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# **Econometric Analysis on Efficiency of Estimator**

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## **Abstract**

This paper investigates the efficiency of an alternative to ratio estimator under the super population model with uncorrelated errors and a gamma-distributed auxiliary variable. Comparisons with usual ratio and unbiased estimators are also made.

**Key words:** Bias, Mean Square Error, Ratio Estimator Super Population.

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## **1. Introduction**

It is well known that the ratio method of estimation occupies an important place in sample surveys. When the study variate  $y$  and the auxiliary variate  $x$  is positively (high) correlated, the ratio method of estimation is quite effective in estimating the population mean of the study variate  $y$  utilizing the information on auxiliary variate  $x$ .

Consider a finite population with  $N$  units and let  $x_i$  and  $y_i$  denote the values for two positively correlated variates  $x$  and  $y$  respectively for the  $i$ th unit in this population,  $i=1,2,\dots,N$ . Assume that the population mean  $\bar{X}$  of  $x$  is known. Let  $\bar{x}$  and  $\bar{y}$  be the sample means of  $x$  and  $y$  respectively based on a simple random sample of size  $n$  ( $n < N$ ) units drawn without replacement scheme. Then the classical ratio estimator for  $\bar{Y}$  is defined by

$$\bar{y}_r = \bar{y}(\bar{X} / \bar{x}) \quad (1.1)$$

The bias and mean square error (MSE) of  $\bar{y}_r$  are, up to second order moments,

$$B(\bar{y}_r) = \lambda(R S_x^2 - S_{yx}) / \bar{X} \quad (1.2)$$

$$M(\bar{y}_r) = \lambda(S_y^2 + R^2 S_x^2 - 2R S_{yx}), \quad (1.3)$$

where  $\lambda = (N - n) / (nN)$ ,

$$R = \bar{Y} / \bar{X}, \quad S_y^2 = (N - 1)^{-1} \sum_{i=1}^N (y_i - \bar{Y})^2, \quad S_x^2 = (N - 1)^{-1} \sum_{i=1}^N (x_i - \bar{X})^2,$$

$$\text{and } S_{yx} = (N - 1)^{-1} \sum_{i=1}^N (y_i - \bar{Y})(x_i - \bar{X}).$$

It is clear from (1.3) that  $M(\bar{y}_r)$  will be minimum when

$$R = S_{yx} / S_x^2 = \beta, \quad (1.4)$$

where  $\beta$  is the regression coefficient of  $y$  on  $x$ . Also for  $R = \beta$ ,

the bias of  $\bar{y}_r$  in (1.2) is zero. That is,  $\bar{y}_r$  is almost unbiased for  $\bar{Y}$ .

Let  $E(\bar{y}|\bar{x}) = \alpha + \beta \bar{x}$  be the line of regression of  $\bar{y}$  on  $\bar{x}$ , where E denotes averaging over all possible sample design simple random sampling without replacement (SRSWOR). Then  $\beta = S_{yx}/S_x^2$  and  $\bar{Y} = \alpha + \beta \bar{X}$  so that, in general,

$$R = (\alpha / \bar{X}) + \beta \quad (1.5)$$

It is obvious from (1.4) and (1.5) that any transformation that brings the ratio of population means closer to  $\beta$  will be helpful in reducing the mean square error (MSE) as well as the bias of the ratio estimator  $\bar{y}_r$ . This led Srivenkataramana and Tracy (1986) to suggest an alternative to ratio estimator  $\bar{y}_r$  as

$$\bar{y}_a = \bar{z}(\bar{X}/\bar{x}) + A = \bar{y}_r - A\{(\bar{X}/\bar{x}) - 1\} \quad (1.6)$$

which is based on the transformation

$$\bar{z} = \bar{y} - A, \quad (1.7)$$

where  $E(\bar{z}) = \bar{Z} (= \bar{Y} - A)$  and A is a suitably chosen scalar.

In this paper exact expressions of bias and MSE of  $\bar{y}_a$  are worked out under a super population model and compared with the usual ratio estimator.

