\ 5\ 3) = y_3$; $y_3S = (2\ 2\ 3) = y_4$;
$y_4S = (2\ 1\ 3) = y_5$; $y_5S = (5\ 4\ 3) = y_6$;
$y_6S = (5\ 1\ 0) = y_7$; $y_7S = (2\ 1\ 3) = y_8\ (= y_5)$.

We see x_1 is a limit cycle with (2, 1, 3) as the limit cycle.

$x_2 = (0\ 2\ 0)$. To find the effect of x_2 on S.

$x_2S = (0, 0, 0)$ is a realized fixed point.

Let $x_3 = (0, 0, 3) \in$ P.

To find the effect of x_3 on S.

$x_3S = (0, 3, 3) = y_1$; $y_1S = (3, 0, 3) = y_2$;
$y_2S = (3, 3, 0) = y_3$; $y_3S = (0, 3, 3) = y_4\ (=y_1)$.

Thus x_3 gives a realized limit cycle.

$$x_{10} = (1, 2, 3) = x_1 + x_2 + x_3$$

$$= (1\ 0\ 0) + (0\ 2\ 0) + (0\ 0\ 3)$$
$$= (1, 2, 3)\ (1\ 2\ 3)\ S \to (4, 2, 0)$$

is the realized fixed point (1 0 0) S $\to$ (2 1 3) is a MOD limit cycle.

(0 2 0)S = (0 0 0) is a realized fixed point.
(0 0 3)S = (0 3 3) is a limit cycle.

Sum of
(1 0 0) S + (0 2 0) S + (0 0 3) S = (2 1 3) + (0 0 0) + (0 3 3) = (2 4 0).

Thus we see effect of sum of three elements in P is not the resultant sum.

This is clearly shown by x_{10} = (1, 2, 3) $\in$ P.

Consider x_4 = (2 1 2) $\in$ P; to find the effect of x_4 on S.

x_4S = (3 3 0) = y_1 ; y_1S = (0 3 3) = y_2 ;
y_2S = (3 0 3) = y_3 ; y_3S = (3 3 0) = y_4 (=y_1).

The resultant is a realized limit cycle.

Let x_1 = (2 0 0) $\in$ P.

To find the effect of x_1 on S.

x_1S = (2 4 0) = y_1 ; y_1S = (2 4 0) = y_2

is realized fixed point.

Let x_2 = (0 1 0) $\in$ P; to find the effect of x_2 on S.

x_2S = (3 3 0) = y_1 ; y_1S = (0 3 3) = y_2 ;
y_2S = (3 0 3) = y_3 ; y_3S = (3 3 0) = y_4 (=y_1).

The resultant is a limit cycle.

Let x_3 = (0 0 2) $\in$ P, to find the effect of x_3 on S.

$x_3S = (4\ 2\ 0) = y_1$; $y_1S = (4\ 2\ 0) = y_2$ $(=y_1)$.

The resultant is a realized fixed point.

x_4S gives the limit cycle as (3, 3, 0)
x_1S is a fixed point (2, 4, 0)
x_2S is a limit cycle (3, 3 0)
x_3S is a fixed point (4, 2 0).

Sum of $x_1S + x_2S + x_3S$
$= (2, 4, 0) + (3, 3, 0) + (4, 2, 0)$
$= (3, 3, 0)$.

Here the resultant sum is the sum of the resultant.

Next we make the fixed point of MOD matrix operators using the state vectors as 0 or 1 tuples with operators of updating and threshold the state vectors.

This will be illustrated by the following example.

Example 3.6: Let

$$S = \begin{bmatrix} 3 & 2 & 1 & 4 & 0 \\ 5 & 0 & 2 & 1 & 4 \\ 1 & 3 & 0 & 2 & 1 \\ 0 & 4 & 3 & 0 & 5 \\ 2 & 1 & 4 & 5 & 2 \end{bmatrix}$$

be the MOD matrix operator with entries from Z_6.

Let $P = \{(x_1, x_2, x_3, x_5) \mid x_i \in \{0, 1\}; 1 \leq i \leq 5\}$ (where $x_i = 1$ it implies the state vector is on state if $x_i = 0$ then it is off state.

In this working the state vector at each stage will be updated and thresholded.

Let x = (1 0 1 0 0) $\in$ P.

To find the effect of x on S.

xS = (4 5 1 0 1) after updating and thresholding

xS = (4 5 1 0 1) $\to$ (1 1 1 0 1) = y_1;

y_1S = (5 0 1 5 1) after updating and threshold.

y_1S $\to$ (1 0 1 1 1) = y_2
y_2S = (0 4 2 5 2) $\to$ (1 1 1 1 1) = y_3 ;
y_3S = (5 4 4 0 0) $\to$ (1 1 1 0 0) = y_4 ;
y_4S = (3, 5, 3, 1, 5) $\to$ (1 1 1 1 1) = y_5 ;
y_5 S $\to$ (1 1 1 0 0).

Thus the resultant is a limit cycle.

Let x_1 = (0 1 1 0 0) $\in$ P.

To find the effect of x_1 on S.

x_1S = (0, 3, 2, 3, 5) $\to$ (0, 1, 1, 1 1) = y_1 ;
y_1S = (2, 2, 3, 2, 0) $\to$ (1, 1, 1, 1, 0) = y_2 ;
y_2S = (3, 3, 0, 1, 4) $\to$ (1, 1, 1, 1, 1) = y_3 ;
y_3S = (5 4 4 0 0) $\to$ (1 1 1 0 0) = y_4 ;
y_4S = (3 5 3 1 5) = (1, 1, 1, 1, 1) = y_5 ;
y_5S $\to$ (1 1 1 0 0) = y_6 (=y_4).

Once again the resultant of (0 1 1 0 0) is only a limit cycle.

Let x_3 = (0 0 0 1 1) $\in$ P.

To find the effect of x_3 on S.

x_3S = (2, 5, 1, 5, 1) $\to$ (1 1 1 1 1) = y_1 ;

y_1S = (5 4 4 0 0) $\to$ (1 1 1 1 1) = y_2 (=y_1).

Thus the resultant of x_3 is fixed point.

Let $z_1 = (1\ 0\ 0\ 0\ 0)$ and $z_2 = (0\ 0\ 0\ 1\ 0) \in P$.

We will find the effect of z_1, z_2 and $z_1 + z_2$ on the MOD matrix operator S.

$z_1S = (3, 2, 1, 4, 0) \to (1, 1, 1, 1, 0) = y_1$
$y_1S = (3\ 3\ 0\ 1\ 4) \to (1\ 1\ 0\ 1\ 1) = y_2$
$y_2S = (4\ 1\ 4\ 4\ 5) \to (1, 1, 1, 1, 1) = y_3$
$y_3S = (5\ 4\ 4\ 0\ 0) \to (1\ 1\ 1\ 0\ 0) = y_4$
$y_4S = (3\ 5\ 3\ 1\ 5) \to (1, 1, 1, 1, 1) = y_5$
$y_5S = (1, 1, 1, 0, 0) = y_6\ (=y_4)$.

Thus the resultant is a limit cycle.
$(1\ 1\ 1\ 1\ 1) \to (1\ 1\ 1\ 0\ 0) \to (1\ 1\ 1\ 1\ 1)$ --I

Consider the resultant of z_2 on S ($z_2 = (0\ 0\ 0\ 1\ 0)$)
$z_2S = (0\ 4\ 3\ 0\ 5) \to (0, 1, 1, 1, 1) = y_1$
$y_1S = (2\ 2\ 3\ 2\ 0) \to (1, 1, 1, 1, 0) = y_2$
$y_2S = (3, 3, 0, 1, 4) \to (1, 1, 0, 1, 1) = y_3$
$y_3S = (4\ 1\ 4\ 4\ 1) \to (1, 1, 1, 1, 1) = y_4$
$y_4S = (5, 4, 4, 0, 0) \to (1, 1, 1, 0, 0) = y_5$
$y_5S = (3\ 5\ 3\ 1\ 5) \to (1, 1, 1, 1, 1) = y_6\ (=y_4)$.

Thus the limit point of $z_2 = (0\ 0\ 0\ 1\ 0)$ is also the limit cycle given by
$(1, 1, 1, 1, 1) \to (1, 1, 1, 0, 0) \to (1, 1, 1, 1, 1)$ -- II

From I and II it is clear the resultant is the limit cycle.

Let $z_1 + z_2 = (1\ 0, 0, 1, 0)$, to find the effect of $z_1 + z_2$ on S.

$(z_1 + z_2)\ S = (3\ 0\ 4\ 4\ 5) \to (1\ 0\ 1\ 1\ 1) = y_1$;
$y_1S = (0\ 4\ 2\ 5\ 2) \to (1, 1, 1, 1, 1) = y_2$
$y_2S = (5, 4, 4, 0\ 0) \to (1, 1, 1, 0, 0) = y_3$
$y_3S = (3\ 5\ 3\ 1\ 5) \to (1, 1, 1, 1, 1) = y_4\ (= y_2)$.

Thus the resultant of $z_1 + z_2$ is also a limit cycle
$(1, 1, 1, 1, 1) \to (1, 1, 1, 0, 0) \to (1, 1, 1, 1, 1)$ --- III

Thus I, II and III are the same.

Hence the result of z_1, z_2 and their sum $z_1 + z_2$ is a limit cycle, which is the same limit cycle evident from I, II and III.

Let $x_1 = (0\ 0\ 1\ 0\ 0)$ and $x_2 = (0\ 0\ 1\ 0\ 1)$ to find the resultant of x_1, x_2 and $x_1 + x_2$ on the MOD matrix operator S.

$x_1S = (1\ 3\ 0\ 2\ 1) \to (1, 1, 1, 1, 1) = y_1$
$y_1S = (5, 4, 4, 0\ 0) \to (1, 1, 1, 0, 0) = y_2$
$y_2S = (3\ 5\ 3\ 1\ 5) \to (1, 1, 1, 1, 1) = y_3\ (=y_1)$
is a realized limit cycle.

$(1, 1, 1, 1, 1) \to (1, 1, 1, 0, 0) \to (1, 1, 1, 1, 1)$

Consider the effect of $x_2 = (0\ 0\ 1\ 0\ 1)$ on S.

$x_2S = (3\ 4\ 4\ 1\ 3) \to (1, 1, 1, 1, 1) = y_1$
$y_1S = (5, 4, 4, 0, 0) \to (1, 1, 1, 0, 0) = y_2$
$y_2S = (3, 5, 3, 1, 5) \to (1, 1, 1, 1, 1) = y_3$
$y_3S = (5, 4, 4, 0, 0) \to (1, 1, 1, 0, 0) = y_4\ (=y_2)$.

Thus we see the resultant of x_2 is a limit cycle given by
$(1, 1, 1, 1, 1) \to (1, 1, 1, 0, 0) \to (1, 1, 1, 1, 1)$.

Consider $x = x_1 + x_2 = (0\ 0\ 1\ 0\ 0) + (0\ 0\ 1\ 0\ 1) = (0\ 0\ 1\ 0\ 1)$.

So is again a limit cycle with same cycle.

Consider $x_1 = (0\ 0\ 0\ 0\ 1)$ and $x_2 = (0\ 0\ 0\ 1\ 0) \in P$.

To find the effect of x_1, x_2 and $x_1 + x_2$ on S.

$x_1S = (2\ 1\ 4\ 5\ 2) \to (1, 1, 1, 1, 1) = y_1$

$y_1S = (5, 4, 4, 0\ 0) \to (1, 1, 1, 0, 0) = y_2$
$y_2S = (3\ 5\ 3\ 1\ 5) \to (1, 1, 1, 1, 1) = y_3\ (=y_1)$.

Thus the resultant is a limit cycle.

Consider $x_2 = (0\ 0\ 0\ 1\ 0) \in P$.

To find the effect of x_2S.

$x_2S = (0\ 4\ 3\ 0\ 5) \to (0\ 1\ 1\ 1\ 0) = y_1$
$y_1S = (0\ 1\ 5\ 3\ 4) \to (0\ 1\ 1\ \ 1\ 1) = y_2$
$y_2S = (2\ 1\ 3\ 2\ 0) \to (1, 1, 1, 1, 0) = y_3$
$y_3S = (3, 3, 0, 1, 4) \to (1\ 1\ 0\ 1\ 1) = y_4$
$y_4S = (4\ 1\ 4\ 4\ 5) \to (1, 1, 1, 1, 1) = y_5$
$y_5S = (5, 4\ 4\ 0\ 0) \to (1, 1, 1, 0, 0) = y_6$
$y_6S = (3\ 5\ 3\ 1\ 5) \to (1, 1, 1, 1, 1) = y_7\ (=y_5)$.

Thus is again a limit cycle $x_1 + x_2 = (0\ 0\ 0\ 1, 1) = x$.

To find the effect of x on S.

$xS = (3\ 5\ 1\ 5\ 1) \to (1, 1, 1, 1, 1) = y_1$
$y_1S = (5, 4, 4, 0, 0) \to (1, 1, 1, 0, 0) = y_2$
$y_2S = (3, 5, 3, 1, 5) \to (1, 1, 1, 1, 1) = y_3\ (=y_1)$.

Thus this is also a limit cycle; infact the same limit cycle.

Next we proceed onto study effect of this same S using

$B = \{(a_1, a_2, a_3, a_4, a_5) \mid a_i \in Z_6;\ 1 \leq i \leq 6\}$.

We do not update or threshold the vectors in B.

Take $x_1 = (1\ 0\ 1\ 0\ 0) \in B$.

To find the resultant of x_1 S.

$$x_1S = (4, 5, 1, 0, 1) = y_1 \text{ (say)}$$

$y_1S = (4, 0, 2, 0, 5) = y_2$ (say)
$y_2S = (0, 1, 0, 3, 0) = y_3$
$y_3S = (5, 0, 1, 1, 3) = y_4$
$y_4S = (4, 2, 2, 1, 0) = y_5$
$y_5S = (0, 0, 5, 4, 3) = y_6$
$y_6S = (5, 0, 0, 1, 5) = y_7$
$y_7S = (1, 1, 4, 3, 3) = y_8$
$y_8S = (0, 5, 0, 4, 5) = y_9$
$y_9S = (5, 3, 0, 0, 2) = y_{10}$
$y_{10}S = (4, 0, 1, 3, 4) = y_{11}$
$y_{11}S = (3, 3, 5, 2, 0) = y_{12}$
$y_{12}S = (5, 5, 3, 1, 3) = y_{13}$
$y_{13}S = (1, 2, 0, 4, 4) = y_{14}$
$y_{14}S = (3, 4, 3, 2, 0) = y_{15}$
$y_{15}S = (2, 5, 5, 4, 3) = y_{16}$
$y_{16}S = (0, 2, 0, 2, 3) = y_{17}$
$y_{17}S = (4, 5, 4, 5, 0) = y_{18}$
$y_{18}S = (5, 4, 5, 3, 1) = y_{19}$
$y_{19}S = (0, 2, 2, 3, 2) = y_{20}$
$y_{20}S = (4, 2, 3, 4, 3) = y_{21}$
$y_{21}S = (1, 0, 2, 3, 1) = y_{22}$
$y_{22}S = (1, 3, 2, 1, 1) = y_{23}$
$y_{23}S = (4, 1, 2, 4, 3) = y_{24}$
$y_{24}S = (1, 3, 0, 0, 2) = y_{25}$
$y_{25}S = (4, 4, 3, 5, 4) = y_{26}$
$y_{26}S = (1, 5, 1, 4, 4) = y_{27}$
$y_{27}S = (1, 1, 3, 7, 1) = y_{28}$
$y_{28}S = (1, 4, 4, 4, 2) = y_{29}$
$y_{29}S = (1, 2, 5, 2, 2) = y_{30}$
$y_{30}S = (2, 3, 1, 2, 3) = y_{31}$
$y_{31}S = (4, 5, 2, 4, 1) = y_{32}$
$y_{32}S = (5, 1, 0, 0, 2) = y_{33}$.

We have not reached realized a fixed point or a realized limit even after 32 iterations.

However we are sure by $(6^5 - 1)$ iterations we will reach a realized fixed point or a limit cycle. For B is only a finite set hence the claim.

Next we find the effect of $x_2 = (0\ 0\ 0\ 0\ 1) \in B$ on S.

$x_2S = (2, 1, 4, 5, 2) = y_1$
$y_1S = (1, 2, 5, 3, 1) = y_2$
$y_2S = (2, 0, 0, 3, 0) = y_3$
$y_3S = (0, 4, 5, 2, 3) = y_4$
$y_4S = (1, 2, 2, 5, 1) = y_5$
$y_5S = (5, 5, 0, 3, 1) = y_6$
$y_6S = (5, 5, 5, 0, 1) = y_7$
$y_7S = (5, 3, 1, 4, 3) = y_8$ and so on.

However we will get the resultant as realized fixed point or a realized limit cycle with in $6^5 - 1$ iterations.

Next we proceed onto study the effect of a lower triangular MOD matrix operator.

Example 3.7: Let

$$S = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 2 & 0 & 0 \\ 3 & 3 & 2 & 0 \\ 2 & 2 & 2 & 2 \end{bmatrix}$$

be the lower triangular MOD matrix operator with entries from Z_4.

$P = \{(a_1\ a_2\ a_3\ a_4) \mid a_i \in Z_4;\ 1 \leq i \leq 4\}$ be the 4^4 number of state vectors.

Consider $x_1 = (1\ 0\ 0\ 0) \in P$.

$x_1S = (1\ 0\ 0\ 0) = x_1$ is a classical fixed point.

Let $x_2 = (2\ 0\ 0\ 0) \in P$ $x_2S = x_2$ is again a classical fixed point.

Let $x_3 = (3\ 0\ 0\ 0) \in P$.

$x_3S = x_3$ is the classical fixed point.

Thus this MOD tower triangular matrix operator has classical fixed points.

Let $x_4 = (1,1,1,1) \in P$.

$x_4S = (0\ 3\ 0\ 2) = y_1$

$y_1S = (2\ 2\ 0\ 0) = y_2$
$y_2S = (2\ 0\ 0\ 0) = y_3$
$y_3S = (2\ 0\ 0\ 0) = y_4$ $(=y_3)$ is only a realized fixed point.

Let $x_5 = (2\ 0\ 0\ 2) \in P$.

To find the effect of x_5 on S.

$x_5S = (2\ 0\ 0\ 0) = x_5$;

x_5 is also a classical fixed point.

Let $x_6 = (0, 1, 0, 0)$.

To find the effect of x_6 on S.

$x_6S = (2, 2, 0, 0) = y_1$
$y_1S = (2, 0, 0, 0) = y_3$ $(=y_2)$.

Thus (0 1 0 0) is only a realized fixed point and not a classical fixed point.

Let $x_7 = (0\ 0\ 1\ 0) \in P$.

To find the effect of x_7 on S.

$x_7S = (3\ 3\ 2\ 0) = y_1$
$y_1S = (2\ 3\ 0\ 0) = y_2$

$y_2S = (0\ 2\ 0\ 0) = y_3$
$y_3S = (0\ 0,\ 0,\ 0) = y_4\ (= y_3)$ is a realized fixed point.

Consider $x_8 = (0\ 0\ 0\ 1) \in P$.

The effect of x_8 is on S is as follows.

$x_8S = (2\ 2\ 2\ 2) = y_1$
$y_1S = (0\ 2\ 0\ 0) = y_2$
$y_2S = (0\ 0\ 0\ 0)$ is a realized fixed point

Now $x_4 = (1, 1, 1, 1) = x_1 + x_6 + x_7 + x_8 =$
$(1, 0, 0, 0) + (0, 1, 0\ 0) + (0, 0, 1, 0) + (0, 0, 0, 1)$.

x_4S gives (2, 0, 0, 0) as a realized fixed point.

x_1 is a classical fixed point (1 0, 0, 0)
x_6 is a realized fixed point (2, 0, 0, 0)
x_7 and x_8 are realized fixed point (0 0 0 0).

However $x_4S \neq x_1S + x_6S + x_7S + x_8S$

$(2000) \neq (1000) + (2, 0, 0, 0) + (0\ 0\ 0\ 0) + (0\ 0\ 0\ 0)$.

Hence the concept of sum of the resultant is a resultant sum is not true in general.

Let $x = (1\ 2\ 3\ 1) \in P$.

To find the effect of x on S.

$xS = (0, 3, 0\ 2) = y_1$
$y_1S = (2\ 2\ 0\ 0) = y_2$
$y_2S = (2\ 0\ 0\ 0) = y_3$
$y_3S = (2\ 0\ 0\ 0) = y_4\ (=y_3)$ is a realized fixed point.

Next we study MOD symmetric matrix operators.

Example 3.8: Let

$$S = \begin{bmatrix} 2 & 1 & 0 & 3 \\ 1 & 0 & 2 & 1 \\ 0 & 2 & 1 & 0 \\ 3 & 1 & 0 & 3 \end{bmatrix}$$

be the symmetric MOD matrix operator with entries from Z_4.

$P = \{(a_1, a_2, a_3, a_4) \mid a_i \in Z_4 \;\; 1 \leq i \leq 4\}$ be collection of state vectors.

Let $x_1 = (1\ 0\ 0\ 0) \in P$.

To find the effect of x on S.

$x_1S = (2\ 1\ 0\ 3) = y_1$
$y_1S = (2\ 1\ 2\ 0) = y_2$
$y_2S = (1\ 2\ 2\ 3) = y_3$
$y_3S = (1\ 0\ 2\ 2) = y_4$
$y_4S = (0\ 3\ 2\ 1) = y_5$
$y_5S = (2\ 1\ 0\ 2) = y_6$
$y_6S = (3\ 0\ 2\ 2) = y_7$
$y_7S = (0\ 1\ 2\ 3) = y_8$
$y_8S = (2\ 3\ 0\ 2) = y_9$
$y_9S = (1\ 0\ 2\ 3) = y_{10}$
$y_{10}S = (2\ 0\ 2\ 0) = y_{11}$
$y_{11}S = (0\ 2\ 2\ 2) = y_{12}$
$y_{12}S = (0\ 2\ 2\ 2) = y_{13}\ (=y_{12})$.

Thus x_1 is a realized fixed point given by (0 2 2 2).

Consider $x_2 = (0\ 1\ 0\ 0) \in P$.

The effect of x_2 on S is as follows.

$x_2S = (1\ 0\ 2\ 1) = y_1$
$y_1S = (1\ 2\ 2\ 2) = y_2$

$y_2S = (2\ 3\ 2\ 3) = y_3$
$y_3S = (1\ 1\ 0\ 2) = y_4$
$y_4S = (1\ 3\ 2\ 2) = y_5$
$y_5S = (3\ 3\ 0\ 0) = y_6$
$y_6S = (1\ 3\ 2\ 0) = y_7$
$y_7S = (1\ 1\ 0\ 2) = y_8$
$y_8S = (1\ 3\ 2\ 2) = y_9$
$y_9S = (3\ 3\ 0\ 0) = y_{10}\ (=y_6)$.

Thus the resultant is a realized limit cycle.

Let $x_3 = (0, 0, 1, 0) \in P$.

To find the effect of x_3 on S.
$x_3S = (0\ 2\ 1\ 0) = y_1$
$y_1S = (2\ 2\ 1\ 2) = y_2$
$y_2S = (0\ 2\ 1\ 2) = y_3$
$y_3S = (0\ 0\ 1\ 0) = y_4\ (=x_3)$.

Thus x_3 is only a realized fixed point of S.

Let $x_4 = (0\ 0\ 0\ 1) \in P$.

To find the effect of x_4 on S.

$x_4S = (3\ 1\ 0\ 3) = y_1$
$y_1S = (0\ 2\ 2\ 3) = y_2$
$y_2S = (3\ 3\ 2\ 3) = y_3$
$y_3S = (2\ 2\ 0\ 1) = y_4$
$y_4S = (1\ 3\ 0\ 3) = y_5$
$y_5S = (2\ 0\ 2\ 3) = y_6$
$y_6S = (1\ 1\ 2\ 3) = y_7$
$y_7S = (0\ 0\ 0\ 1) = y_8\ (= x_4)$.

Thus it is a realized fixed point as only after seven iterations we get x_4.

Let $x = x_1 + x_2 + x_3 + x_4$

$$(1, 1, 1, 1) = (1, 0, 0, 0) + (0, 1, 0, 0) + (0, 0, 1, 0) + (0, 0, 0, 1)$$
$$= (1, 1, 1, 1) \in P.$$

To find the effect of x on S.

$xS = (2\ 0\ 3\ 3) = y_1$
$y_1S = (1\ 3\ 3\ 3) = y_2$
$y_2S = (2, 2, 1, 3) = y_3$
$y_3S = (3\ 3\ 1\ 1) = y_4$
$y_4S = (2\ 2\ 3\ 3) = y_5$
$y_5S = (3\ 3\ 3\ 1) = y_6$
$y_6S = (0\ 2\ 1\ 3) = y_7$
$y_7S = (3\ 1\ 1\ 3) = y_8$
$y_8S = (0\ 0\ 3\ 3) = y_9$
$y_9S = (1\ 1\ 3\ 1) = y_{10}$
$y_{10}S = (2\ 0\ 1\ 3) = y_{11}$
$y_{11}S = (1\ 3\ 1\ 3) = y_{12}$
$y_{12}S = (2\ 2\ 3\ 3) = y_{13}\ (=y_5)$.

The resultant is only a realized limit cycle.

Consider $x = (1\ 3\ 1\ 3) \in P$.

To find the effect of x on S.

$xS = (2\ 2\ 3\ 3) = y_1$
$y_1S = (3\ 3\ 3\ 3) = y_2$
$y_2S = (2\ 0\ 1\ 1) = y_3$
$y_3S = (3\ 3\ 1\ 1) = y_4$
$y_4S = (0\ 2\ 3\ 3) = y_5$
$y_5S = (3\ 1\ 3\ 3) = y_6$
$y_6S = (0\ 0\ 1\ 3) = y_7$
$y_7S = (1\ 1\ 2\ 1) = y_8$
$y_8S = (2\ 2\ 0\ 3) = y_9$
$y_9S = (3\ 1\ 0\ 1) = y_{10}$
$y_{10}S = (2\ 0\ 2\ 1) = y_{11}$
$y_{11}S = (3\ 3\ 2\ 1) = y_{12}$

$y_{12}S = (0, 0, 0\ 3) = y_{13}$
$y_{13}S = (1\ 3\ 0\ 1) = y_{14}$
$y_{14}S = (0\ 2\ 2\ 1) = y_{15}$
$y_{15}S = (1\ 1\ 2\ 1) = y_{16}$
$y_{16}S = (2\ 2\ 0\ 3) = y_{17}$
$y_{17}S = (3\ 1\ 0\ 1) = y_{18}\ (=y_{10})$.

Thus the resultant is a realized limit cycle of S.

Consider x = (3, 1, 3, 1) $\in$ P.

To find the effect of x on S

$xS = (0\ 2\ 1\ 1) = y_1$
$y_1S = (1\ 3\ 1\ 1) = y_2$
$y_2S = (0\ 0\ 3\ 1) = y_3$
$y_3S = (3\ 3\ 3\ 3) = y_4$
$y_4S = (2\ 0\ 1\ 1) = y_5$
$y_5S = (3\ 3\ 1\ 1) = y_6$
$y_6S = (0\ 2\ 3\ 3) = y_7$
$y_7S = (3\ 1\ 3\ 3) = y_8$
$y_8S = (0\ 0\ 1\ 3) = y_9$
$y_9S = (1\ 1\ 1\ 1) = y_{10}$
$y_{10}S = (2\ 0\ 3\ 3) = y_{11}$
$y_{11}S = (1\ 3\ 3\ 3) = y_{12}$
$y_{12}S = (2\ 2\ 1\ 3) = y_{13}$
$y_{13}S = (3\ 3\ 1\ 1) = y_{14}\ (= y_6)$.

Thus the resultant of x is a realized limit point of S.

Now we have worked with symmetric MOD matrix operators and lower triangle MOD matrix operator of S.

Now we proceed onto study the resultant of column vectors on MOD matrix operators.

***Example 3.9**:* Let

$$A^{\perp} = \left\{\begin{pmatrix}0\\0\end{pmatrix}, \begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix}, \begin{pmatrix}2\\0\end{pmatrix}, \begin{pmatrix}0\\2\end{pmatrix}, \begin{pmatrix}1\\2\end{pmatrix}, \begin{pmatrix}2\\1\end{pmatrix}, \begin{pmatrix}1\\1\end{pmatrix}, \begin{pmatrix}2\\2\end{pmatrix}\right\}$$

be the set of state column vectors given in example 3.1.

Let

$$M = \begin{bmatrix}2 & 1\\1 & 2\end{bmatrix}$$

be the same MOD operator matrix as in example 3.1.

$$\text{Let } x = \begin{bmatrix}1\\0\end{bmatrix};$$

$$Mx = \begin{bmatrix}2 & 1\\1 & 2\end{bmatrix}\begin{bmatrix}1\\0\end{bmatrix} = \begin{bmatrix}2\\1\end{bmatrix} = y_1$$

$$My_1 = \begin{bmatrix}2 & 1\\1 & 2\end{bmatrix}\begin{bmatrix}2\\1\end{bmatrix} = \begin{bmatrix}2\\1\end{bmatrix} = y_2\ (= y_1).$$

Thus y_2 is a realized fixed point of M.

$$\text{Let } Z = \begin{bmatrix}2\\2\end{bmatrix} \in A^{\perp}.$$

To find the resultant of Z on M.

$$MZ = \begin{bmatrix}0\\0\end{bmatrix} \text{ is the fixed point.}$$

However if x = (1, 0), then xM is a realized fixed point (2, 1) = y.

Now for x = (2, 2), x is a realized fixed point (0, 0).

We give another example.

Example 3.10: Let

$$M = \begin{bmatrix} 3 & 2 & 1 & 5 \\ 4 & 0 & 3 & 1 \\ 2 & 1 & 1 & 0 \\ 1 & 0 & 4 & 2 \end{bmatrix}$$

be the MOD matrix operator.

Let $A^{\perp} = \left\{ \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{bmatrix} \mid a_i \in Z_6; 1 \leq i \leq 4 \right\}$ be the state vectors.

$$\text{Let } x = \begin{bmatrix} 3 \\ 0 \\ 2 \\ 0 \end{bmatrix} \in A^{\perp}.$$

$$Mx = \begin{bmatrix} 3 & 2 & 1 & 5 \\ 4 & 0 & 3 & 1 \\ 2 & 1 & 1 & 0 \\ 1 & 0 & 4 & 2 \end{bmatrix} \begin{bmatrix} 3 \\ 0 \\ 2 \\ 0 \end{bmatrix} = \begin{bmatrix} 5 \\ 0 \\ 2 \\ 5 \end{bmatrix} = y_1$$

$$My_1 = \begin{bmatrix} 3 & 2 & 1 & 5 \\ 4 & 0 & 3 & 1 \\ 2 & 1 & 1 & 0 \\ 1 & 0 & 4 & 2 \end{bmatrix} \begin{bmatrix} 5 \\ 0 \\ 2 \\ 5 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 5 \end{bmatrix} = y_2$$

$$My_2 = \begin{bmatrix} 3 & 2 & 1 & 5 \\ 4 & 0 & 3 & 1 \\ 2 & 1 & 1 & 0 \\ 1 & 0 & 4 & 2 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \\ 5 \end{bmatrix} = \begin{bmatrix} 3 \\ 5 \\ 1 \\ 4 \end{bmatrix} = y_3$$

$$My_3 = \begin{bmatrix} 3 & 2 & 1 & 5 \\ 4 & 0 & 3 & 1 \\ 2 & 1 & 1 & 0 \\ 1 & 0 & 4 & 2 \end{bmatrix} \begin{bmatrix} 3 \\ 5 \\ 1 \\ 4 \end{bmatrix} = \begin{bmatrix} 4 \\ 1 \\ 0 \\ 3 \end{bmatrix} = y_4$$

$$My_4 = \begin{bmatrix} 3 & 2 & 1 & 5 \\ 4 & 0 & 3 & 1 \\ 2 & 1 & 1 & 0 \\ 1 & 0 & 4 & 2 \end{bmatrix} \begin{bmatrix} 4 \\ 1 \\ 0 \\ 3 \end{bmatrix} = \begin{bmatrix} 2 \\ 5 \\ 3 \\ 5 \end{bmatrix} = y_5$$

$$My_5 = \begin{bmatrix} 3 & 2 & 1 & 5 \\ 4 & 0 & 3 & 1 \\ 2 & 1 & 1 & 0 \\ 1 & 0 & 4 & 2 \end{bmatrix} \begin{bmatrix} 2 \\ 5 \\ 3 \\ 5 \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \\ 0 \\ 5 \end{bmatrix} = y_6$$

$$My_6 = \begin{bmatrix} 3 & 2 & 1 & 5 \\ 4 & 0 & 3 & 1 \\ 2 & 1 & 1 & 0 \\ 1 & 0 & 4 & 2 \end{bmatrix} \begin{bmatrix} 2 \\ 4 \\ 0 \\ 5 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ 2 \\ 5 \end{bmatrix} = y_7$$

$$My_7 = \begin{bmatrix} 3 & 2 & 1 & 5 \\ 4 & 0 & 3 & 1 \\ 2 & 1 & 1 & 0 \\ 1 & 0 & 4 & 2 \end{bmatrix} \begin{bmatrix} 3 \\ 1 \\ 2 \\ 5 \end{bmatrix} = \begin{bmatrix} 2 \\ 5 \\ 3 \\ 3 \end{bmatrix} = y_8$$

$$My_8 = \begin{bmatrix} 3 & 2 & 1 & 5 \\ 4 & 0 & 3 & 1 \\ 2 & 1 & 1 & 0 \\ 1 & 0 & 4 & 2 \end{bmatrix} \begin{bmatrix} 2 \\ 5 \\ 3 \\ 3 \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \\ 0 \\ 2 \end{bmatrix} = y_9$$

$$My_9 = \begin{bmatrix} 3 & 2 & 1 & 5 \\ 4 & 0 & 3 & 1 \\ 2 & 1 & 1 & 0 \\ 1 & 0 & 4 & 2 \end{bmatrix} \begin{bmatrix} 4 \\ 2 \\ 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 4 \\ 2 \end{bmatrix} = y_{10}$$

$$My_{10} = \begin{bmatrix} 3 & 2 & 1 & 5 \\ 4 & 0 & 3 & 1 \\ 2 & 1 & 1 & 0 \\ 1 & 0 & 4 & 2 \end{bmatrix} \begin{bmatrix} 2 \\ 0 \\ 4 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \\ 2 \\ 4 \end{bmatrix} = y_{11}$$

$$My_{11} = \begin{bmatrix} 3 & 2 & 1 & 5 \\ 4 & 0 & 3 & 1 \\ 2 & 1 & 1 & 0 \\ 1 & 0 & 4 & 2 \end{bmatrix} \begin{bmatrix} 2 \\ 4 \\ 2 \\ 4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 4 \\ 0 \end{bmatrix} = y_{12}$$

$$My_{12} = \begin{bmatrix} 3 & 2 & 1 & 5 \\ 4 & 0 & 3 & 1 \\ 2 & 1 & 1 & 0 \\ 1 & 0 & 4 & 2 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 4 \\ 0 \end{bmatrix} = \begin{bmatrix} 4 \\ 0 \\ 4 \\ 4 \end{bmatrix} = y_{13}$$

$$My_{13} = \begin{bmatrix} 3 & 2 & 1 & 5 \\ 4 & 0 & 3 & 1 \\ 2 & 1 & 1 & 0 \\ 1 & 0 & 4 & 2 \end{bmatrix} \begin{bmatrix} 4 \\ 0 \\ 4 \\ 4 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \\ 0 \\ 4 \end{bmatrix} = y_{14}$$

$$My_{14} = \begin{bmatrix} 3 & 2 & 1 & 5 \\ 4 & 0 & 3 & 1 \\ 2 & 1 & 1 & 0 \\ 1 & 0 & 4 & 2 \end{bmatrix} \begin{bmatrix} 0 \\ 2 \\ 0 \\ 4 \end{bmatrix} = \begin{bmatrix} 0 \\ 4 \\ 2 \\ 2 \end{bmatrix} = y_{15}$$

$$My_{15} = \begin{bmatrix} 3 & 2 & 1 & 5 \\ 4 & 0 & 3 & 1 \\ 2 & 1 & 1 & 0 \\ 1 & 0 & 4 & 2 \end{bmatrix} \begin{bmatrix} 0 \\ 4 \\ 2 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ 0 \\ 0 \end{bmatrix} = y_{16}$$

$$My_{16} = \begin{bmatrix} 3 & 2 & 1 & 5 \\ 4 & 0 & 3 & 1 \\ 2 & 1 & 1 & 0 \\ 1 & 0 & 4 & 2 \end{bmatrix} \begin{bmatrix} 2 \\ 2 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \\ 0 \\ 2 \end{bmatrix} = y_{14}\ (= y_{10}).$$

Thus the resultant of x is a realized limit cycle. However from example 3.2 for $x^t = (3\ 0\ 2\ 0)$.

We get the resultant of the row vector x^t on the MOD matrix operator M is a realized limit cycle just after second iteration.

But x as a column state vector on the same MOD matrix operator M attain a realized limit cycle after 16 iterations and the values are transpose of each other.

So the following problems are thrown open.

Conjecture 3.7: Let M be the MOD, $n \times n$ matrix operator with entries from Z_m.

$P = \{(a_1, \ldots, a_n) \mid a_i \in Z_m\ 1 \leq i \leq n\}$ be the collection of row state vectors.

$$P^{\perp} = \left\{ \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} \mid a_i \in Z_m;\ 1 \leq i \leq n \right\}$$

be the collection of column state vectors.

i) If $x \in P$ and $x^t \in P^{\perp}$ be the row state vector and column state vector which has same entries then will xM and Mx^t result in same resultant that is y_1 is the resultant of xM. Then y_1^t is the resultant of Mx^t with same number of iterations.

ii) Will classical fixed points of row vectors x on M also be the classical fixed points of the column vectors x^t of M?

iii) Does there exist a MOD matrix operator M in which (i) and (ii) are true?

From the example 3.2 the questions proposed in the conjecture need not in general be true.

One has to however characterize those MOD matrix operators in which such results are true.

Example 3.11: Now consider the MOD matrix operator S given in example 3.3 of the chapter.

$$S = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{bmatrix}$$

is the MOD matrix operator operating on column state vectors.

$$B = \left\{ \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{bmatrix} \mid a_i \in \{0, 1\} = Z_2, 1 \leq i \leq 4 \right\}.$$

Take $y = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{bmatrix} \in B$.

To find the resultant of y on S

$$Sy = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} = y_1;$$

$$Sy_1 = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{bmatrix} = y.$$

Hence y on S as a column state vector behaves in the same way as $x = y^t$ as the row state vector.

$$\text{Let on s} = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix} \in \text{B}.$$

To find the effect of s on S

$$\text{Ss} = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} = s_1,$$

$$\text{Ss}_1 = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix} = s_2 \, (= s).$$

s and s^t of B and P behave in the same way on S for $s_2 = x_1^t$ (refer example 3.3).

$$\text{Let } t_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix} \in \text{B}.$$

To find the effect of t_2 on S.

$$\text{St}_2 = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} = t_3$$

$$St_3 = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix} = t_4 \; (= t_2).$$

Thus $t_2 = x_2^t$ behaves in the same manner.