## 2. The Super Population Model

Following Durbin (1959) and Rao (1968) it is assumed that the finite population under consideration is itself a random sample from a super population and the relation between x and y is of the form:

$$y_i = \alpha + \beta x_i + u_i ; \quad (i = 1, 2, \dots, N) \quad (1.8)$$

where  $\alpha$  and  $\beta$  are unknown real constants;  $u_i$ 's are uncorrelated random errors with conditional (given  $x_i$ ) expectations

$$E(u_i | x_i) = 0 \quad (1.9)$$

$$E(u_i^2 | x_i) = \delta x_i^g \quad (1.10)$$

( $i=1, 2, \dots, N$ ),  $0 < \delta < \infty$ ,  $0 \leq g \leq 2$  and  $x_i$  are independently identically

distributed (i.i.d.) with a common gamma density

$$G(\theta) = e^{-x} x^{\theta-1} / \Gamma(\theta), \quad x > 0, \quad 2 < \theta < \infty \quad (2.1)$$

We will write  $E_x$  to denote expectation operator with respect to the common distribution of  $x_i$  ( $i=1, 2, 3, \dots, N$ ) and  $E_x E_c$ , as the over all expectation operator for the model. We denote a design by  $p$  and the design expectation  $E_p$ , for instance, see Chaudhuri and Adhikary (1983,89) and Shah and Gupta (1987). Let 's' denote a simple random sample of  $N$  distinct labels chosen without replacement out of  $i=1, 2, 3, \dots, N$ . Then

$$X(=N \bar{X}) = \sum_{i \in s} x_i + \sum_{i \notin s} x_i \quad (2.2)$$

Following Rao and Webster (1966) we will utilize the distributional properties of  $x_j / x_i$ ,  $\sum_{i \in s} x_i$ ,  $\sum_{i \notin s} x_i$ ,  $\sum_{i \in s} x_i / \sum_{i \notin s} x_i$  in our subsequent derivations.

### 3. The bias and mean square error

The estimator  $\bar{y}_a$  in (1.6) can be written as

$$\bar{y}_a = \left[ (1/n) \left( \sum_{i \in S} y_i \right) \frac{\binom{N}{n}}{\binom{N}{i}} - A \left\{ \frac{\binom{N}{n}}{\binom{N}{i}} - 1 \right\} \right] \quad (3.1)$$

based on a simple random sample of n distinct labels chosen without replacement out of  $i = 1, 2, \dots, N$ .

The bias

$$B = E_p ( \bar{y}_a - \bar{Y} ) \quad (3.2)$$

of  $\bar{y}_a$  has model expectation  $E_m(B)$  which works out as follows:

$$\begin{aligned} E_m ( B ( \bar{y}_a ) ) &= E_p E_x E_c \left[ \left\{ \alpha + \beta (1/n) \left( \sum_{i \in S} x_i \right) + \bar{u} \right\} \frac{n \sum_{i=1}^N x_i}{n \sum_{i \in S} x_i} \right. \\ &\quad - A \left\{ \frac{n \left( \sum_{i=1}^N x_i \right) - 1}{N \left( \sum_{i \in S} x_i \right)} \right\} - \\ &\quad \left. - E_x E_c ( \alpha + \beta \bar{x} + \bar{U} ) \right] \\ &= E_p E_x E_c \\ &\quad \left[ \alpha \left( \frac{n \sum_{i=1}^N x_i}{N \sum_{i \in S} x_i} \right) + \beta (1/N) \left( \sum_{i=1}^N x_i \right) + \left( \sum_{i \in S} u_i \right) \left( \frac{\sum_{i=1}^N x_i}{N \sum_{i \in S} x_i} \right) - A \left\{ \left( \frac{n \sum_{i=1}^N x_i}{N \sum_{i \in S} x_i} \right) - 1 \right\} \right] \end{aligned}$$

$$- E_x E_c (\alpha + \beta \bar{X})$$

$$= E_p E_x \left[ \alpha \left( n \sum_{i=1}^N x_i \right) / \left( N \sum_{i \in S} x_i \right) + \beta \bar{X} - A \left\{ \left( n \sum_{i=1}^N x_i \right) / \left( N \sum_{i \in S} x_i \right) - 1 \right\} \right] - \alpha - \beta E_x (\bar{X})$$

$$= E_x \left[ \alpha (n/N) \left( 1 + \sum_{i \notin S} x_i / \sum_{i \in S} x_i \right) - A \left\{ (n/N) \left( 1 + \sum_{i \notin S} x_i / \sum_{i \in S} x_i \right) - 1 \right\} \right] - \alpha$$