Since S happens to be a symmetric MOD operator the result is obtained in this manner.

Hence we now consider a non symmetric MOD matrix operator with entries from Z_2 in the following example.

Example 3.12: Let

$$S = \begin{bmatrix} 1 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 \end{bmatrix}$$

be the MOD matrix operator with entries from Z_2.

Let $P = \{(a_1, a_2, a_3, a_4, a_5) \mid a_i \in Z_2 \;\; 1 \leq i \leq 5\}$ and

$$P^{\perp} = \left\{ \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \end{bmatrix} \mid a_i \in Z_2; 1 \leq i \leq 5 \right\}$$

be the row state vectors and column state vectors respectively.

Let x = (1 0 0 0 0) and $x^t = y = \begin{bmatrix}1\\0\\0\\0\\0\end{bmatrix}$ be two initial state vectors from P and $P^\perp$ respectively.

To find the effect of x on S.

$$xS = (1\ 1\ 0\ 0\ 1) = x_1$$
$$x_1S = (1\ 1\ 0\ 0\ 1) = x_2\ (= x_1).$$

Thus x is a realized fixed point of the MOD matrix operator S.

$$Sy = \begin{bmatrix}1\\0\\1\\0\\0\end{bmatrix} = y_1; \qquad Sy_1 = \begin{bmatrix}1\\0\\0\\1\\0\end{bmatrix} = y_2;$$

$$Sy_2 = \begin{bmatrix}1\\1\\1\\1\\1\end{bmatrix} y_3; \qquad Sy_3 = \begin{bmatrix}1\\0\\0\\0\\0\end{bmatrix} = y_4\ (=y).$$

Thus the resultant of y the column vector on S is a realized fixed point given as three iteration as y itself.

However for $y^t = x$ the resultant of the row vector is a realized fixed point different from x.

Let x = (0 1 0 0 0) and $y = x^t = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$ be the row state vector and column state vectors of P and $P^\perp$ respectively.

$$\begin{aligned} xS &= (0\ 1\ 0\ 1\ 0) = x_1 \\ x_1S &= (0\ 1\ 1\ 0\ 0) = x_2 \\ x_2S &= (1\ 0\ 1\ 1\ 1) = x_3 \\ x_3S &= (0\ 1\ 0\ 0\ 0) = x_4\ (= x). \end{aligned}$$

Thus the resultant is a realized fixed point same as that of x.

Consider $y = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$ to find the effect of y on the MOD matrix operator S.

$$Sy = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \\ 1 \end{bmatrix} = y_1; \qquad Sy_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \\ 1 \end{bmatrix} = y_2;$$

$$Sy_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \\ 0 \end{bmatrix} = y_3; \qquad Sy_3 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} = y_4;$$

$$Sy_4 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \\ 1 \end{bmatrix} y_5 \ (= y_1).$$

Thus the resultant is not a realized fixed point but a realized limit cycle given by

$$\begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \\ 1 \end{bmatrix} \to \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \\ 1 \end{bmatrix} \to \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \\ 0 \end{bmatrix} \to \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} \to \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \\ 1 \end{bmatrix}.$$

Thus in this case of x = (0 1 0 0 0) and $y = x^t = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$

we see x results in a realized fixed point which is x itself where as y is the realized limit cycle given by $\begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \\ 1 \end{bmatrix}$ which is not y.

Thus we see if x is a realized fixed point $x^t = y$ can be a realized limit cycle and so on.

Let x = (0 0 1 0 0) and $x^t = y = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$ to find the effect of x and y on the MOD matrix operator S.

$$xS = (1\ 1\ 1\ 0\ 1) = x_1$$
$$x_1S = (0\ 0\ 1\ 0\ 0) = x_2\ (= x_3)$$

is the realized fixed point after one iteration yielding x itself.

$Sy = (0\ 0\ 1\ 1\ 0) = y_1$; $Sy_1 = (0\ 1\ 1\ 0\ 1) = y_2$;
$Sy_2 = (0\ 1\ 1\ 1\ 1) = y_3$; $Sy_3 = (0\ 0\ 1\ 0\ 0) = y_4\ (= y)$.

Thus in this case the resultant of y is a realized fixed point after three iterations yielding $y = x^t$.

Hence in this case only the number of iteration vary for x and $x^t = y$.

Let x = (0 0 0 1 0) $\in$ P and $y = x^t = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \in P^{\perp}$.

To find the effect of x and y on S.

The effect x on S is as follows.

$xS = (0\ 0\ 1\ 1\ 0) = x_1$; $x_1S = (1\ 1\ 0\ 1\ 1) = x_2$;
$x_2S = (1\ 1\ 1\ 1\ 1) = x_3$; $x_3S = (0\ 0\ 0\ 1\ 0) = x_4$ (=x).

Thus the resultant is a realized fixed point after three iterations the resultant is x.

Let us now find the resultant of y on S.

$$Sy = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \\ 1 \end{bmatrix} = y_1; \qquad Sy_1 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} = y_2 \ (= y).$$

The resultant of y is also a realized fixed point giving the same y after one iteration.

Let x = (0 0 0 0 1) and $x^t = y = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$ be the row and column

state vector respectively from P and $P^{\perp}$.

To find the effect of x and S.

$$xS = (0\ 1\ 0\ 1\ 0) = x_1$$
$$x_1S = (0\ 1\ 1\ 0\ 0) = x_2$$
$$x_2S = (1\ 0\ 1\ 1\ 1) = x_3$$
$$x_3S = (0\ 1\ 0\ 0\ 0) = x_4$$
$$x_4S = (0\ 1\ 0\ 1\ 0) = x_5\ (=x_1).$$

Thus the resultant of x on the MOD matrix operator S is a limit cycle given by

$$(0\ 1\ 0\ 1\ 0) \to (0\ 1\ 1\ 0\ 0) \to (1\ 0\ 1\ 1\ 1) \to$$
$$(0\ 1\ 0\ 0\ 0) \to (0\ 1\ 0\ 1\ 0).$$

Now we find the effect of $y = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$ on S.

$$Sy = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} = y_1; \qquad Sy_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} = y_2;$$

$$Sy_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = y_3; \qquad Sy_3 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} = y_4;$$

$$Sy_4 = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} = y_5\ (=y_1).$$

Thus the resultant of y on the MOD matrix operator S is a realized limit cycle given after 3 iteration.

$$\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \to \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \to \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \to \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \to \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \to \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}.$$

However the vectors are not the transpose of each other.

Now we find the sum of the state vectors x = (1, 1, 1, 1, 1) and

$$y = x^t = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \text{ of P and } P^{\perp} \text{ respectively.}$$

$xS = (1\ 0\ 0\ 0\ 0) = x_1$
$x_1S = (1\ 1\ 0\ 0\ 1) = x_2$
$x_2S = (1\ 1\ 0\ 0\ 1) = x_3\ (= x_3)$ is a realized fixed point.

Now the effect of y = $\begin{bmatrix}1\\1\\1\\1\\1\end{bmatrix}$ on the MOD matrix operator S is given in the following.

$$Sy = \begin{bmatrix}1\\0\\0\\0\\0\end{bmatrix} = y_1; \qquad Sy_1 = \begin{bmatrix}1\\0\\1\\0\\0\end{bmatrix} = y_2;$$

$$Sy_2 = \begin{bmatrix}1\\0\\0\\1\\0\end{bmatrix} = y_3; \qquad Sy_3 = \begin{bmatrix}1\\1\\1\\1\\1\end{bmatrix} = y_4\ (=y).$$

Thus the resultant of y = $\begin{bmatrix}1\\1\\1\\1\\1\end{bmatrix}$ is a realized fixed point leading to same y after three iterations.

Now having seen examples the following result is mandatory.

THEOREM 3.2: *Let S be a n × n symmetric matrix with entries from Z_m be the MOD matrix operator,*

$$P = \{(a_1, a_2, \ldots, a_n) / a_i \in Z_\mu, 1 \le i \le n\} \text{ and}$$

$$P^\perp = \{\begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} / a_i \in Z_m; 1 \le i \le n\}$$

be the row state vectors and column state vectors respectively.

The resultant of the row vector x on S be a, then resultant of x^t on S is a^t and vice versa.

Proof follows from the simple fact that the MOD matrix operator is a symmetric operator.

If the MOD matrix operator s is not symmetric the predictions are different.

It is an open conjecture to find the classical fixed points of S for both P and $P^\perp$.

We provide one more example to this effect.

Example 3.13: Let

$$S = \begin{bmatrix} 3 & 1 & 2 & 0 & 6 \\ 1 & 0 & 1 & 1 & 0 \\ 2 & 1 & 4 & 0 & 5 \\ 0 & 1 & 0 & 1 & 0 \\ 6 & 0 & 5 & 0 & 2 \end{bmatrix}$$

MOD symmetric matrix with entries from Z_7.

$$\text{Let } x = (2\ 1\ 0\ 0\ 0) \in P \text{ and } y = \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \in P^{\perp}.$$

The effect of x on S is as follows.

$xS = (0\ 2\ 5\ 1\ 5) = x_1$; $x_1S = (0\ 6\ 5\ 3\ 0) = x_2$;
$x_2S = (2\ 1\ 5\ 2\ 4) = x_3$; $x_3S = (6\ 2\ 3\ 3\ 3) = x_4$;
$x_4S = (2\ 5\ 6\ 5\ 1) = x_5$; $x_5S = (1\ 6\ 3\ 3\ 2) = x_6$;
$x_6S = (6\ 0\ 2\ 2\ 4) = x_7$; $x_7S = (4\ 3\ 5\ 2\ 5) = x_8$;
$x_8S = (6\ 4\ 0\ 5\ 3) = x_9$; $x_9S = (5\ 4\ 3\ 2\ 0) = x_{10}$;
$x_{10}S = (2\ 3\ 5\ 6\ 3) = x_{11}$; $x_{11}S = (2\ 6\ 0\ 2\ 1) = x_{12}$;
$x_{12}S = (4\ 4\ 1\ 1\ 0) = x_{13}$; $x_{13}S = (4\ 6\ 2\ 5\ 1) = x_{14}$;
$x_{14}S = (0\ 6\ 4\ 4\ 1) = x_{15}$; $x_{15}S = (1\ 3\ 5\ 1\ 4) = x_{16}$;
$x_{16}S = (5\ 0\ 3\ 4\ 4) = x_{17}$; $x_{17}S = (3\ 5\ 0\ 4\ 4) = x_{18}$;
$x_{18}S = (3\ 0\ 3\ 2\ 5) = x_{19}$; $x_{19}S = (3\ 1\ 1\ 2\ 1) = x_{20}$;
$x_{20}S = (4\ 6\ 2\ 3\ 4) = x_{21}$; $x_{21}S = (4\ 2\ 0\ 2\ 0) = x_{22}$;
$x_{22}S = (0\ 6\ 3\ 4\ 3) = x_{23}$; $x_{23}S = (2\ 0\ 5\ 3\ 0) = x_{24}$;
$x_{24}S = (2\ 3\ 3\ 3\ 2) = x_{25}$; $x_{25}S = (6\ 1\ 1\ 6\ 3)$ and so on.

We see we are not in a position to arrive at the resultant, however before or at the end of $6^5 - 2$ iterations we will certainly get the resultant.

Now we try to find the effect of $y = x^t$ on this symmetric.

MOD-matrix operator S.

$$x^t = y = \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \in P^{\perp}.$$

To find the effect of y on S.

$$Sy = \begin{bmatrix} 0 \\ 2 \\ 5 \\ 1 \\ 5 \end{bmatrix} = y_1.$$

Clearly $y_1 = x_1^t$ so the first iteration is the transpose of the first iteration of x.

$$Sy_1 = \begin{bmatrix} 0 \\ 6 \\ 5 \\ 3 \\ 0 \end{bmatrix} = y_2 \text{ (also } y_2 = x_2^t\text{)}; \quad Sy_2 = \begin{bmatrix} 2 \\ 1 \\ 5 \\ 2 \\ 4 \end{bmatrix} = y_3 \text{ (also } y_3 = x_3^t\text{)}$$

$$Sy_3 = \begin{bmatrix} 6 \\ 2 \\ 3 \\ 3 \\ 3 \end{bmatrix} = y_4 \; (y_4 = x_4^t); \qquad Sy_4 = \begin{bmatrix} 2 \\ 5 \\ 6 \\ 5 \\ 1 \end{bmatrix} = y_5 \; (y_5 = x_5^t);$$

$$Sy_5 = \begin{bmatrix} 1 \\ 6 \\ 3 \\ 3 \\ 2 \end{bmatrix} = y_6 \; (y_6 = x_6^t); \qquad Sy_6 = \begin{bmatrix} 6 \\ 0 \\ 2 \\ 2 \\ 4 \end{bmatrix} = y_7 \; (x_7^t = y_7);$$

$$Sy_7 = \begin{bmatrix} 4 \\ 3 \\ 5 \\ 2 \\ 5 \end{bmatrix} = y_8 \; (x_8^t = y_7); \qquad Sy_8 = \begin{bmatrix} 6 \\ 5 \\ 0 \\ 5 \\ 3 \end{bmatrix} = y_9 \; (x_9^t = y_9);$$

$$Sy_9 = \begin{bmatrix} 5 \\ 4 \\ 3 \\ 2 \\ 0 \end{bmatrix} = y_{10} \; (x_{10}^t = y_{10}); \qquad Sy_{10} = \begin{bmatrix} 2 \\ 3 \\ 5 \\ 6 \\ 3 \end{bmatrix} = y_{11} \; (x_{11}^t = y_{11});$$

$$Sy_{11} = \begin{bmatrix} 2 \\ 6 \\ 0 \\ 2 \\ 1 \end{bmatrix} = y_{12} \; (x_{12}^t = y_{12}); \qquad Sy_{12} = \begin{bmatrix} 4 \\ 4 \\ 1 \\ 1 \\ 0 \end{bmatrix} = y_{13} \; (x_{13}^t = y_{13});$$

$$Sy_{13} = \begin{bmatrix} 4 \\ 6 \\ 2 \\ 5 \\ 1 \end{bmatrix} = y_{14} \ (x_{14}^t = y_{14}); \quad Sy_{14} = \begin{bmatrix} 0 \\ 4 \\ 6 \\ 4 \\ 1 \end{bmatrix} = y_{15} \ (x_{15}^t = y_{15})$$

$$Sy_{15} = \begin{bmatrix} 1 \\ 3 \\ 5 \\ 1 \\ 4 \end{bmatrix} = y_{16} \ (x_{16}^t = y_{16}); \quad Sy_{16} = \begin{bmatrix} 5 \\ 0 \\ 3 \\ 4 \\ 4 \end{bmatrix} = y_{17} \ (x_{17}^t = y_{17})$$

$$Sy_{17} = \begin{bmatrix} 3 \\ 5 \\ 0 \\ 4 \\ 4 \end{bmatrix} = y_{18} \ (y_{18} = x_{18}^t); \quad Sy_{18} = \begin{bmatrix} 3 \\ 0 \\ 3 \\ 2 \\ 5 \end{bmatrix} = y_{19} \ (y_{19} = x_{19}^t);$$

$$Sy_{19} = \begin{bmatrix} 3 \\ 1 \\ 1 \\ 2 \\ 1 \end{bmatrix} = y_{20} \ (y_{19} = x_{18}^t); \quad Sy_{20} = \begin{bmatrix} 4 \\ 6 \\ 2 \\ 3 \\ 4 \end{bmatrix} = y_{21} \ (y_{20} = x_{20}^t);$$

$$Sy_{21} = \begin{bmatrix} 4 \\ 2 \\ 0 \\ 2 \\ 0 \end{bmatrix} = y_{22}\ (y_{21} = x_{21}^t); \quad Sy_{22} = \begin{bmatrix} 0 \\ 6 \\ 3 \\ 4 \\ 3 \end{bmatrix} = y_{23}\ (y_{22} = x_{22}^t)$$

$$Sy_{23} = \begin{bmatrix} 2 \\ 0 \\ 5 \\ 3 \\ 0 \end{bmatrix} = y_{24}\ (y_{23} = x_{23}^t); \quad Sy_{24} = \begin{bmatrix} 2 \\ 3 \\ 3 \\ 3 \\ 2 \end{bmatrix} = y_{25}\ (x_{24}^t = x_{24})$$

$$Sy_{25} = \begin{bmatrix} 6 \\ 1 \\ 1 \\ 0 \\ 3 \end{bmatrix} = y_{26}\ (x_{25}^t = y_{25}).$$

We see at each stage the value of $Sy_t = (x_7S)^t$.

This effect is from the fact the MOD symmetric matrix operator.

Next we give an example of a MOD-matrix operator S for which we use only row state vector and column state vectors taking entries from {0, 1} and we at each stage update and threshold the state vector.

Example 3.14: Let

$$S = \begin{bmatrix} 3 & 2 & 1 & 4 & 0 \\ 1 & 0 & 2 & 1 & 2 \\ 0 & 1 & 0 & 2 & 1 \\ 1 & 4 & 3 & 0 & 2 \\ 2 & 3 & 4 & 1 & 3 \end{bmatrix}$$

be the MOD matrix operator with elements from Z_5.

Let $P = \{(a_1, a_2, a_3, a_4, a_5) \mid a_i \in \{0, 1\}; 1 \leq i \leq 5\}$ and

$$P^{\perp} = \left\{ \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \end{bmatrix} \mid a_i \in \{0, 1\}, 1 \leq i \leq 5 \right\}$$

be the collection of state vectors which state on or off state.

For the first time we work with vectors from P and $P^{\perp}$.

Consider $x = (1\ 0\ 0\ 0\ 0) \in P$

$xS = (3\ 2\ 1\ 4\ 0) \to (1, 1, 1, 1, 0) = x_1;$
$x_1S = (0\ 2\ 1\ 2\ 0) \to (1, 1, 1, 1, 0) = x_2.$

Thus the resultant is a realized fixed point of S.

Consider $x^t = y = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$ we find the effect of y on S.

$$Sy = \begin{bmatrix} 3 \\ 1 \\ 0 \\ 1 \\ 2 \end{bmatrix} \to \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \\ 1 \end{bmatrix} = y_1; \quad Sy_1 = \begin{bmatrix} 2 \\ 4 \\ 0 \\ 1 \\ 2 \end{bmatrix} \to \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \\ 1 \end{bmatrix} = y_2 \; (=y_1).$$

The resultant is a MOD realized fixed point and $x_2^t \neq y_2$ and so on.

Let $y = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$ and x = (0 1 0 0 0 0)

be the column state vector and row state vector respectively.

$xS = (1\ 0\ 2\ 1\ 2) \to (1\ \ 1\ \ 1\ \ 1\ \ 1) = x_1$
$x_1S = (2\ 0\ 0\ 3\ 3) \to (1\ \ 1\ \ 0\ \ 1\ \ 1) = x_2$
$x_2S = (2\ 4\ 0\ 1\ 2) \to (1\ \ 1\ \ 0\ \ 1\ \ 1) = x_3\ (=x_2).$

Thus it is a realized fixed point.

$$Sy = \begin{bmatrix} 2 \\ 0 \\ 1 \\ 4 \\ 3 \end{bmatrix} \to \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = y_1 \; ; \qquad Sy_1 = \begin{bmatrix} 0 \\ 1 \\ 4 \\ 0 \\ 3 \end{bmatrix} \to \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \\ 1 \end{bmatrix} = y_2;$$

$$Sy_2 = \begin{bmatrix} 3 \\ 4 \\ 2 \\ 4 \\ 0 \end{bmatrix} \to \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = y_3; \qquad Sy_3 = \begin{bmatrix} 0 \\ 4 \\ 3 \\ 3 \\ 0 \end{bmatrix} \to \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \\ 0 \end{bmatrix} = y_4;$$

$$Sy_4 = \begin{bmatrix} 2 \\ 3 \\ 3 \\ 2 \\ 3 \end{bmatrix} \to \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = y_5 \; (= y_1).$$

Thus the resultant is a realized limit point so x is a realized fixed point but x^t is a realized limit cycle.