$$= \alpha (n/N) \{ 1 + (N-n)\theta / (n\theta - 1) \}$$

$$- A \{ (n/N) (1 + (N-n)\theta / (n\theta - 1)) - 1 \} - \alpha$$

$$= \alpha \left[ (n/N - 1) + \{ n(N-n)\theta / N(n\theta - 1) \} \right]$$

$$- A \left[ - (N-n) / N + \{ (N-n)n\theta / N(n\theta - 1) \} \right]$$

$$= (N-n) (\alpha - A) / N(n\theta - 1) \quad (3.3)$$

For SRSWOR sampling scheme , the mean square error

$$M (\bar{y}_a) = E_p (\bar{y}_a - \bar{Y})^2 \quad (3.4)$$

of  $\bar{y}_a$  has the following formula for model expectations

$$E_m (M (\bar{y}_a)) :$$

$$E_m (M (\bar{y}_a)) = [E_m (M (\bar{y}_r)) + (N-n)(Nn\theta + 2N - 2n)(A^2 - 2A\alpha) / N^2 (n\theta - 1)(n\theta - 2)]$$

$$(3.5)$$

where

$$M(\bar{y}_r) = E_p(\bar{y}_r - \bar{Y})^2 \quad (3.6)$$

is the MSE of  $\bar{y}_r$  under SRSWOR scheme has the model expectation

$$E_m(M(\bar{y}_r)) = \left\{ (N-n)/N^2 \right\} \left[ \left\{ \frac{(Nn\theta + 2N - 2n)\alpha^2}{(n\theta - 1)(n\theta - 2)} \right\} + \frac{\delta \left\{ (n\theta + g - 1)(n\theta + g - 2) + n\theta(N\theta - n\theta + 1) \right\}}{(n\theta + g - 1)(n\theta + g - 2)} \frac{\Gamma(\theta + g)}{\Gamma\theta} \right] \quad (3.7)$$

[See, Rao(1968, p.439)]

Further, we note that for SRSWOR sampling scheme, the bias

$$B(\bar{y}_r) = E_p(\bar{y}_r - \bar{Y}) \quad (3.8)$$

of usual ratio estimator has the model expectation

$$E_m(B(\bar{y}_r)) = (N-n)\alpha / (n\theta - 1) \quad (3.9)$$

We note from (3.3) and (3.9) that

$$|E_m(B(\bar{y}_a))| < |E_m(B(\bar{y}_r))|$$

if

$$|(\alpha - A)| < |\alpha|$$

or if

$$(\alpha - A)^2 < \alpha^2$$

or if

$$\alpha < A < 2\alpha \quad (3.10)$$



Further we have from (3.5) that

$$E_m(M(\bar{y}_a)) - E_m(M(\bar{y}_r)) < 0$$

if

$$(A^2 - 2A\alpha) < 0$$

or if

$$0 < A < 2\alpha \quad (3.11)$$

which is the same as in (3.10).

Thus we state the following theorem:

Theorem 3.1 : The estimator  $\bar{y}_a$  is less biased as well as more efficient than usual ratio estimator  $\bar{y}_r$  if

$$0 < A < 2\alpha \quad (\alpha \neq 0)$$

i. e. when A lies between 0 and  $2\alpha$ .

Therefore, when intercept term  $\alpha (\neq 0)$  in the model (2.1) is sizable, there will be sufficient flexibility in picking A.

It is to be noted that for  $\alpha = 0$ ,  $\bar{y}_r$  is unbiased and efficient than  $\bar{y}_a$ .

The minimization of (3.5) with respect to A leads to

$$A = \alpha = A_{\text{opt}} \text{ (say)} \quad (3.12)$$

Substitution of (3.12) in (3.5) yields the minimum value of

$$E_m(M(\bar{y}_a)) \text{ as}$$

$$\min. E_m(M(\bar{y}_a)) = \frac{(N-1)}{N^2} \frac{\delta[(n\theta + g - 1)(n\theta + g - 2) + n\theta(N\theta - n\theta + 1)] \Gamma(\theta + g)}{(n\theta + g - 1)(n\theta + g - 2) \Gamma\theta}$$

(3.13)

which equals to  $E_m(M(\bar{y}_r))$  when  $\alpha = 0$ .

It is interesting to note that when  $A = \alpha, \bar{y}_a$  is unbiased and attained its minimum average MSE in model (2.1).