Let x = (0 0 1 0 0) and $x^t = y = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$ be two state vectors.

We now study the effect of them on S.

$xS = (0\ 1\ 0\ 2\ 1) \to (0\ 1\ 1\ 2\ 1) = x_1$
$x_1S = (4\ 3\ 4\ 4\ 3) \to (1\ 1\ 1\ 1\ 1) = x_2$

$$x_2S = (2\ 0\ 0\ 3\ 3) \to (1\ 0\ 1\ 1\ 1) = x_3$$
$$x_3S = (1\ 0\ 3\ 3\ 1) \to (1\ 0\ 1\ 1\ 1) = x_4\ (= x_3).$$

Thus the resultant of (0 0 1 0 0) is a realized fixed point.

$$\text{Let } y = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}.$$

Consider

$$Sy = \begin{bmatrix} 1 \\ 2 \\ 0 \\ 3 \\ 4 \end{bmatrix} \to \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = y_1; \qquad Sy_1 = \begin{bmatrix} 0 \\ 1 \\ 4 \\ 0 \\ 3 \end{bmatrix} \to \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \\ 1 \end{bmatrix} = y_2;$$

$$Sy_2 = \begin{bmatrix} 3 \\ 4 \\ 2 \\ 4 \\ 0 \end{bmatrix} \to \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = y_3; \qquad Sy_3 = \begin{bmatrix} 0 \\ 4 \\ 3 \\ 3 \\ 0 \end{bmatrix} \to \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \\ 0 \end{bmatrix} = y_4;$$

$$Sy_4 = \begin{bmatrix} 2 \\ 3 \\ 3 \\ 2 \\ 3 \end{bmatrix} \to \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = y_5 = (y_1).$$

Thus the resultant is a realized limit cycle.

Here we see (0 0 1 0 0) is a realized fixed points where as

$$\begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

is a realized limit cycle.

Let

$$x = (0\ 0\ 0\ 1\ 0) \in P.$$

To find the effect of x_1 on S.

$$\begin{aligned}
xS &= (1\ 4\ 3\ 0\ 2) \to (1\ 1\ 1\ 1\ 1) = x_1 \\
x_1S &= (2\ 0\ 0\ 3\ 3) \to (1\ 0\ 0\ 1\ 1) = x_2 \\
x_2S &= (1\ 4\ 3\ 0\ 0) \to (1\ 1\ 1\ 0\ 0) = x_3 \\
x_3S &= (4\ 3\ 3\ 2\ 3) \to (1\ 1\ 1\ 1\ 1) = x_4\ (= x_1).
\end{aligned}$$

Thus the resultant of (0 0 0 1 0) is the realized limit cycle given by (1, 1, 1, 1, 1).

$$\text{Consider } y = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \in P^{\perp}.$$

To find the effect of y on S.

$$Sy = \begin{bmatrix} 4 \\ 1 \\ 2 \\ 0 \\ 1 \end{bmatrix} \to \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = y_1; \qquad Sy_1 = \begin{bmatrix} 0 \\ 1 \\ 4 \\ 0 \\ 3 \end{bmatrix} \to \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = y_2;$$

$$Sy_2 = \begin{bmatrix} 2 \\ 0 \\ 4 \\ 4 \\ 1 \end{bmatrix} \to \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \\ 1 \end{bmatrix} = y_3; \qquad Sy_3 = \begin{bmatrix} 3 \\ 1 \\ 3 \\ 1 \\ 0 \end{bmatrix} \to \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 0 \end{bmatrix} = y_4;$$

$$Sy_4 = \begin{bmatrix} 0 \\ 4 \\ 3 \\ 3 \\ 0 \end{bmatrix} \to \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \\ 0 \end{bmatrix} = y_5; \qquad Sy_5 = \begin{bmatrix} 2 \\ 3 \\ 3 \\ 2 \\ 3 \end{bmatrix} \to \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = y_6\ (=y_1).$$

Thus the resultant is a realized limit cycle.

Let $x = (0\ 0\ 0\ 0\ 1) \in P$; to find the effect of x on S.

$$xS = \begin{bmatrix} 2 \\ 3 \\ 4 \\ 1 \\ 3 \end{bmatrix} \to \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = x_1; \qquad x_1S = \begin{bmatrix} 0 \\ 1 \\ 4 \\ 0 \\ 3 \end{bmatrix} \to \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \\ 1 \end{bmatrix} = x_2;$$

$$x_2S = \begin{bmatrix} 3 \\ 4 \\ 2 \\ 4 \\ 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = x_3 \; (= x_1).$$

Thus we have the resultant of $\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$ is a realized limit cycle

given by $\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$.

Let $x = (1\ 0\ 0\ 0\ 1) \in P$.

To find the effect of x on S.

$xS = (0\ 0\ 0\ 0\ 3) \rightarrow (1\ 0\ 0\ 0\ 1)$.

Thus x is a classical fixed point on x.

Let $y = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \in P^{\perp}$.

To find the effect of y on S.

$$Sy = \begin{bmatrix} 3 \\ 3 \\ 1 \\ 3 \\ 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = y_1$$

$$Sy_1 = \begin{bmatrix} 0 \\ 1 \\ 4 \\ 0 \\ 3 \end{bmatrix} \rightarrow \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \\ 1 \end{bmatrix} = y_2; \qquad Sy_2 = \begin{bmatrix} 1 \\ 0 \\ 2 \\ 0 \\ 2 \end{bmatrix} \rightarrow \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 1 \end{bmatrix} = y_3;$$

$$Sy_3 = \begin{bmatrix} 4 \\ 0 \\ 2 \\ 1 \\ 4 \end{bmatrix} \rightarrow \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \\ 1 \end{bmatrix} = y_4; \qquad Sy_4 = \begin{bmatrix} 3 \\ 1 \\ 3 \\ 1 \\ 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = y_5\ (=y_1).$$

Thus the resultant is a realized limit cycle.

Hence (1 0 0 0 1) is a classical fixed point but $\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$ is not a classical fixed point only a realized limit cycle.

Thus we see the state row vectors behave many a times differently for in the case of state column vector evident from this example.

Example 3.15: Let

$$H = \begin{bmatrix} 6 & 1 & 0 & 8 \\ 0 & 7 & 1 & 6 \\ 5 & 0 & 2 & 1 \\ 1 & 2 & 3 & 0 \end{bmatrix}$$

be the MOD matrix operator with entries from Z_9.

We consider

$P = \{(x_1, x_2, x_3, x_4) \mid x_i \in Z_9\ 1 \leq i \leq 4\}$ and

$$P^{\perp} = \{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \mid x_i \in Z_9\ 1 \leq i \leq 4\}$$

to be the row state vectors and $P^{\perp}$ is the column state vectors.

To find the effect of $x = (1\ 0\ 0\ 0) \in P$ on H is as follows.

$xH = (6\ 1\ 0\ 8) \to (1, 1, 0, 1) = x_1$
$x_1H = (7\ 1\ 4\ 5) \to (1, 1, 1, 1) = x_2$
$x_2H = (3\ 1\ 6\ 6) \to (1, 1, 1\ 1) = x_3\ (= x_2)$.

Thus the resultant x is a realized fixed point (1, 1, 1, 1).

Let $y = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \in P^{\perp}$, to find the effect of y on H.

$$Hy = \begin{bmatrix} 6 \\ 0 \\ 5 \\ 1 \end{bmatrix} \to \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix} = y_1; \qquad Hy_1 = \begin{bmatrix} 5 \\ 7 \\ 8 \\ 4 \end{bmatrix} \to \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = y_2$$

$$Hy_3 = \begin{bmatrix} 6 \\ 5 \\ 8 \\ 6 \end{bmatrix} \to \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = y_3 \; (= y_2).$$

Thus this is again not a classical fixed point but only a realized fixed point.

Let $x = (1\ 1\ 0\ 0) \in P$
$xH = (6\ 8\ 1\ 5) \to (1\ 1\ 1\ 1) = x_1$
$x_1H = (3\ 1\ 6\ 6) \to (1\ 1\ 1\ 1) = x_2\ (= x_1)$.

Thus it is a realized fixed point of H.

Now we proceed onto propose a few problems for the reader.

Problems

1. Study the special features enjoyed by MOD-matrix operators.

2. Characterize those MOD-matrix operators which has every row vector to be a classical fixed point.

3. Does such MOD-matrix operator exist?

4. Let $S = \begin{bmatrix} 3 & 7 & 2 & 0 & 1 \\ 1 & 5 & 0 & 8 & 3 \\ 0 & 1 & 3 & 9 & 2 \\ 5 & 0 & 1 & 2 & 7 \\ 1 & 2 & 3 & 4 & 0 \end{bmatrix}$ be the MOD-matrix operator

   with entries from Z_{10}.

   Let $P = \{(a_1, a_2, a_3, a_4, a_5) \mid a_i \in Z_{10}; 1 \leq i \leq 5\}$ be the state row vectors and

   $P^{\perp} = \{\begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \end{bmatrix} \mid a_i \in Z_{10}; 1 \leq i \leq 5\}$ be state column vectors.

   i) Find all classical fixed points of S in P and $P^{\perp}$.
   ii) Can we say if x is the classical fixed point of S then $x^t \in P^{\perp}$ be the classical fixed point of S?
   iii) Find all limit points of S in P.
   iv) Compare these limits points of S in $P^{\perp}$.
   v) Show in general if x and y in P have x_t and y_t as the resultants in P. Then the resultant of $x + y \neq x_t + y_t$; that $(x + y)_t \neq x_t + y_t$.
   vi) Characterize all those points in P in which (v) is true; that is $(x + y)_t = x_t + y_t$.
   vii) Obtain any other special feature enjoyed by M.

5. Let S = $\begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 0 & 7 & 8 & 9 & 1 & 1 \\ 0 & 0 & 1 & 3 & 4 & 2 \\ 0 & 0 & 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 & 6 & 0 \\ 0 & 0 & 0 & 0 & 0 & 5 \end{bmatrix}$ be the MOD-matrix operator

   with entries in Z_{12}.

   Study questions (i) to (vii) of problem (4) for this S.

6. Let T = $\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 & 0 \\ 3 & 2 & 1 & 0 & 0 \\ 4 & 3 & 2 & 1 & 0 \\ 5 & 4 & 3 & 2 & 1 \end{bmatrix}$ be MOD matrix operator with

   entries from Z_6.

   Study questions (i) to (vii) of problem (4) for this T.

7. Let N = $\begin{bmatrix} 3 & 1 & 2 & 0 & 3 & 4 & 5 \\ 1 & 0 & 1 & 2 & 0 & 1 & 1 \\ 2 & 1 & 3 & 6 & 2 & 0 & 6 \\ 0 & 2 & 6 & 1 & 1 & 4 & 0 \\ 3 & 0 & 2 & 1 & 0 & 2 & 7 \\ 4 & 1 & 0 & 4 & 2 & 1 & 0 \\ 5 & 1 & 6 & 0 & 7 & 0 & 5 \end{bmatrix}$ be the MOD-symmetric

   matrix operator with entries in Z_8.

i) Study questions (1) to (vii) of problem (4) for this N.
ii) Obtain any other special feature associated with N.

8. Let M = $\begin{bmatrix} 3 & 1 & 2 & 4 & 3 \\ 4 & 0 & 2 & 1 & 0 \\ 3 & 3 & 1 & 1 & 2 \\ 1 & 4 & 4 & 2 & 1 \\ 2 & 0 & 3 & 4 & 3 \end{bmatrix}$ be the MOD skew symmetric

matrix operator with entries from Z_5.

i) Study questions (i) to (vii) of problem 4 for this M.
ii) Compare N of problem 7 with this M.
iii) Obtain all the distinct features associated with M.

9. Let S = $\begin{bmatrix} 0 & 5 & 0 & 1 & 5 \\ 2 & 0 & 6 & 4 & 2 \\ 1 & 1 & 0 & 3 & 8 \\ 7 & 0 & 7 & 0 & 6 \\ 0 & 5 & 2 & 3 & 0 \end{bmatrix}$ be the MOD-matrix operator.

i) Study questions (i) to (vii) of problem 4 for this S.
ii) Does the diagonal elements being zero contribute to any other special feature?

10. Let $W = \begin{bmatrix} 0 & 1 & 2 & 3 & 4 \\ 5 & 0 & 1 & 2 & 3 \\ 4 & 5 & 0 & 1 & 2 \\ 3 & 4 & 5 & 0 & 1 \\ 2 & 3 & 4 & 5 & 0 \end{bmatrix}$ and $W^{\perp} = \begin{bmatrix} 0 & 5 & 4 & 3 & 2 \\ 1 & 0 & 5 & 4 & 3 \\ 2 & 1 & 0 & 5 & 4 \\ 3 & 2 & 1 & 0 & 5 \\ 4 & 3 & 2 & 1 & 0 \end{bmatrix}$ be

   the MOD-matrix operators.

   i) Study questions (i) to (vii) of problem 4 for this S.
   ii) Compare the resultants of state vectors of W and $W^{\perp}$.

11. Let $P = \begin{bmatrix} 0 & 1 & 2 & 0 & 1 \\ 1 & 5 & 0 & 2 & 3 \\ 2 & 0 & 1 & 5 & 0 \\ 0 & 2 & 5 & 0 & 3 \\ 1 & 3 & 0 & 3 & 2 \end{bmatrix}$ and $P_1 = \begin{bmatrix} 0 & 1 & 2 & 0 & 1 \\ 1 & 5 & 9 & 2 & 3 \\ 2 & 0 & 1 & 5 & 0 \\ 0 & 2 & 5 & 6 & 3 \\ 1 & 3 & 9 & 3 & 2 \end{bmatrix}$ be

   two MOD matrix operators with entries from Z_{12}.

   i) Characterize all those state vectors which has same resultants in both P and P_1.
   ii) Will classical fixed points of P be classical fixed points of P_1?
   iii) Can a classical fixed point of P yield a different resultant by P_1 and vice versa?
   Justify your claim by examples.
   iv) Study questions (i) to (vii) of problem (4) for this P and P_1.
   v) Characterize those state vectors in P and P_1 which yield same resultants.

12. Let $W = \begin{bmatrix} 3 & 7 & 2 & 1 & 0 & 5 & 1 \\ 1 & 2 & 0 & 3 & 5 & 1 & 2 \\ 0 & 5 & 3 & 1 & 2 & 3 & 4 \\ 5 & 0 & 6 & 7 & 1 & 2 & 6 \\ 1 & 2 & 6 & 0 & 3 & 4 & 1 \\ 3 & 4 & 5 & 6 & 6 & 7 & 0 \\ 4 & 2 & 3 & 1 & 1 & 0 & 7 \end{bmatrix}$ be the MOD matrix

operator with entries from Z_8.

Let $B = \{(x_1, x_2, \ldots, x_7) \mid x_i \in \{0, 1\}; 1 \leq i \leq 7\}$ and

$B^{\perp} = \{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_7 \end{bmatrix} \mid x_i \in \{0, 1\}; 1 \leq i \leq 7\}$ be the state vectors

which signifies only the on or off state.

i) Study questions (i) to (vii) of problem 4 for this W and W^t.

ii) Characterize all classical fixed points of w and $W^{\perp}$. Do they coincide or are different?

iii) Can a classical fixed point of $W^{\perp}$ be a realized fixed point or a limit cycle of W? Justify?

iv) What is the resultant of x = (1 0 0 0 1 0 0) and

$$x^t = y\begin{bmatrix}1\\0\\0\\0\\1\\0\\0\end{bmatrix} \text{ on W and } W^{\perp}?$$

v) If x_1 = (1 0 1 0 0 1 0) and x_2 = (0 1 0 0 0 0 1) $\in$ B.

Find the resultant of x_1, x_2 and $x_1 + x_2$.

Are these resultants related or no relation exists.

vi) Let

$$y_1\begin{bmatrix}1\\1\\0\\0\\0\\0\\0\end{bmatrix} \text{ and } y_2\begin{bmatrix}0\\0\\0\\1\\1\\0\\0\end{bmatrix} \in B^{\perp}.$$

Find the resultant of y_1, y_2 and $y_1 + y_2$ on W.

Are they related or not related with each other?

13. Let M = $\begin{bmatrix} 3 & 0 & 1 & 2 & 5 & 7 & 6 & 2 \\ 1 & 2 & 3 & 0 & 4 & 2 & 1 & 6 \\ 6 & 1 & 2 & 3 & 0 & 5 & 3 & 1 \\ 1 & 2 & 3 & 4 & 5 & 0 & 6 & 7 \\ 0 & 9 & 2 & 3 & 4 & 5 & 6 & 1 \\ 7 & 2 & 0 & 4 & 5 & 2 & 1 & 3 \\ 1 & 2 & 3 & 1 & 2 & 3 & 6 & 0 \\ 2 & 3 & 5 & 6 & 7 & 2 & 0 & 5 \end{bmatrix}$ be the MOD matrix

operator with entries from Z_{10}.

Let P = $\{(x_1, x_2, .., x_8) \mid x_i \in Z_{10}\ 1 \leq i \leq 8\}$,

B = $\{(a_1, a_2, \ldots, a_8) \mid a_i \in \{0, 1\};\ 1 \leq i \leq 8\}$,

$P^{\perp} = \{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_8 \end{bmatrix} \mid x_i \in Z_{10};\ 1 \leq i \leq 8\}$ and

$B^{\perp} = \{\begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ \vdots \\ a_8 \end{bmatrix} \mid a_i \in \{0, 1\};\ 1 \leq i \leq 8\}$ be the state row vectors

and column vectors.

i) Study questions (i) to (vii) of problem (4) for this M

ii) Study questions (ii) and (iii) of problem (12) for this M.
iii) If x = (1 1 0 0 0 0 0 1) ∈ P find the resultant of x on M.
iv) If x_1 = (1 1 0 0 0 0 0 1) ∈ B find the resultant of x on M as a on and off state vector
v) Compare the resultants in (iii) and (iv).

14. Let $M_1 = \begin{bmatrix} 3 & 1 & 0 & 2 & 1 \\ 0 & 4 & 1 & 2 & 3 \\ 1 & 1 & 2 & 0 & 1 \\ 2 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 2 \end{bmatrix}$ be the MOD-matrix operator

with entries from Z_5.

Let $M_2 = \begin{bmatrix} 0 & 1 & 2 & 3 & 0 \\ 2 & 1 & 0 & 0 & 4 \\ 1 & 0 & 2 & 1 & 0 \\ 0 & 2 & 1 & 0 & 2 \\ 1 & 1 & 0 & 0 & 0 \end{bmatrix}$ be the MOD matrix operator

with entries in Z_5,

Let P = {$(x_1, x_2, x_3, x_4, x_5) \mid x_i \in Z_5; 1 \le i \le 5$} and

$P^\perp = \{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} \mid x_i \in z_5, 1 \le i \le 5\}$ be the state vectors.

i) If $x_1 \in P$ and a_1 is its resultant with respect to M_1 and a_2 its resultant with respect to M_2.
Will $a_1 + a_2$ be the resultant on the MOD operator matrix sum $M_1 + M_2$? Justify your claim.

ii) Characterize all those $x \in P$ and $x^t \in P^\perp$ such that (i) is true

iii) Will they be related or no relation exists?

15. Let $S = \begin{bmatrix} 3 & 1 & 1 & 0 & 0 & 2 & 3 \\ 0 & 1 & 0 & 2 & 1 & 0 & 3 \\ 1 & 0 & 1 & 0 & 2 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 2 & 0 & 2 & 0 & 2 & 0 & 2 \\ 1 & 1 & 0 & 0 & 2 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 & 2 & 0 \end{bmatrix}$ be the MOD matrix operator with entries from Z_4.

i) Let $x = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \in P^\perp$ be the state vector.

Find the resultant of x on S.

ii) If $x_1 = \begin{bmatrix} 2 \\ 2 \\ 2 \\ 2 \\ 2 \\ 2 \\ 2 \end{bmatrix} \in P^{\perp}$ find its resultant on S.

Are the resultants of x and x_1 related?

16. Let $M = \begin{bmatrix} 3 & 1 \\ 2 & 5 \end{bmatrix}$ and $M_1 = \begin{bmatrix} 1 & 2 \\ 0 & 4 \end{bmatrix}$ be two MOD matrix operators with entries from Z_6.

Let $x_1 = (0, 2)$ and $x_2 = (1, 3)$ be two initial state vector.

i) Find the resultants of x_1 and x_2 on M_1.
ii) Find the resultants of x_1 and x_2 on M_2
iii) Find the resultant of (1, 5) on M_1 and M_2.
iv) Find the resultant of x_1 and x_2 on $M = \begin{bmatrix} 4 & 3 \\ 2 & 3 \end{bmatrix}$.
v) Compare all the above results. Does these exist any relation between them?

17. Let $P_1 = \begin{bmatrix} 3 & 1 & 2 \\ 0 & 4 & 5 \\ 6 & 0 & 3 \end{bmatrix}$ and $P_2 = \begin{bmatrix} 3 & 0 & 6 \\ 1 & 4 & 0 \\ 2 & 5 & 3 \end{bmatrix}$ be two MOD matrix operators with entries from Z_7.

Let $x_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ and $x_2 \begin{bmatrix} 4 \\ 2 \\ 1 \end{bmatrix} \in P^{\perp}$.

i) Find the resultants of x_1 on P_1 and P_2.
ii) Find the resultant of x_2 on P_1 and P_2.

iii) Find the resultant of x_1 on $\begin{bmatrix} 6 & 1 & 1 \\ 1 & 1 & 5 \\ 1 & 5 & 6 \end{bmatrix} = P_3$

Compare the results in (i) with this resultant on P_3.
iv) Find the resultant of x_1 and x_2 on P_3.
v) Can we say the resultants of x_2 on P_1, P_2 and P_3 are in any way related?

18. Find all special features enjoyed by MOD matrix operators.

19. Can one characterize all those MOD matrix operators which give only limit cycle as the resultant?

20. Characterize those MOD matrix operators whose resultants are only classical fixed points.

21. Characterize all those MOD matrix operators whose resultants are only realized fixed points.

22. Let $B = \begin{bmatrix} 1 & 2 & 0 & 0 & 0 \\ 6 & 3 & 4 & 0 & 0 \\ 0 & 5 & 2 & 3 & 0 \\ 0 & 0 & 3 & 4 & 1 \\ 0 & 0 & 0 & 5 & 2 \end{bmatrix}$ be the MOD matrix operator

with entries from Z_8.

i) Mention all the special features enjoyed by B.

ii) If x = (3 1 2 0 6) and y = $\begin{bmatrix} 3 \\ 1 \\ 2 \\ 0 \\ 6 \end{bmatrix}$ find the resultant of x and y on B.

iii) If $x_1 = (1\ 2\ 3\ 4\ 5)$ and $x_2 = (5\ 4\ 3\ 2\ 1)$.

Find the resultant of x_1, x_2 and $x_1 + x_2$ on B.

Are these resultants related in any way?

iv) Find the resultants of $x_1^{\perp}, x_2^{\perp}$ and $x_1^{\perp} + x_2^{\perp}$ on B.

Are these resultants related in any way?

23. Let x = $\begin{bmatrix} 5 & 0 & 0 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 7 & 0 & 0 \\ 0 & 0 & 0 & 0 & 8 & 0 \\ 0 & 0 & 0 & 0 & 0 & 6 \end{bmatrix}$ be the MOD matrix operator with entries from Z_9.

Let P and $P^{\perp}$ be the state row vectors and state column vectors.

i) Study the special features associated with $\times$.
ii) Are all the resultant fixed points?

iii) Find the resultant of a = $\begin{bmatrix} 3 \\ 0 \\ 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$ and b = (4 3 2 1 0 3)

iv) Characterized all classical fixed points of $\times$.
v) Characterize all realized fixed points of $\times$.
vi) Can there by state row vectors and state column vectors whose resultants are realized limit cycles?
vii) Obtain all special features associated with diagonal matrix operators.

24. Let M = $\begin{bmatrix} 0 & 1 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 4 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 5 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 6 \end{bmatrix}$ be the MOD matrix

operator with entries from Z_7.

i) Study all the special features associated with this MOD matrix operators?
ii) If M^t be the MOD matrix operator, characterize all those state vectors of P and $P^{\perp}$?
  a) Which are classical fixed points?
  b) Which are realized fixed points.
  c) Which are realized limit cycles.

Chapter Four

FIXED POINTS OF MOD-MATRIX OPERATORS DEFINED ON $\langle Z_n \cup I \rangle$, $C(Z_n)$, $\langle Z_n \cup g \rangle$, $\langle Z_n \cup h \rangle$ AND $\langle Z_n \cup K \rangle$

In this chapter for the first time we study the MOD matrix operators using modulo neutrosophic numbers $\langle Z_n \cup I \rangle$ finite complex modulo integers $C(Z_n)$, dual modulo integers $\langle Z_n \cup g \rangle$; $g^2 = 0$, and so on.

We find the fixed points associated with them. Each of them behave in a very different way.

All these will be illustrated by examples.

Example 4.1: Let

$$S = \begin{bmatrix} 1+I & 0 & 2+3I \\ 2 & I & 2+I \\ I & 2+2I & 3 \end{bmatrix}$$

be a neutrosophic MOD matrix operator with entries from $\langle Z_4 \cup I\rangle = \{a + bI \mid a, b \in Z_4, I^2 = I\}$.

Let $P = \{(x_1, x_2, x_3) \mid x_i \in \langle Z_4 \cup I\rangle; 1 \leq i \leq 3\}$ be the collection of MOD neutrosophic state vectors.

To find the effect of $x = (1, 0, 1) \in P$.

$xS = (1 + 2I, 2 + 2I, 1 + 3I) = y_1;$
$y_1S = (1 + I, 2 + 2I, 1 + 2I) = y_2;$
$y_2S = (1 + 2I, 2 + 2I, 1) = y_3;$
$y_3S = (1 + I, 2 + 2I, 1 + I) = y_4;$
$y_4S = (1 + I, 2 + 2I, 3 + 3I) = y_5;$
$y_5S = (1 + I, 2 + 2I, 1 + I) = y_6 (= y_4).$

Thus the resultant of $x = (1, 0, 1)$ is a realized limit cycle.

Consider $y = (0, 1, 0) \in P$.

To find the effect of y on S.

$yS = (2, I, 2 + I) = y_1;$ $y_1S = (2 + 3I, I, 2) = y_2;$
$y_2S = (2, I, 2) = y_3;$ $y_3S = (2 + 2I, I, 2) = y_4;$
$y_4S = (2 + 2I, I, 3I + 2) = y_5;$ $y_5S = (1 + 2I, I, 0) = y_6;$
$y_6S = (2, I, 3I) = y_7;$ $y_7S = (2 + 3I, I, 2I) = y_8;$
$y_8S = (2, I, 2I) = y_9;$ $y_9S = (2 + 2I, I, 2I) = y_{10};$
$y_{10}S = (2 + 2I, I, 2I) = y_{11} (=y_{10}).$

Thus the resultant of $y = (0, 1, 0)$ is a realized fixed point given by $(2 + 2I, I, 2I)$.

Consider $x + y = (1, 0, 1) + (0, 1, 0) = (1, 1, 1) = t$.

To find the resultant of t on S.

$tS = (3 + 2I, 2 + 3I, 3) = t_1;$ $t_1S = (3, 2 + 3I, 3 + 2I) = t_2;$
$t_2S = (3 + 2I, 2 + 3I, 3 + 2I) = t_3;$
$t_3S = (3 + 2I, 2 + 3I, 3 + I) = t_4;$
$t_4S = (3 + 2I, 2 + 3I, 3 + I) = t_5 (= t_4).$

Thus the resultant of $x + y = (1, 1, 1)$ is a realized fixed point given by $(3 + 2I, 2 + 3I, 3 + I)$.

However the sum of the resultant of x and y is

$(1 + I, 2 + 2I, 1 + I) + (2 + 2I\ \ I\ \ 2I) = (3 + 3I, 2 + 3I\ \ 1 + 3I)$.

They are not related for in the first place $x = (1, 0, 1)$ gives a resultant which is a limit cycle what as that of $x + y = (1, 1, 1)$ is a realized fixed point.

Let $x = (1 + I, 0, 0) \in P$ to find the effect of x on S.

$xS = (1 + 3I, 0, 2) = y_1$; $\quad y_1S = (1 + I, 0, 2I) = y_2$;
$y_2S = (1 + I, 0, 2) = y_3$; $\quad y_3S = (1 + I, 0, 0) = y_4$;
$y_4S = (1 + 3I, 0, 2) = y_5\ (= y_1)$.

The resultant is a realized limit cycle.

Let $x = (1 + 2I, 1 + I, 2 + 3I) \in P$.

To find the effect of x on S.

Consider

$xS = (3, 2I, 2 + 3I) = y_1$; $\quad y_1S = (3, 2I, 0) = y_2$;
$y_2S = (3 + 3I, 2I, 2 + 3I) = y_3$; $\quad y_3S = (3 + 2I, 2I, 3I) = y_4$;
$y_4S = (3I, 2I, 2 + 2I) = y_5$; $\quad y_5S = (2I, 2I, 2 + 3I) = y_6$;
$y_6S = (2I, 2I, 3I) = y_7$; $\quad y_7S = (3I, 2I, I) = y_8$;
$y_8S = (3I, 2I, 0) = y_9$; $\quad y_9S = (2I, 2I, 0) = y_{11}\ (= y_{10})$.

The resultant is a fixed point given by $(2I, 2I, 0)$.

Next we give examples of neutrosophic MOD matrix operator.

Example 4.2: Let M = $\begin{bmatrix} 3I+2 & 0 & 0 & 0 \\ 0 & 4+I & 0 & 0 \\ 0 & 0 & 5I+2 & 0 \\ 0 & 0 & 0 & 4+3I \end{bmatrix}$

be the neutrosophic MOD matrix operator with entries from $\langle Z_6 \cup I \rangle$.

Let x = (3 + 2I, 0, 0, 2+5I) $\in$ P = $\{(a_1, a_2, a_3, a_4) \mid a_i \in \langle Z_6 \cup I\rangle; 1 \leq i \leq 4\}$.

xM = (3I 0, 0, 2 + 5I) = y_1;

y_1M = (3I, 0, 0, 2 + 5I) = y_2 (= y_1) is a realized fixed point of M.

Let x = (0, 1+I, 3+I, 0) $\in$ P.

To find the effect of x on M.

xM = (0, 4, 4I, 0) = y_1; y_1M = (0, 4 + 4I, 4I, 0) = y_2;
y_2M = (0, 4, 4I, 0) = y_3 (=y_1).

Thus the resultant is a realized limit cycle.

Consider x = (3 + 2I, 1 + 4I, 2 + 3I, 4 + I) $\in$ P.

To find the effect of x on M.

xM = (I, 3I + 4, 4 + I, 4 + I) = y_1;
y_1M = (5I, 4 + I, 3I + 2, 4 + I) = y_2;
y_2M = (I, 4 + 3I, 4 + I, 4 + I) = y_3;
y_3M = (5I, 4 + I, 3I + 2, 4 + I) = y_4 (= y_2).

Thus we see the resultant is a limit cycle.

Example 4.3: Let

$$S = \begin{bmatrix} 3+2I & 4+3I \\ 1+2I & 4+2I \end{bmatrix}$$

be the neutrosophic MOD matrix operator.

Let P = {(x, y) | x, y ∈ $\langle Z_5 \cup I\rangle$ = {a + bI | a, b ∈ Z_5}}.

Let x = (3 + 2I, 1 + 4I) ∈ P.

To find the effect of x on S.

xS = (0, I + 4) = y_1; y_1S = (4 + I, 1 + 3I) = y_2;
y_2S = (3 + 3I, 4I) = y_3; y_3S = (2 + 3I, 2+4I) = y_4;
y_4S = (3, 1) = y_5; y_5S = (3I, 1 + I) = y_6;
y_6S = (1, 4 + 4I) = y_7; y_7S = (2 + 2I, 0) = y_8;
y_8S = (1 + 4I, 3) = y_9; y_9S = (1 + 3I, 1 + 2I) = y_{10}

and so on.

However we will have a realized fixed point or a limit cycle as the set P is finite.

Let us consider x = (I, I) ∈ P.

To find the effect of x on S.

xS = (3I, 3I) = y_1; y_1S = (4I, 4I) = y_2;
y_2S = (2I, 2I) = y_3; y_3S = (I, I) = y_4 (= x).

Thus the resultant is a realized fixed point which is x itself after 3 iterations.

Now we will see the MOD matrix neutrosophic operator when the matrix is symmetric, skew symmetric upper triangular and super diagonal by an example each.

Example 4.4: Let S = $\begin{bmatrix} 0 & 3+2I & 0 & 0 \\ 2I & 0 & 1+3I & 0 \\ 0 & 0 & 0 & 2+2I \\ 0 & 0 & 2+6I & 0 \end{bmatrix}$

be the neutrosophic MOD matrix operator with entries in $\langle Z_7 \cup I\rangle$.

P = $\{(a_1, a_2, a_3, a_4) \mid a_i \in \langle Z_7 \cup I\rangle; 1 \leq i \leq 4\}$ be the collection of neutrosophic state vectors.

Let x = (3 + 4I, 4I, 3I, 2) $\in$ P.

The effect of x on S is

xS = (I, 6I, 4, 5I) = y_1.

$$y_1S = (I, 6I, 4, 5I)\begin{bmatrix} 0 & 3+2I & 0 & 0 \\ 2I & 0 & 1+3I & 0 \\ 0 & 0 & 0 & 2+2I \\ 0 & 0 & 2+6I & 0 \end{bmatrix}$$

= (5I, 5I, I, 1 + I) = y_2

y_2S = (3I, I, 2 + 3I, 4I) = y_3; y_3S = (2I, I, I, 4) = y_4;
y_4S = (2I, 3I, 1, 4I) = y_5;
y_5S = (2I, 3I, 2I, 2 + 2I) = y_6;
y_6S = (6I, 3I, 4, I) = y_7; y_7S = (6I, 2I, I, I) = y_8;
y_8S = (4I, 2I, 2I, 4I) = y_9; y_9S = (4I, 6I, 5I, I) = y_{10}
and so on.

However we will reach a realized fixed point or a limit cycle as P is a finite set.

Example 4.5: Let $S = \begin{bmatrix} 3+2I & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 3I & 0 \\ 0 & 1+4I & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 \\ 0 & 0 & 2+I & 0 & 0 \end{bmatrix}$ be the neutrosophic MOD matrix operator with entries from $\langle Z_6 \cup I\rangle$.

Let $P = \{(a_1, a_2, a_3, a_4, a_5) \mid a_i \in \langle Z_6 \cup I\rangle; 1 \leq i \leq 5\}$ be the collection of state vectors.

To find the effect of

$x = (3 + 2I, I, 4, 2, 3+I) \in P.$
$xS = (3 + 4I, 4 + 4I, 0, 3I, 4) = y_1;$
$y_1S = (3 + 2I, 0, 2 + 4I, 0, 0) = y_2;$
$y_2S = (3 + 4I, 2 + 4I, 0, 0, 0) = y_3;$
$y_3S = (3 + 2I, 0, 0, 0, 0) = y_4;$
$y_4S = (3 + 4I, 0, 0, 0, 0) = y_5;$
$y_5S = (3 + 2I, 0, 0, 0, 0) = y_6 \ (= y_4).$

Thus the resultant is a realized limit cycle.

Example 4.6: Let $S = \begin{bmatrix} 3+I & 2I & 0 & 1 \\ 2I & 0 & 1+I & 0 \\ 0 & 1+I & 2 & I \\ 1 & 0 & I & 1+3I \end{bmatrix}$

be the neutrosophic MOD symmetric matrix operator with entries from $\langle Z_4 \cup I\rangle$.

Let $x = (2 + I, 3, 0, 1) \in P = \{(a_1, a_2, a_3, a_4) \mid a_i \in \langle Z_4 \cup I\rangle;$
$1 \leq i \leq 4\}.$

The effect of x on S is as follows.

$xS = (3, 2I, 3, 3) = y_1;$
$y_1S = (3I, 3+I\ 2+2I, 2) = y_2;$
$y_2S = (2I, 2, 3+3I, 2+I) = y_3;$
$y_3S = (2+I, 3 + I, 3I, 2) = y_4;$
$y_4S = (2I, 0, 3 + I, 2I) = y_5;$
$y_5S = (2I, 2I, 2, 2I) = y_6;$
$y_6S = (2I, 2 + 2I, 2I, 0) = y_7;$
$y_7S = (0, 0, 2, 0) = y_8;$
$y_8S = (0, 2 + 2I, 0, 2I) = y_9;$
$y_9S = (2I, 0, 2, 0) = y_{10};$
$y_{10}S = (0\ 2+2I, 0, 0) = y_{11};$
$y_{11}S = (0, 2 + 2I, 0, 0) = y_{12} (= y_{11}).$

Thus the resultant is a realized fixed point after 10 iterations.

Let x = (3, 1, 0, 0) ∈ P to find the effect of x on the MOD operator

$$xS = (3, 1, 0, 0)\begin{bmatrix} 3+I & 2I & 0 & 1 \\ 2I & 0 & 1+I & 0 \\ 0 & 1+I & 2 & I \\ 1 & 0 & I & 1+3I \end{bmatrix}.$$

$$= (1 + I, 2I, 1 + I, 3) = y_1;$$

$y_1S = (2 + I, 1 + 3I, 1 + 2I, 0) = y_2;$
$y_2S = (2 + 2I, 1 + I, 3 + 3I, 2) = y_3;$
$y_3S = (2I, 3 + I, 3 + 3I, 2I) = y_4;$
$y_4S = (2I, 3+I, 1 + I, 0) = y_5;$
$y_5S = (0, 1 + 3I, 1 + 3I, 0) = y_6;$
$y_6S = (0, 1 + 3I, 3 + 3I, 0) = y_7;$
$y_7S = (0, 3+I, 3 + I, 2I) = y_8;$
y_8S and so on.

This certainly we will arrive at a realized fixed point or a realized limit cycle.

Example 4.7: Let $B = \begin{bmatrix} 0 & 1+I & 0 & 0 \\ 2I & 0 & 1+2I & 0 \\ 0 & 2+I & 0 & 1+2I \\ 0 & 0 & 2+I & 0 \end{bmatrix}$

be the neutrosophic MOD matrix with entries from $\langle Z_3 \cup I\rangle$.

Let $x = (2I, 0, 0, 2) \in P = \{(a_1, a_2, a_3, a_4) \mid a_i \in \langle Z_3 \cup I\rangle; 1 \leq i \leq 4\}$.