In practice the value of  $\alpha$  will have to be assessed, at the estimation stage, to be used as  $A$ . To assess  $\alpha$ , we may use scatter diagram of  $y$  versus  $x$  for data from a pilot study, or a part of the data from the actual study and judge the  $y$ -intercept of the best fitting line.

From (3.7) and (3.13) we have

$$E_m(M(\bar{y}_r)) - \min.E_m(M(\bar{y}_a)) = \left\{ (N-n)(Nn\theta + 2N - 2n)\alpha^2 \right\} / \left\{ N^2(n\theta - 1)(n\theta - 2) \right\} > 0 \quad (3.14)$$

which shows that  $\bar{y}_a$  is more efficient than ratio estimator when  $A = \alpha$  is known exactly. For  $\alpha = 0$

$$\min.E_m(M(\bar{y}_a)) = E_m(M(\bar{y}_r)) \quad (3.15)$$

For SRSWOR, the variance

$$V(\bar{y}) = E_p(\bar{y} - \bar{Y})^2 \quad (3.16)$$

of usual unbiased estimator has the model expectation:

$$E_m(V(\bar{y})) = (N-n) \left[ \beta^2 \theta + \{ \delta \Gamma(\theta + g) / \Gamma \theta \} \right] / nN \quad (3.17)$$

The expressions of  $E_m(M(\bar{y}_a))$  and  $E_m(V(\bar{y}))$  are not easy task to compare algebraically. Therefore in order to facilitate the comparison, denoting

$$E_1 = 100 E_m(V(\bar{y})) / E_m(M(\bar{y}_a)) \quad \text{and} \quad E_2 = 100 E_m(V(\bar{y}_r)) / E_m(M(\bar{y}_a)),$$

we present below in tables 1,2,3, the values of the relative efficiencies of

$\bar{y}_a$  with respect to  $\bar{y}$  and  $\bar{y}_r$  for a few combination of the parametric values under the model (2.1). Values are given for  $N = 60$ ,  $\delta = 2.0, \theta = 8, \alpha = 0.5, 1.0, 1.5$ ,  $\beta = 0.5, 1.0, 1.5$  and  $g = 0.0, 0.5, 1.0, 1.5, 2.0$ . The ranges of A, for  $\bar{y}_a$  to be better than  $\bar{y}_r$  for given  $\alpha = 0.5, 1.0, 1.5$  are respectively (0,1), (0,2), (0,3). This clearly indicates that as the size of  $\alpha$  increases the range of A for  $\bar{y}_a$  to be better than  $\bar{y}_r$  increases i.e. flexibility of choosing A increases.

We have made the following observations from the tables 1,2 and 3 :

- (i) As  $g$  increases both  $E_1$  and  $E_2$  decrease. When  $n$  increases  $E_1$  increases while  $E_2$  decreases.
- (ii) As  $\alpha$  increases ( i.e. if the intercept term  $\alpha$  departs from origin in positive direction) relative efficiency of  $\bar{y}_a$  with respect to  $\bar{y}$  decreases while  $E_2$  increases.
- (iii) As  $\beta$  increases  $E_1$  increases for fixed  $g$  while  $E_2$  is unaffected.
- (iv) The maximum gain in efficiency is observed over  $\bar{y}$  as well as over  $\bar{y}_r$  if A coincide with the value of  $\alpha$ . Finally, the estimator  $\bar{y}_a$  is to be preferred when the intercept term  $\alpha$  departs substantially from origin.

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**Table 1: Relative efficiencies of  $\bar{y}_a$  with respect to  $\bar{y}$  and  $\bar{y}_r$**

$\alpha = 0.5$							
g	$\beta$	n = 10					
		E <sub>1</sub>			E <sub>2</sub>		
		A			A		
		0.30	0.60	0.90	0.30	0.60	0.90
0.0	0.5	192.86	193.23	191.40	101.34	101.54	100.57
	1.0	482.16	483.16	478.09	101.34	101.54	100.57
	1.5	964.32	966.17	956.98	101.34	101.54	100.57
0.5	0.5	132.67	132.77	132.30	100.49	100.56	100.21
	1.0	237.82	237.99	237.16	100.49	100.56	100.21
	1.5	413.08	413.36	411.93	100.49	100.56	100.21
1.0	0.5	111.06	111.08	110.95	10.17	100.19	100.07
	1.0	148.08	148.11	147.93	10.17	100.19	100.07
	1.5	209.78	209.83	209.57	10.17	100.19	100.07
1.5	0.5	103.99	104.00	103.96	100.06	100.07	100.03
	1.0	116.64	116.65	116.60	100.06	100.07	100.03
	1.5	137.71	137.72	137.66	100.06	100.07	100.03
2.0	0.5	102.23	102.23	102.22	100.02	100.02	100.01
	1.0	106.43	106.43	106.42	100.02	100.02	100.01
	1.5	113.43	113.43	113.42	100.02	100.02	100.01