The effect of x on B.

$$xB = (2I, 0, 0, 2) \begin{bmatrix} 0 & 1+I & 0 & 0 \\ 2I & 0 & 1+2I & 0 \\ 0 & 2+I & 0 & 1+2I \\ 0 & 0 & 2+I & 0 \end{bmatrix}$$

$= (0, I, 1+2I, 0) = y_1;$
$y_1B = (2I, 2+I, 1+2I, 0) = y_2;$
$y_2B = (I, 2 + 2I, 2 + I, 1 + 2I) = y_3;$
$y_3B = (2I, 1 + 2I, 1 + 2I, 2 + I) = y_4;$
$y_4B = (0, 2 + 2I, 1 + 2I, 1 + 2I)$ and so on.

However certainly at one stage that is after only finite number of iterations we may be arrive at a realized fixed point of a realized limit cycle.

Let $x = (1, 0, 0, 0) \in P$.

To find the effect of x on B.

$xB = (0, 1 + I, 0, 0) = y_1;$ $y_1B = (I, 0, 1 + 2I, 0) = y_2;$
$y_2B = (0, 2I, 0, 0) = y_3;$ $y_3B = (I, 0, 0, 0) = y_4;$
$y_4B = (0, 2I, 0, 0) = y_5 (= y_4).$

Thus the resultant is a realized limit cycle.

Let $x = (0, 0, 0, 1) \in P$.

To find the effect of x on B.

$xB = (0, 0, 2 + I, 0) = y_1$; $y_1B = (0\ 1 + 2I, 0, 2 + I) = y_2$;
$y_2B = (0, 0, 2 + I, 0) = y_3$;
$y_3B = (0\ 1 + 2I, 0, 2 + I) = y_4\ (= y_2)$.

Thus the resultant of (0, 0, 0, 1) is a realized limit cycle.

Characterizing all classical fixed points of B, realized fixed points of B and realized limit cycle of B happens to be a difficult problem.

Example 4.8: Let $A = \begin{bmatrix} 2I & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & 1+I \end{bmatrix}$

be the MOD-neutrosophic diagonal matrix of A with entries from $\langle Z_3 \cup I\rangle$.

Let $x = (1, 0, 0) \in P = \{(x_1, x_2, x_3) \mid x_i \in \langle Z_3 \cup I\rangle, 1 \leq i \leq 3\}$

To effect of x on A is;

$xA = (2I, 0, 0) = y_1$; $y_1A = (I, 0, 0) = y_2$;
$y_2A = (2I, 0, 0) = y_3\ (= y_1)$.

Thus the resultant is a realized limit cycle.

Let $x = (0, 1, 0) \in P$. The effect of x on a A is;

$xA = (0, I, 0) = y_1$; $y_1A = (0, I, 0) = y_2\ (= y_1)$.

The resultant is a realized limit cycle of A.

Let x = (1, 0, I) ∈ P.

The effect of x on A is;

$xA = (2I, 0, 2I) = y_1$; $y_1A = (I, 0, I) = y_2$;
$y_2A = (2I, 0, 2I) = y_3$ $(= y_1)$.

Thus the resultant is a realized limit cycle of A.

Example 4.9: Let S = $\begin{bmatrix} 1 & 0 & 0 & I & 1+2I \\ 0 & 3I & 0 & 0 & 1+3I \\ 0 & 0 & 1+I & 0 & 0 \\ I & 0 & 0 & 2I & 0 \\ 1+2I & 1+3I & 0 & 0 & 1 \end{bmatrix}$

be a MOD-neutrosophic matrix with entries from $\langle Z_4 \cup I\rangle$.

Let x = (1, 0, 0, 0, 0) ∈ P = $\{(x_1, x_2, x_3, x_4, x_5) \mid x_i \in \langle Z_4 \cup I\rangle; 1 \leq i \leq 5\}$

To find the effect of x on S.

$xS = (1, 0, 0, I, 1 + 2I) = y_1$;
$y_1S = (2+I\ \ 1 + 3I, 0, 3I, 2) = y_2$;
$y_2S = (0, 2 + 2I, 0, I, 3 + 2I) = y_3$;
$y_3S = (2 + 3I, 3 + 3I, 0, 2I, 3 + 2I) = y_4$;
$y_4S = (1 + I, 3 + 3I, 0, I, 0) = y_5$;
$y_5S = (1 + 2I, 2I, 0, 0, 2I) = y_6$;
$y_6S = (1, 2I, 0, 3I, 1 + 2I) = y_7$.

However after a finite number of iterations we will arrive at a realized fixed point or a realized limit cycle.

The main observation from this study is the following theorem.

THEOREM 4.1: *Let M be any n × n neutrosophic matrix with entries from $\langle Z_m \cup I \rangle$.*

> *i)* $P_I = \{(a_1, a_2, \ldots, a_n) / a_i \in Z_mI = \{aI / a \in Z_m\}; 1 \leq i \leq n\}$ *be the pure neutrosophic state vectors. If* $x \in P_I$ *then the resultant is always in* P_I.
>
> *ii)* $P_R = \{(a_1, a_2, \ldots, a_n) / a_i \in Z_m; 1 \leq i \leq n\}$ *be the collection of real state vectors. If* $x \in P_R$ *the resultant in general need not be in* P_R.
>
> The proof follows from simple arguments.

Next we proceed onto study the MOD complex modulo integer matrix.

This we will represent by some examples.

Example 4.10: Let S = $\begin{bmatrix} 2+i_F & 0 & i_F \\ 0 & 1+i_F & 1 \\ 2 & 1+2i_F & 0 \end{bmatrix}$ be the MOD

complex modulo integer matrix with entries from $C(Z_3)$.

Let P = $\{(a_1, a_2, a_3) \mid a_i = a + bi_F \in C(Z_3);\ a, b \in Z_3;\ i_F^2 = 2\}$ be the state vectors.

Let x = (1, 2, 0) ∈ P.

To find the effect of x on S;

$xS = (2 + i_F, 2 + 2i_F, 2 + i_F) = y_1;$

$$y_1S = (2 + i_F, 2 + 2i_F, 2 + i_F)\begin{bmatrix} 2+i_F & 0 & i_F \\ 0 & 1+i_F & 1 \\ 2 & 1+2i_F & 0 \end{bmatrix}$$

$= (1, 0, 1 + i_F) = y_2$

$y_2S = (1, 0, 1 + i_F) = y_3$ $(= y_2)$.

Thus the resultant is a realized fixed point.

Let $x = (1, 0, i_F) \in P$.

To find the effect of x on S.

$xS = (2 \;\; 1 + i_F \;\; i_F) = y_1$; $y_1S = (1 + i_F, 1, 1) = y_2$;
$y_2S = (0, 2, i_F) = y_3$; $y_3S = (2i_F, 0, 2) = y_4$;
$y_4S = (2 + i_F, 2 + i_F, 1) = y_5$;
$y_5S = (i_F + 2, 2 + 2i_F, 1) = y_6$.

We will however arrive at a realized fixed point or a realized limit cycle after finite number of iterations.

Let $x = (i_F, 2i_F, 0) \in P$.

To find the effect of x on S.

$xS = (1, 1 + 2i_F, 2 + 2i_F) = y_1$; $y_1S = (2i_F, 2, 1) = y_2$

$$y_2S = (2i_F, 2, 1)\begin{bmatrix} 2+i_F & 0 & i_F \\ 0 & 1+i_F & 1 \\ 2 & 1+2i_F & 0 \end{bmatrix} = (i_F, i_F, 0) = y_3;$$

$y_3S = (2i_F + 2, i_F + 2, i_F + 2) = y_4$;
$y_4S = (0, 1, 0) = y_5$;
$y_5S = (0, 1 + i_F, 1) = y_6$;
$y_6S = (2, 1 + i_F, 1 + i_F) = y_7$;

$y_7S = (i_F, 2 + 2i_F, 1) = y_8$;
$y_8S = (2i_F + 1, 1, 1 + 2i_F) = y_9$;
$y_9S = (2, 1 + 2i_F, 2 + i_F) = y_{10}$;
$y_{10}S = (2 + i_F, 2 + 2i_F, 1 + i_F) = y_{11}$.

Certainly after a finite number of iterations we will arrive at a realized limit cycle or a realized fixed point.

So even in case of symmetric complex MOD operators we don't see any symmetry or symmetric behavior of the state vector.

Further as in case of pure neutrosophic state vectors whose resultant is also pure neutrosophic we see in case of only complex state vectors that is (ai_F, bi_F, ci_F) the resultant in general is a mixed one.

This is the marked difference between the MOD neutrosophic matrix operators and MOD complex matrix operators.

In view of all these observations on MOD complex matrix operators we give the following theorem.

THEOREM 4.2: *Let S = (a_{ij}) be a MOD complex modulo integer $p \times p$ matrix MOD operator with entries from $C(Z_n)$;*

$$i_F^2 = (n-1)$$

If x = ($a_1 i_F$, …, $a_p i_F$); $a_i \in Z_n$; $1 \leq i \leq p$ be any initial only complex number state vector. The resultant of x on S in general is not a only complex number state vector.

Proof follows from several illustrated examples.

It is left as a open conjecture to characterize both the matrices S as well as x so that

i) the resultant is pure complex number.
ii) Characterize those state vectors whose resultant is real.
iii) Characterize those state vector so that the resultant is a mixed one.

However this will impose conditions also on S.

We finalize MOD complex modulo integer matrix operators with these examples.

Example 4.11: Let $S = \begin{bmatrix} 0 & i_F & 2i_F & 0 & i_F \\ i_F & 4i_F & 0 & i_F & 0 \\ 2i_F & 0 & 3i_F & 0 & 4i_F \\ 0 & i_F & 0 & i_F & 0 \\ 2i_F & 0 & i_F & 3i_F & i_F \end{bmatrix}$ be the MOD complex modulo integer matrix operator with entries from $\langle Z_5 \cup I\rangle$.

We call S of this from as pure complex MOD matrix operators.

We study the effect of x = (1, 2, 3, 0, 4) on S.

$$xS = (1, 2, 3, 0, 4) \begin{bmatrix} 0 & i_F & 2i_F & 0 & i_F \\ i_F & 4i_F & 0 & i_F & 0 \\ 2i_F & 0 & 3i_F & 0 & 4i_F \\ 0 & i_F & 0 & i_F & 0 \\ 2i_F & 0 & i_F & 3i_F & i_F \end{bmatrix}$$

$= (i_F, 4i_F, 0, 2i_F, 2i_F) = y_1;$

$y_1S = (2, 3, 1, 3, 2) = y_2;$
$y_2S = (4i_F, 2i_F, 4i_F, 2i_F, 3i_F) = y_3;$
$y_3S = (4, 1, 2, 2, 2) = y_3;$
$y_3S = (4i_F, 0, i_F, 4i_F, 4i_F) = y_4;$
$y_4S = (0, 2, 0, 4, 3) = y_5;$
$y_5S = (3i_F, 2i_F, 3i_F, 0, 3i_F) = y_6;$
$y_6S = (1, 4, 1, 4, 2) = y_7;$

$y_7S = (3i_F, i_F, i_F, 4i_F, 2i_F) = y_8$;
$y_8S = (3, 4, 4, 4, 1) = y_9$;
$y_9S = (3i_F, 3i_F, 4i_F, i_F, 0) = y_{10}$ and so on.

However the resultant will be only a realized limit cycle as we see if the first iteration is a pure complex number when the state vector is a real number and the real and complex occur alternatively so the resultant can only be a realized limit cycle.

Consider $x = (i_F, 3i_F, i_F, 2i_F, 0)$ be a initial state vector which is pure complex.

To find the effect of x on S.

$xS = (0, 0, 0, 0, 0) = y_1$; $\quad y_1S = (0, 0, 0, 0, 0) = y_2$.

Thus the resultant is realized fixed point yielding (0, 0, 0, 0, 0).

Next we find the resultant of $x = (i_F, 0, i_F, 0, i_F)$ on the complex MOD-matrix operator S.

$xS = (2, 1, 1, 2, 2) = y_1$; $\quad y_1S = (2i_F, 3i_F, 4i_F, 3i_F) = y_2$;

$y_2S = (3, 2, 1, 4, 4) = y_2$ and so on.

For this pure complex modulo integer state vector we see the first iteration is real the second iteration is complex, complex and real occur alternatively so the final resultant is only a realized limit cycle.

Finally we see if x is a mixed complex number then certainly the resultant can be complex.

However there is little chance to be pure complex or pure real but depending on the MOD complex number matrix operator.

Example 4.12: Let M = $\begin{bmatrix} 1 & 2 & 0 & 1 \\ 3 & 0 & 1 & 0 \\ 0 & 1 & 0 & 3 \\ 2 & 0 & 3 & 0 \end{bmatrix}$ be the MOD complex number matrix operator with entries from $C(Z_4)$.

To find the effect of x = (1, 2, 1, 3) ∈ P = $\{(x_1, x_2, x_3, x_4) \mid x_i \in C(Z_4) = \{a + bi_F / a, b \in Z_4;\ i_F^2 = 3\}\ 1 \leq i \leq 4\}$ on M.

$xM = (1, 3, 3, 0) = y_1$; $y_1M = (2, 1, 3, 2) = y_2$;
$y_2M = (1, 3, 3, 3) = y_3$; $y_3M = (0, 1, 0, 2) = y_4$;
$y_4M = (3, 0, 3, 0) = y_5$; $y_5M = (3, 1, 0, 0) = y_6$;
$y_6M = (2, 2, 1, 3) = y_7$; $y_7M = (2, 1, 3, 1) = y_8$;
$y_8M = (3, 3, 0, 3) = y_9$; $y_9M = (2, 2, 3, 3) = y_{10}$;
$y_{10}M = (2, 3, 0, 3) = y_{11}$; $y_{11}M = (3, 0, 0, 2) = y_{12}$;
$y_{12}M = (3, 2, 2, 3) = y_{13}$ and so on.

Thus the resultant will be realized fixed point which will only be a real or it may be a realized limit cycle but it will also be real.

Let $x = (2i_F, i_F, 0, 3i_F) \in P$.

To find the effect of x on M.

$xM = (3i_F, 0, 2i_F, 2i_F) = y_1$; $y_1M = (3i_F, 0, 2i_F, i_F) = y_2$;

$y_2M = (i_F, 0, 3i_F, i_F)$ and so on.

Thus if the initial state vector is pure complex the resultant will be a realized fixed point which is pure complex or a realized limit cycle which will be only pure complex.

Next we study the dual number MOD-matrix operator with entries from $\langle Z_n \cup g\rangle = \{a + bg \mid a_i \in Z_n, g^2 = 0\}$.

We will illustrate this situation by some examples.

Example 4.13: Let $S = \begin{bmatrix} 2+g & g & 0 & 0 & 2 \\ 0 & 0 & 1+g & g & 0 \\ 1+g & 3 & 0 & 0 & 3g \\ 0 & 0 & 2+3g & 1 & 0 \\ 2+3g & 1+2g & 0 & 0 & 1+2g \end{bmatrix}$

be the dual number MOD-matrix operator with entries from $\langle Z_4 \cup g\rangle = \{a + bg \mid a, b \in Z_4, g^2 = 0\}$.

Let $x = (g, 2g, 0, 3g, g) \in P = \{(a_1, a_2, a_3, a_4, a_5) \mid a_i \in \langle Z_4 \cup g\rangle, 1 \leq i \leq 5\}$.

To find the effect of x on S.

$xS = (0, g, 0, 3g\ 3g) = y_1$;
$y_1S = (3g, 3g, 3g, 3g, 3g) = y_2$;
$y_2S = (3g, 0, g, 3g\ g) = y_3$;
$y_3S = (g, 0, 2g, 3g, 3g) = y_4$;
$y_4S = (3g, g, 2g, 3g, g) = y_5$ and so on.

However it can be easily verified that the resultant of x will be realized fixed point or realized limit cycle which will only be a pure dual number.

We call x a pure dual number if $x = (a_1g, a_2g, \ldots, a_5g)$ where $a_i \in Z_4$.

Thus the resultants of all pure dual number will only be pure dual number if the MOD dual number matrix operator has its entries from $\langle Z_4 \cup g\rangle$.

Example 4.14: Let M = $\begin{bmatrix} g & 2g & 3g & 0 & 4g & 5g \\ 0 & g & 2g & 4g & 0 & g \\ 2g & 0 & 0 & 2g & g & 0 \\ 0 & 3g & 4g & 0 & 0 & 5g \\ g & 0 & 2g & 4g & 5g & 0 \\ 3g & g & 0 & g & 0 & 2g \end{bmatrix}$

be the MOD-dual number matrix operator with entries from $\langle Z_6 \cup g\rangle = \{a + bg \mid a, b \in Z_6, g_2 = 0\}$.

P = $\{(a_1, a_2, \ldots, a_6) \mid a_i \in \langle Z_6 \cup g\rangle;\ 1 \leq i \leq 6\}$ be the collection of all dual number state vectors.

Let x = (1, 2, 3, 0, 1, 0) $\in$ P.

To find the effect of x on M.

$$xM = (2g, 4g, 3g, 0, 0\ 2g) = y_1;$$
$$y_1M = (0, 0, 0, 0, 0, 0) = y_2.$$

Thus after one iteration a pure real state vector is zero.

In fact let x = (1 + g, 2g + 3, 2 + g, 0, 3 + 2g, 1 + 2g) $\in$ P be the initial state vector.

To find the effect of x on M.

$$xM = (3g, 3g, 3g, 5g, 3g, 4g) = y_1;$$
$$y_1M = (0, 0, 0, 0, 0, 0).$$

Thus the resultant is a realized fixed point.

If x = $(a, g, a_2g, \ldots, a_6g) \in$ P be a pure dual number.

The effect of x on M is (0, 0, 0, 0, 0, 0) is a realized fixed point.

Let us give another example of the MOD-dual number matrix operator by an example.

Example 4.15: Let A = $\begin{bmatrix} 3 & 2 & 0 & 0 & 1 \\ 0 & 0 & 4 & 3 & 0 \\ 2 & 1 & 0 & 2 & 1 \\ 0 & 0 & 3 & 0 & 2 \\ 2 & 3 & 1 & 1 & 0 \end{bmatrix}$

be the MOD-dual number matrix operator with entries from $\langle Z_5 \cup I\rangle = \{a + bg \mid a, b \in Z_5, g^2 = 0\}$.

Let $x = (1, 2, 3, 1, 0) \in P = \{(a_1, a_2, a_3, a_4, a_5) \mid a_i \in \langle Z_5 \cup g\rangle, 1 \leq i \leq 5\}$ be the initial state vector.

To find the effect of x on A,

$xA = (4, 0, 1, 2, 1) = y_1$; $y_1A = (1, 2, 2, 3, 4) = y_2$;
$y_2A = (3, 1, 1, 4, 4) = y_3$; $y_3A = (4, 4, 0, 4, 2) = y_4$;
$y_4A = (1, 4, 0, 4, 2) = y_5$; $y_5A = (0, 2, 0, 4, 4) = y_6$;
$y_6A = (3, 2, 4, 0, 3) = y_7$; $y_7A = (3, 4, 1, 2, 2) = y_8$;
$y_8A = (0, 3, 4, 1, 3) = y_9$.

and so on.

If we start with a real state vector the resultant is again a real vector.

However if x = (g, 2g, 3g, 0, 4g) be a state vector to find the effect of x on A.

$xA = (2g, 2g, 2g, 0, 4g) = y_1$;
$y_1A = (3g, 3g, 2g, 4g, 4g) = y_2$;
$y_2A = (g, 0, 3g, 2g, 3g) = y_3$;
$y_3A = (0, 4g, 4g, 4g, 3g) = y_4$;
$y_4A = (4g, 3g, g, 3g, 2g) = y_5$ and so on.

After a finite number of iterations we arrive at a realized fixed point or a realized limit cycle which is a pure dual number.

Thus by this MOD-matrix real operator real state vectors' resultant is real, similarly pure dual number state vector resultant is a pure dual number.

Now we work with the mixed dual number initial state vector $x = (2 + g, 1 + 2g, g, 4, 3g + 3) \in P$.

$$xA = (2 + g, 1 + 2g, g, 4, 3 + 3g) \begin{bmatrix} 3 & 2 & 0 & 0 & 1 \\ 0 & 0 & 4 & 3 & 0 \\ 2 & 1 & 0 & 2 & 1 \\ 0 & 0 & 3 & 0 & 2 \\ 2 & 3 & 1 & 1 & 0 \end{bmatrix}$$

$$= (2 + g, 3 + 2g, 4 + g, 1 + g, 2g) = y_1;$$

$y_1A = (4 + 4g, 3 + 4g, 3g, 2, 3 + 4g) = y_2;$
$y_2A = (3 + g, 2 + 3g, 1, 2 + 2g, 3 + 2g) = y_3;$
$y_3A = (2, 1 + 3g, 2, 1 + g, 3) = y_4;$
$y_4A = (1, 0, 1, 4g, 2g + 1) = y_5;$
$y_5A = (4g + 1, 1 + g, 4g + 1, 2g + 3, 2 + 3g) = y_6;$
$y_6A = (4 + g, 4 + g, 3g, 2 + 4g, 2g + 3) = y_7;$
$y_7A = (3 + 3g, 2 + g, 3g, 4g, 3 + g) = y_8.$

Thus the resultant is realized limit cycle or a realized fixed point which is a mixed dual number.

We first give some related results of the MOD-dual number operators.

THEOREM 4.3: *Let $S = \{n \times n$ matrix with entries from $Z_m\}$ where $P = \{(a_1, a_2, \ldots, a_n) \mid a_i \in \langle Z_m \cup g\rangle = \{a + bg \mid a, b \in Z_m, g^2 = 0\}; 1 \leq i \leq n\}$ be the initial state vector.*

i) If $x = (a_1, \ldots, a_n)$; $a_i \in Z_m$; $1 \leq i \leq n$ be the real state vector then the resultant of x on S is only real.
ii) If $x = (a_1g, a_2g, \ldots, a_ng)$ $a_i \in Z_m$; $1 \leq i \leq n$ be the pure dual number state vector. Resultant of x on S is only a pure dual number vector.
iii) If $x = (a_1, a_2, \ldots, a_n)$; $b_i + c_ig = a_i \in \langle Z_n \cup g\rangle$ be a mixed dual number;

The resultant of x on S can be a pure real state vector or a pure dual state vector or mixed dual number state vector.

Proof is direct and hence left as an exercise to the reader.

THEOREM 4.4: *Let $M = (a_{ij})$ a $p \times p$ matrix with entries from Z_n, $g^2 = 0$.*

i) If $x = (a_1, a_2, \ldots, a_p)$; $a_i \in Z_ng$ be the pure dual number state vector, then the resultant of x on M is realized fixed point always a zero vector $(0, 0, \ldots, 0)$ after the first iteration.
ii) If $x = (a_1, \ldots, a_p)$, $a_i \in Z_n$; $1 \leq i \leq p$ be the real state vector the resultant is always a realized fixed point after two iterations given by $(0, 0, 0, \ldots, 0)$.
iii) If $x = (x_1, x_2, \ldots, x_p)$; $x_i \in \langle Z_n \cup g\rangle$; $1 \leq i \leq p$ be the initial state vector the resultant is a realized fixed point or a realized limit cycle.

Proof is direct and hence left as an exercise to the reader.

Next we study using MOD special dual like number matrix operators by examples.

Example 4.16: Let M = $\begin{bmatrix} 3+h & h & 2 & 2h+1 \\ 0 & 1+3h & 0 & h \\ h+1 & 2h & 2h & 0 \\ 2 & 1+3h & 0 & 1 \end{bmatrix}$

be the MOD special dual like matrix operator with entries from $\langle Z_n \cup h\rangle = \{a + bh \mid a,\ b \in Z_4, h^2 = h\}$.

Let x = (h, 0, 2h, 3h) be the state vector whose entries are pure special dual like numbers.

To find the effect of x on M.

xM = (2h, h, 2h, 0) = y_1; y_1M = (0, 2h, 0, 3h) = y_2;
y_2M = (2h, 0, 0, 3h) = y_3; y_3M = (2h, 2h, 0, h) = y_4;
y_4M = (2h, 2h, 0, h) = y_5 (= y_4).

Thus the resultant of x is a realized fixed point of M.

Let x = (1, 2, 3, 0) $\in$ P be the initial state vector.

To find the effect of x on M.

xM = (3, 2 + h, 2 + 2h, 1) = y_1;
y_1M = (1 + h, 3, 2, h) = y_2;
y_2M = (1 + h, 3 + 3h, 2 + 2h, 1 + h) = y_3;
y_3M = (3 + h, 2h, 2 + 2h, 2) = y_4.

We would after a finite number of iterations will arrive at a realized fixed point or a realized limit cycle.

Consider x = (1 + h, 2h + 1, 0, 0) a state vector.

To find the effect of x on M.

xM = (3 + h, 1 + h, 2 + 2h, 1) = y_1;
y_1M = (1 + h, 2, 2 + 2h, 3 + 3h) = y_2.

We will arrive after a finite number of iterations the realized fixed point or a realized limit cycle.

Example 4.17: Let S = $\begin{bmatrix} h & 0 & 2h & 3h & 0 \\ 0 & 4h & 0 & 2h & h \\ 2h & 0 & 4h & 0 & 2h \\ 0 & 5h & 0 & 4h & 0 \\ 3h & h & 2h & 0 & 2h \end{bmatrix}$

be the MOD special dual like number matrix operator with entries from $\langle Z_6 \cup h\rangle = \{a + bh, ab \in Z_6, h^2 = h\}$.

Let x = (2, 1, 3, 4, 0) be a state vector in
$P = \{(a_1, a_2, a_3, a_4, a_5) \mid a_i \in \langle Z_6 \cup h\rangle; 1 \leq i \leq 5\}$.

To find the effect of x on S.

$xS = (2h, 0, 4h, 0, h) = y_1;$
$y_1S = (h, h, 4h, 0, 4h) = y_2;$
$y_2S = (3h, 2h, 2h, 5h, 3h) = y_3;$
$y_3S = (4h, 0, 2h, 3h, 4h) = y_4;$
$y_4S = (2h, h, 0, 0, 0) = y_5;$
$y_5S = (2h, 4h, 4h, 2h, h) = y_6;$
$y_6S = (h, 3h, 4h, h, 2h) = y_7;$
$y_7S = (3h, h, 4h, 3h, 3h) = y_8;$
$y_8S = (2h, 4h, 4h, 5h, 3h) = y_9.$

Thus the resultant will be a realized limit cycle or a realized fixed point but it will be a pure special dual like number vector.

So even all real state vectors has the resultant to be only a pure special dual like number vector.

Let x = (h, 2h, 3h, 0, 0) be the initial state vector.

To find the effect of x on M.

$xS = (h, 2h, 2h, h, 2h) = y_1;$

$y_1S = (5h, 3h, 2h, 5h, 2h) = y_2;$
$y_2S = (3h, 3h, 2h, 5h, 5h) = y_3;$
$y_3S = (4h, 0, 0, 5h, 5h) = y_4;$
$y_4S = (h, 0, 0, 2h, 4h) = y_5;$
$y_5S = (h, 2h, 4h, 5h, 2h) = y_6;$
$y_6S = (3h, 5h, 4h, h, h) = y_7$ and so on.

Thus the pure special dual like number state vector.

Let $x = (1 + h, 2 + 3h, 1 + 3h, 4 + h, 0)$ be the state vector.

To find the effect of x on S is given by the following way;

$xS = (4h, 3h, 2h, 0, h) = y_1;$
$y_1S = (5h, h, 0, 0, 3h) = y_2;$
$y_2S = (2h, h, 4h, 5h, h) = y_3;$
$y_3S = (h, 0, 4h, 4h, 0) = y_4.$

However we see after a finite number of iterations we will get the resultant which is only a pure special dual like number vector what ever be the state vector be real or pure special dual like number or a mixed one all of them have the resultant to be only a pure special dual like number.

In view of this we have the following theorem.

THEOREM 4.5: *Let $A = (a_{ij})_{p\times p}$ where $a_{ij} \in Z_nh$, $h^2 = h$ be the MOD-special dual like number matrix operator.*

i) All real state vectors $x \in \{(a_1, a_2, \ldots, a_p) / a_i \in Z_n;$ $1 \leq i \leq p\}$ yields the resultant to be always a pure special dual like number vector.
ii) All state vectors $x = \{(x_1, x_2, \ldots, x_p) / a_i \in Z_nh;$ $1 \leq i \leq p\}$ yields the resultant to be always a pure special dual like number state vector.
iii) All initial state vectors mixed numbers also yield the resultant to be only a pure special dual like number vector.

Proof is direct and hence left as an exercise to the reader.

Now we give examples of real MOD matrix operator on special dual like number vectors.

Example 4.18: Let $M = \begin{bmatrix} 3 & 2 & 0 & 1 & 4 \\ 1 & 0 & 2 & 0 & 3 \\ 0 & 1 & 0 & 4 & 0 \\ 2 & 0 & 3 & 0 & 2 \\ 0 & 1 & 0 & 4 & 0 \end{bmatrix}$

be the MOD special dual like number matrix operator with entries from $\langle Z_5 \cup h \rangle = \{a + bh \mid a, b \in Z_5, h^2 = h\}$.

$P = \{(x_1, x_2, \ldots, x_5) / x_i \in \langle Z_5 \cup g \rangle; 1 \leq i \leq 5\}$.

To find the effect of $x \in P$ on M.

Let $x = (3, 1, 2, 0, 4) \in P$.

To find the effect of x on M.

$xM = (0, 2, 2, 2, 0) = y_1$; $y_1M = (0, 2, 0, 3, 0) = y_2$;
$y_2M = (3, 0, 3, 0, 2) = y_3$; $y_3M = (4, 1, 4, 3, 2) = y_4$;
$y_4M = (4, 4, 1, 3, 0) = y_5$; $y_5M = (2, 4, 2, 3, 4) = y_6$;
$y_6M = (1, 2, 2, 1, 1) = y_7$ and so.

Certainly the resultant is also only a real state vector.

Let $x = (h, 2h, 0, h, 0)$ be the state vector.

To find the effect of x on M.

$xM = (2h, 2h, 2h, h, 2h) = y_1$;
$y_1M = (4h, 3h, 2h, 3h, h) = y_2$;
$y_2M = (h, h, 0, h, h) = y_3$;

$y_3M = (h, 3h, 0, 0, 4h) = y_4$;
$y_4M = (h, 0, h, 2h, 3h) = y_5$ and so on.

We see certainly the resultant is only a pure special dual like number.

Let $x = (1 + h, 2h + 1, 2h + 3, 0, h + 1)$ be the state vector.

The resultant of x on M is as follows.

$xM = (4, 1, 4h + 2, 2 + 3h, 2) = y_1$;
$y_1M = (2 + h, 2 + 4h, 3 + 4h, 2h, 3 + h)$;
$y_2M = (3 + h, 2h, 4 + 4h, 1 + h, 4)$ and so on.

The resultant is a state vector from P.

Thus in view of all these the following results can be proved.

THEOREM 4.6: *Let $M = (m_{ij})_{n\times n}$ matrix whose entries are from $Z_m \subseteq \langle Z_m \cup h\rangle = \{a + bh / a, b \in Z_m; h^2 = h\}$ the MOD special dual like number operator and $P = \{(a_1, a_2, .., a_n) / a_i \in \langle Z_m \cup h\rangle; 1 \leq i \leq n\}$ be the collection of special dual like number state vectors.*

i) Every $x = (x_1, \ldots, x_n)$ (where $x_i \in Z_m$ the real state vector has its resultant on M to be only a real state vector.
ii) Every $x = (y_1, y_2, \ldots, y_n)$ ($y_i \in Z_mh$) the pure special dual like number state vector has its resultant on M to be only a pure special dual like number state vector.
iii) If $x = (a_1, a_2, \ldots, a_n)$ $a_i \in \langle Z_m \cup j\rangle$; $1 \leq i \leq n$ then the resultant of x on M can be in P.

Next we give a few more illustration of MOD special dual like number matrix operators M.

Example 4.19: Let M = $\begin{bmatrix} 3+h & 0 & 0 & 0 \\ 0 & 4+2h & 0 & 0 \\ 0 & 0 & 6h+1 & 0 \\ 0 & 0 & 0 & 7h \end{bmatrix}$

be the MOD special dual like number matrix operator with entries from $\langle Z_8 \cup h\rangle = \{a + bh / a, b \in Z_8, h^2 = h\}$.

Let x = (3, 1, 2, 0) be a state vector.

To find the effect of x on M.

$$xM = (1 + 3h, 4 + 2h, 2 + 4h, 0) = y_1;$$
$$y_1M = (5h + 3, 4h, 2, 0) = y_2;$$
$$y_2S = (5h + 3, 0, 2, 0) = y_3;$$
$$y_3S = (5h + 3, 0, 2, 0) = y_4;$$
$$y_4M = (7h + 1, 0, 4h+2, 0) = y_5 (= y_3).$$

Thus the resultant of x is a realized limit cycle.

Let x = (3, 2, 4, 1) be the initial state vector.

To find the effect of x on M.

$$xM = (1 + 3h, 4h, 4, 7h) = y_1;$$
$$y_1M = (3+5h , 0, 4, h) = y_2;$$
$$y_2M = (1+7h, 0, 4, 7h) = y_3;$$
$$y_3M = (3+5h, 0, 4, h) = y_4 (= y_2).$$

Thus this resultant is also a realized limit cycle.

Let x = (h, 2h, 4h, 5h) be the initial state vector.

To find the effect of x on M.

$$xM = (4h, 4h, 4h, 3h) = y_1;$$
$$y_1M = (0, 0, 4h, 5h) = y_2;$$
$$y_2S = (0, 0 4h, 3h) = y_3;$$

$y_3S = (0, 0, 4h, 5h) = y_4 = y_2.$

Thus this resultant is also a realized limit cycle which is only a pure special dual like number state vector.

So even if the MOD-special dual like number operator matrix is a diagonal matrix we see if x the initial state vector is a pure special dual like number then so is the resultant.

Example 4.20: Let M = $\begin{bmatrix} 0 & 0 & 6 & 2h+1 \\ 0 & 0 & 1+h & 6h \\ 2+3h & 4h & 0 & 0 \\ 3h+1 & 5 & 0 & 0 \end{bmatrix}$

be the MOD special dual like number matrix operator with entries from $\langle Z_6 \cup h\rangle = \{a + bh \mid a, b \in Z_7, h^2 = h\}$.

Let x = (1, 2, 3, 4) be the initial state vector.

To find the effect of x on M.

$xM = (3, 6 + 5h, 1 + h, 1) = y_1;$
$y_1M = (4h+3, 5h + 5, 3 + 2h, 3) = y_2;$
$y_2M = (2, 1 + 6h, 4h + 3, 2h+2) = y_3;$
$y_3S = (1+h, 3h +3, 6 + 6h, 4h+2) = y_4$ and so on.

Thus after a finite number of iterations we will arrive at a realized fixed point or a realized limit cycle.

However if x is a real number vector with entries in Z_7 still the resultant can be a mixed row vector.

Consider x = (h, 2h, h, 0) be the state vector which is a pure special dual like number state row vector.

To find the effect of x on M.

$xM = (5h, 3h, 3h, h) = y_1$;

$y_1M = (5h, 3h, 3h, h) = y_2$ is a realized fixed point which is a pure special dual like number state row vector.

Low $x = (1 + h, 2 + 2h, 0, 3h + 1)$ be the initial state vector.

To find the effect of x on M.

$xM = (h + 1, h + 5, 1 + 5h, h + 1) = y_1$ and so on.

After a finite number of iterations one may get a realized limit cycle or a realized fixed point.

Next we study by illustrative examples the MOD-special quasi dual number matrix operator with entries from $\langle Z_n \cup k\rangle = \{a + bk \mid a, b \in Z_n ; k^2 = (n – 1) k\}$

Example 4.21: Let $P = \begin{bmatrix} 3+k & k & 0 \\ 2k & 0 & 1+2k \\ 1+k & 1 & 4k \end{bmatrix}$

be the MOD-special quasi dual number matrix operator with entries from $\langle Z_5 \cup k\rangle = \{a + bk \mid a, b \in Z_5, k^2 = 4k\}$.

Let $x = (1, 0, 2)$ be the initial state vector.

To find the effect of x on P.

$$xP = (3k, k + 2, 3k) = y_1 ;$$
$$y_1P = (3k\ 4k, 2) = y_2;$$
$$y_2P = (2, 2+2k, 4k) = y_3 \text{ and so on.}$$

Thus the resultant of pure real row vector can be a mixed special quasi dual number state vector.

Consider $x = (k, 2k, k)$ be the initial state vector.

To find the effect of x on P.

$xP = (4k, 0, k) = y_1$; $\quad y_1P = (0, 2k, k) = y_2$;
$y_2P = (k, k, 4k) = y_3$ and so on.

Thus the resultant of a pure special quasi dual number vector is always a pure special quasi dual number vector only.

Example 4.22: Let $S = \begin{bmatrix} 3k & 0 & 0 & k \\ 0 & 2k & k & 0 \\ k & 0 & 0 & 3k \\ 0 & k & 2k & 0 \end{bmatrix}$

be the special quasi dual number MOD-matrix operator with entries from $\langle Z_4 \cup k\rangle = \{a + bk \mid a, b \in Z_4, k^2 = 3k\}$.

Let $x = (1, 0, 2, 1)$ be the pure real state vector.

Effect of x on S is as follows.

$xS = (k, k, 2k, 3k) = y_1$; $\quad y_1S = (3k, 3k, k, k) = y_2$;
$y_2S = (2k, k, 3k, 2k) = y_3$ and so on.

It is clear that after a finite number of iterations we will arrive at a realized limit cycle or a realized fixed point but the resultant will always be a pure special quasi dual number row vector.

Let $x = (3 + k, k + 2, 2k, 3k + 1)$ be the initial state vector. Certainly the resultant of this state vector will also be only a pure special quasi dual number row vector.

In view of this we prove the following theorem.

THEOREM 4.7: *Let $S = (m_{ij})_{n \times n}$ special quasi dual number MOD matrix operator with entries from $Z_m k = \{ak / k^2 = (m-1) k;\ a \in Z_m\}$.*

$P = \{(a_1, a_2, \ldots, a_n) / a_i \in \langle Z_m \cup k\rangle = \{a + bk / a, b \in Z_m;\ k^2 = (m-1) k\};\ 1 \leq i \leq n\}$ be the collection of all state vectors. For every $x \in P$ the resultant on S is always a pure special quasi dual number row vector in $P_1 = \{(b_1, b_2, \ldots, b_n) \mid b_i \in Z_m k;\ 1 \leq i \leq n\} \subseteq P$.

Proof is direct and hence left as an exercise to the reader.

Example 4.23: Let $M = \begin{bmatrix} 1 & 2 & 3 & 4 & 0 \\ 5 & 0 & 1 & 0 & 2 \\ 0 & 4 & 0 & 5 & 0 \\ 1 & 0 & 2 & 0 & 5 \\ 0 & 1 & 0 & 1 & 0 \end{bmatrix}$ be the MOD-special quasi dual number matrix operator.

$P = \{(a_1, a_2, a_3, a_4, a_5) \mid a_i \in \langle Z_6 \cup k\rangle = \{a + bk \mid a, b \in Z_6,\ k^2 = 5k\};\ 1 \leq i \leq 5\}$ be the collection of all state vector.

Let $x = (1, 0, 2, 0, 3)$ be the initial state vector.

To find the effect of x on M.

$xM = (1, 1, 3, 5, 0) = y_1$; $y_1M = (5, 2, 2, 1, 3) = y_2$;
$y_2M = (4, 3, 1, 3, 3) = y_3$; $y_3M = (2, 3, 3, 0, 3) = y_4$;
$y_4M = (5, 1, 3, 2, 0) = y_5$ and so on.

We see if x is a real row vector so is the resultant.

Next we find the resultant of a pure special quasi dual number row vector.

x = (k, 0, 2k, 0, 3k) be the initial row vector effect of x on M.

xM = (k, 3k, 3k, 5k, 0) and so on.

It can be easily verified that the resultant of a pure special quasi dual number row vector is always a pure special quasi dual number row vector though the MOD matrix operator used is real.

Next we consider the effect of x = (0, 1 + k, 0, 3k + 2, 0) on M, the special quasi dual number row vector.

xM = (2k + 2, 0, 5 + k, 0, 5 + 5k) and so on.

Thus we see we will arrive at a resultant, after a finite number of iterations.

However the resultant of x on M may be a realized fixed point or a realized limit cycle.

In view of this we have the following theorem.

THEOREM 4.8: *Let* $M = (m_{ij})_{m \times n}$ *matrix with* $m_{ij} \in Z_n \subseteq \langle Z_n \cup k\rangle = \{a + bk \,/\, a, b \in Z_n, k^2 = (n-1)k\}$ *be the pure real MOD matrix operator of the special quasi dual numbers.*

$P = \{(a_1, a_2, \ldots, a_m) \,/\, a_i \in \langle Z_n \cup k\rangle, 1 \leq i \leq m\}$

i) *For every x a real state vector of P the resultant of x on M always a real state vector.*

For every pure special quasi dual number state row vector the resultant on M is always a pure special quasi dual number state vector.

Proof is direct and hence left as an exercise to the reader.

Now we proceed onto propose problems based on our study in this chapter. Some of the problems can be treated as open conjecture and some are simple.

Problems

1. What are the special and distinct features enjoyed by MOD - neutrosophic matrix operators?

2. Let M = $\begin{bmatrix} 3+I & 4+2I & 7+4I & 3I \\ 2I+5 & 3I+5I & 0 & 2I \\ 7I+3I & 4+6I & 7I & 0 \\ 12+5I & 0 & 3I & 1+I \end{bmatrix}$ be the MOD - neutrosophic matrix operator with entries from $\langle Z_{13} \cup I\rangle = \{a + bI / a,\ b \in Z_{13}, I^2 = I\}$.

   i) Enumerate all special features enjoyed by M.
   ii) Characterize all classical fixed points of M.
   iii) Characterize all the realized fixed points of M.
   iv) Characterize all realized limit cycles of M.
   v) If x and y are state vectors x ≠ y will the sum of the resultant of x y the same as resultant of x + y.
   vi) Characterize all those state vectors which satisfy (v).

3. Let S = $\begin{bmatrix} 3I & 2I & 0 & 4I & 5I & 6I \\ 0 & 8I & 4I & 0 & 3I & I \\ 7I & 0 & 3I & 4I & 0 & 4I \\ 2I & I & 0 & 0 & I & 0 \\ 6I & 0 & I & 2I & 0 & 3I \\ 0 & 2I & 0 & 0 & 3I & 0 \end{bmatrix}$ be the MOD neutrosophic matrix operator with entries from $\langle Z_{10} \cup I\rangle = \{a + bI \mid a, b \in Z_{10}, I^2 = I\}$.

P = $\{(a_1, a_2, a_3, a_4, a_5, a_6) \mid a_i \in \langle Z_{10} \cup I\rangle; 1 \leq i \leq 6\}$ be the collection of all state vectors.

i) Study questions (i) to (vi) of problem 2 using the operator S.

ii) Can we say resultant of every x in P on S is only a pure neutrosophic row vector?

4. Let M = $\begin{bmatrix} 3 & 7 & 2 & 0 & 1 & 5 & 3 \\ 0 & 2 & 0 & 1 & 0 & 2 & 1 \\ 7 & 0 & 5 & 0 & 8 & 0 & 2 \\ 0 & 1 & 0 & 7 & 0 & 6 & 0 \\ 1 & 0 & 2 & 0 & 5 & 0 & 3 \\ 0 & 5 & 0 & 4 & 0 & 3 & 0 \\ 2 & 0 & 1 & 0 & 7 & 0 & 8 \end{bmatrix}$ be a MOD neutrosophic

matrix operator with entries from $Z_{11} \subseteq \langle Z_{11} \cup I\rangle = \{a + bI \mid a, b \in Z_{11}, I^2 = I\}$.

P = $\{(a_1, a_2, \ldots, a_7) \mid a_i \in \langle Z_{11} \cup I\rangle\ 1 \leq i \leq 7\}$ be the collection of all state row vectors.

i) Study questions (i) to (vi) of problem (2) for this M.

ii) If x is a real number state vector prove the resultant of x is also a real number state vector.

iii) Prove all pure neutrosophic row state vectors have their resultant to be pure neutrosophic resultant to be pure neutrosophic.

5. Let P = $\begin{bmatrix} 3+2I & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2I+7 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 4I+1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 7I+2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 4I+3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 2I \end{bmatrix}$ be

   the MOD neutrosophic diagonal matrix operator with entries from $\langle Z_9 \cup I\rangle$.

   i) Study questions (i) to (vi) of problem (2) using this operator P.

   ii) Find conditions on the state row vectors such that their resultant is a zero row vector.

6. Let T = $\begin{bmatrix} 3g+2 & 0 & 4g+1 & 0 & 2g \\ 0 & 5g+2 & 0 & 7g+1 & 0 \\ 4g & 0 & 4 & 0 & 5+2g \\ 0 & 7g+7 & 0 & 5g+6 & 0 \\ 2+7g & 0 & 5+6g & 0 & 9+2g \end{bmatrix}$ be the

   MOD-dual number matrix operator with entries from $\langle Z_{12} \cup g\rangle = \{a + bg \mid a, b \in Z_{12}, g^2 = 0\}$.

   i) Study questions (i) to (vi) of problem 2 using this T.

   ii) Find all row vectors which will have their resultant to be the zero vector.

7. Let S = $\begin{bmatrix} g & 2g & 3g & 0 & 4g & 5g \\ 0 & g & 2g & 3g & 0 & 4g \\ 5g & 3g & 2g & 3g & 4g & 0 \\ 2g & 0 & 4g & 0 & 5g & g \\ 3g & 4g & 0 & 5g & 0 & 0 \\ 0 & 0 & g & 0 & 2g & 6g \end{bmatrix}$ be the MOD dual

number matrix operator with entries from $\langle Z_8 \cup g\rangle = \{(a + bg \mid a, b \in Z_8,\ g^2 = 0\}$.

i) Prove all pure dual number row initial state vectors resultant are zero vectors after first iteration.

ii) Prove all real row initial state vectors have the resultant on S to be a zero row vector after two iterations.

iii) Obtain all special features associated with this S.

8. Let M = $\begin{bmatrix} 2g+4 & 0 & 0 & 0 & 0 & 0 \\ 0 & 6g+8 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2+6g & 0 & 0 & 0 \\ 0 & 0 & 0 & 4g+6 & 0 & 0 \\ 0 & 0 & 0 & 0 & 8g+4 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$ be

the MOD dual number diagonal matrix operator with entries.

i) Study questions (i) to (vi) of problem 2 for this M.

ii) Find all state vectors x which gives the resultant as a realized fixed point which is the zero row vector.

9. Let

$$S = \begin{bmatrix} 2g & 0 & 0 & 0 & 0 & 0 & 0 \\ 3g & 1+2g & 0 & 0 & 0 & 0 & 0 \\ 4g & 3+2g & 1+2g & 0 & 0 & 0 & 0 \\ 5g & 4g & 3+2g & 1+2g & 0 & 0 & 0 \\ 6g & g & g+1 & 3+2g & 1+2g & 0 & 0 \\ 6 & 2g & 3g & 4g & 3+2g & 1+2g & 0 \\ 4g & g & 2g & g+1 & 4g & 3+2g & 1+2g \end{bmatrix}$$

be the MOD dual number matrix operator with entries from $\langle Z_{10} \cup g\rangle = \{a + bg / a, b \in Z_{10}, g^2 = 0\}$.

i) State all the special features enjoyed by S.
ii) Study questions (i) to (vi) of problem (2) for this S.

10. Can these dual number MOD matrix operator find any special type of applications to real world problems?

11. Let $W = \begin{bmatrix} 3+h & h & 0 & 4h+2 \\ h+1 & 0 & 4+2h & 0 \\ 0 & 3h+1 & 0 & 2h \\ 4h & 0 & 2+3h & 0 \end{bmatrix}$ be the MOD-special dual like number matrix operator.

i) Obtain all the special features associated with W.
ii) Study questions (i) to (vi) of problem (2) for this W.
iii) Characterize all those row vectors which result in a zero row vector as a realized fixed point.
iv) Characterize all classical fixed points of W.
v) Find the maximum number iterations that is needed to make one to arrive at a realized limit cycle or a fixed point.

12. Let $M = \begin{bmatrix} 3h & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 4h+1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 8h & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 4h+1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 3h+2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 4h+7 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & h \end{bmatrix}$

be the MOD-special dual like number matrix operator with entries from $\langle Z_{10} \cup h\rangle = \{a + bh / a, b \in Z_{10}, h^2 = h\}$.

i) Study questions (i) to (vi) of problem (2) for this m.
ii) Enumerate all special features enjoyed by the MOD special dual like number diagonal matrices.

13. Let

$$B = \begin{bmatrix} 9+4h & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 2+3h & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 4h & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 7h+1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 10h & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 4+5h & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & h+2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 6h & 0 & 0 \end{bmatrix}$$

be the MOD special dual like number matrix operator with entries from

$$\langle Z_{11} \cup h\rangle = \{a + bh / a, b \in Z_{11}, h^2 = h\}.$$

i) Study questions (i) to (vi) of problem (2) for this B.
ii) What are special features associated with this operator?

14. Show if

$$X = \begin{bmatrix} 3+I & 2 & 4+I \\ 0 & 3I+2 & 0 \\ 4I & 0 & 2+3I \end{bmatrix}$$

and

$$Y = \begin{bmatrix} 0 & 3+I & 4I \\ 2 & 0 & 3I+2 \\ 2I+4 & 2I & 0 \end{bmatrix}$$

be any two MOD neutrosophic matrix operator with entries from

$$\langle Z_5 \cup I \rangle = \{a + bI / a, b \in Z_5, I^2 = I\}.$$

Let x = (3I + 2, 4I, 2 + I) be the initial state vector.

i) Find xX and xY
ii) Find x (X + Y)
iii) Will xX + xY = x(X + Y)?
iv) Find all those state vector

$$x \in P = \{(a_1, a_2, a_3) / a_i \in \langle Z_5 \cup I \rangle, 1 \leq i \leq 3\}$$

which satisfy (iv).

15. Let

$$M = \begin{bmatrix} 3+4h & 0 & 2+5h & 0 \\ 0 & 2h+4 & 0 & 6h+2 \\ 6h+5 & 0 & 5+2h & 0 \\ 0 & 6h & 0 & 4 \end{bmatrix}$$

and

$$N = \begin{bmatrix} 3 & h+4 & h & 0 \\ 0 & 2h & 0 & 3h+2 \\ 1 & 0 & 4h+1 & 0 \\ 0 & 4h & 0 & 5h+3 \end{bmatrix}$$

be two MOD special dual like number matrix operator will entries from $\langle Z_7 \cup h \rangle$.

Study questions (i) to (v) of problem 14 for this M and N with appropriate changes.

16. Let

$$S = \begin{bmatrix} 4g+2 & 0 & g & 0 \\ 0 & 8g+5 & 0 & 4g \\ 2g+4 & 0 & 7g+2 & 0 \\ 0 & 4g+8 & g+7 & 2g+9 \end{bmatrix}$$

and

$$T = \begin{bmatrix} 2g & 4g+2 & 0 & 7g+1 \\ 5g+4 & 0 & 9+g & 0 \\ 0 & 9g+3 & 0 & 4g+3 \\ 9g & 0 & 2g+1 & 0 \end{bmatrix}$$

be any two MOD dual number matrix operators.

Study questions (i) to (v) of problem 14 for this S and T with appropriate changes.

17. Let

$$W = \begin{bmatrix} 3+2i_F & 4i_F & 2 & 0 \\ 0 & 2+3i_F & 0 & i_F \\ 1+i_F & 0 & 3+2i_F & 0 \\ 0 & 4 & 0 & 3+4i_F \end{bmatrix}$$

and

$$V = \begin{bmatrix} 0 & 0 & 3+2i_F & 4i_F \\ 0 & 0 & 2i_F & 4+i_F \\ i+i_F & 2i_F & 0 & 0 \\ 4+3i_F & 2+4i_F & 0 & 0 \end{bmatrix}$$

be two MOD complex modulo integer matrix operators.

Study questions (i) to (v) of problem (14) for this V and W with appropriate changes.

18. Let

$$A = \begin{bmatrix} 3 & 0 & 4+5i_F & 2 & 0 \\ 0 & 2+i_F & 0 & 0 & 5+i_F \\ 4+i_F & 0 & 2i_F & 7i_F & 0 \\ 0 & 3+4i_F & 0 & 0 & 3+i_F \\ 5+2i_F & 0 & 6+7i_F & 6 & 0 \end{bmatrix}$$

and

$$B = \begin{bmatrix} 3+4i_F & 0 & 0 & 0 \\ 0 & 0 & 0 & 4+2i_F \\ 0 & 9+7i_F & 0 & 0 \\ 0 & 0 & 0 & 8+6i_F \end{bmatrix}$$

be any two MOD complex modulo integer matrix operators.

Study questions (i) to (v) of problem (14) for this A and B with appropriate changes.

19. Let A = (a_{ij}) and B = (b_{ij}) be any two n × n real MOD matrices operators with entries from Z_m.

Study questions (i) to (v) of problem (14) for this A and B with appropriate changes.

20. Let

$$A = \begin{bmatrix} 5k+2 & 0 \\ 3k+3 & 6k \end{bmatrix}$$

and

$$B = \begin{bmatrix} 7k & 0 \\ 0 & 4k+1 \end{bmatrix}$$

be any two special quasi dual number MOD matrix operators.

Study questions (i) to (v) of problem (14) for this A and B with appropriate changes.

21. Let

$$X = \begin{bmatrix} 4I & 2I & 0 & 7+I \\ 0 & 3I+1 & 4I & 0 \\ 6I+3 & 0 & I+2 & 4+3I \\ 0 & 2I+1 & 0 & 4I \end{bmatrix}$$

be the MOD neutrosophic matrix operator with entries from $\langle Z_9 \cup I\rangle$.

Let

$$B = \left\{ \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{bmatrix} \mid a_i \in \langle Z_9 \cup I \rangle = \{a + bI \,/\, a, b \in Z_9, I^2 = I\}; \; 1 \le i \le 4 \right\}.$$

i) Find all $y \in B$ for which Xy gives classified points.
ii) Find all $y \in B$ which gives the resultant as realized fixed points
iii) Find all $y \in B$ which gives the resultant as realized limit cycle.
iv) Let

$$y_1 = \begin{bmatrix} 3 \\ 2+I \\ I \\ 7I \end{bmatrix} \text{ and } y_2 = \begin{bmatrix} 4 \\ 0 \\ 3I+2 \\ 0 \end{bmatrix} \in B.$$

v) Find Xy_1 and Xy_2, that is resultant of y_1 and y_2.
vi) Find X
vii) Is the resultant of $X(y_1 + y_2)$ sum of the resultants $Xy_1 + Xy_2$?

22. Let

$$M = \begin{bmatrix} 3g+4 & 0 & 2g \\ 0 & 4g+1 & 0 \\ 5g+4 & 0 & 2g+3 \end{bmatrix}$$

be the MOD dual number matrix operator with entries from $\langle Z_5 \cup g \rangle$.

$$B = \left\{ \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} \mid a_i \in \langle Z_5 \cup g \rangle = \{a + bg \,/\, a, b \in Z_5, g^2 = 0\} \; 1 \le i \le 3 \right\}$$

i) Find all $y \in B$ such that the resultant of y on M is a classical fixed point.
ii) Find all $y \in B$ such that the resultant of y on M is the realized fixed point $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$.
iii) If $y_1, y_2 \in B$ can the resultant of y_1 and y_2 on M be equal to the sum of the resultant of $y_1 + y_2$?

23. Let

$$N = \begin{bmatrix} 0 & 4g & 0 & 0 \\ 2g & 0 & 5g & 0 \\ 0 & 7g & 0 & 8g \\ 9 & 0 & 4g & g \end{bmatrix}$$

be the MOD dual number matrix operator with entries from

$\langle Z_{10} \cup g\rangle$.

$$B = \left\{ \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{bmatrix} \mid a_i \in \langle Z_{10} \cup g\rangle = \{a + bg / a, b \in Z_{10},\right.$$

$g^2 = 0\}, 1 \leq i \leq 4\}$ be the collection of state vector.

Study questions (i) to (iii) of problem (22) for this N with appropriate changes.

24. Let $M = (a_{ij})_{n\times n}$ matrix with entries from $\langle Z_m \cup k\rangle = \{(a + bk \mid a, bb \in Z_m, k^2 = (m - 1) k\}$ be the MOD special quasi dual number matrix operator.

Let

$$B = \left\{ \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} \mid a_i \in \langle Z_m \cup k \rangle, 1 \leq i \leq n \right\}$$

be the collection of column state vector. $B^{\perp} = \{(a_1, a_2, \ldots, a_n) \mid a_i \in \langle Z_m \cup k \rangle; 1 \leq i \leq n\}$ be the collection of row state vectors.

i) Find columns vectors y in B such that the resultant is z than for the y^t in $B^{\perp}$ the resultant is z^t.
ii) Find those MOD matrix operators for which (1) is true.

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On India's 60th Independence Day, Dr.Vasantha was conferred the Kalpana Chawla Award for Courage and Daring Enterprise by the State Government of Tamil Nadu in recognition of her sustained fight for social justice in the Indian Institute of Technology (IIT) Madras and for her contribution to mathematics. The award, instituted in the memory of Indian-American astronaut Kalpana Chawla who died aboard Space Shuttle Columbia, carried a cash prize of five lakh rupees (the highest prize-money for any Indian award) and a gold medal.

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Dr. Florentin Smarandache is a Professor of Mathematics at the University of New Mexico in USA. He published over 75 books and 200 articles and notes in mathematics, physics, philosophy, psychology, rebus, literature. In mathematics his research is in number theory, non-Euclidean geometry, synthetic geometry, algebraic structures, statistics, neutrosophic logic and set (generalizations of fuzzy logic and set respectively), neutrosophic probability (generalization of classical and imprecise probability). Also, small contributions to nuclear and particle physics, information fusion, neutrosophy (a generalization of dialectics), law of sensations and stimuli, etc. He got the 2010 Telesio-Galilei Academy of Science Gold Medal, Adjunct Professor (equivalent to Doctor Honoris Causa) of Beijing Jiaotong University in 2011, and 2011 Romanian Academy Award for Technical Science (the highest in the country). Dr. W. B. Vasantha Kandasamy and Dr. Florentin Smarandache got the 2012 New Mexico-Arizona and 2011 New Mexico Book Award for Algebraic Structures. He can be contacted at smarand@unm.edu

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