$\alpha = 0.5$							
$g$	$\beta$	$n = 20$					
		$E_1$			$E_2$		
		A			A		
		0.30	0.60	0.90	0.30	0.60	0.90
0.0	0.5	196.58	196.96	195.11	103.33	101.52	100.56
	1.0	491.46	492.39	487.77	103.33	101.52	100.56
	1.5	982.92	984.39	975.53	103.33	101.52	100.56
0.5	0.5	134.37	134.46	134.46	100.48	100.55	100.20
	1.0	240.86	241.02	240.02	100.48	100.55	100.20
	1.5	418.35	418.63	417.20	100.48	100.55	100.20
1.0	0.5	111.76	111.79	111.65	100.17	100.19	100.07
	1.0	149.01	149.05	148.87	100.17	100.19	100.07
	1.5	211.10	211.16	210.90	100.17	100.19	100.07
1.5	0.5	104.00	104.00	103.96	100.06	100.07	100.02
	1.0	116.64	116.65	116.60	100.06	100.07	100.02
	1.5	137.71	137.73	137.67	100.06	100.07	100.02
2.0	0.5	101.60	101.60	101.58	100.02	100.02	100.01
	1.0	105.77	105.77	105.76	100.02	100.02	100.01
	1.5	112.73	112.73	112.73	100.02	100.02	100.01

**Table 2: Relative efficiencies of  $\bar{y}_a$  with respect to  $\bar{y}$  and  $\bar{y}_r$**

$\alpha = 1.0$									
g	$\beta$	n = 10							
		E <sub>1</sub>				E <sub>2</sub>			
		A				A			
		0.50	1.0	1.50	1.90	0.50	1.0	1.50	1.90
0.0	0.5	190.31	193.36	190.31	183.82	104.73	106.41	104.73	101.16
	1.0	475.78	483.40	475.78	459.55	104.73	106.41	104.73	101.16
	1.5	951.55	966.79	951.55	919.10	104.73	106.41	104.73	101.16
0.5	0.5	132.03	132.80	132.03	130.34	101.73	102.32	101.73	100.43
	1.0	236.67	238.05	236.67	233.65	101.73	102.32	101.73	100.43
	1.5	411.07	413.46	411.07	405.82	101.73	102.32	101.73	100.43
1.0	0.5	110.87	111.09	110.87	110.36	100.61	100.82	100.61	100.15
	1.0	147.82	148.12	147.82	147.15	100.61	100.82	100.61	100.15
	1.5	209.42	209.84	209.42	208.46	100.61	100.82	100.61	100.15
1.5	0.5	103.93	104.00	103.93	103.77	100.21	100.28	100.21	100.05
	1.0	116.57	116.65	116.57	116.39	100.21	100.28	100.21	100.05
	1.5	137.63	137.73	137.63	137.41	100.21	100.28	100.21	100.05
2.0	0.5	102.21	102.23	102.21	102.15	100.67	100.09	100.07	100.01
	1.0	106.41	106.43	106.41	106.3	100.67	100.09	100.07	100.01
	1.5	113.41	113.43	113.41	113.35	100.67	100.09	100.07	100.01

$\alpha = 1.0$									
g	$\beta$	n = 20							
		E <sub>1</sub>				E <sub>2</sub>			
		A				A			
		0.50	1.0	1.50	1.90	0.50	1.0	1.50	1.90
0.0	0.5	194.01	197.08	194.01	187.47	104.67	106.33	104.67	101.14
	1.0	485.03	492.70	485.03	468.68	104.67	106.33	104.67	101.14
	1.5	970.06	985.40	970.06	937.36	104.67	106.33	104.67	101.14
0.5	0.5	133.73	134.49	133.73	132.05	101.70	102.28	101.70	100.08
	1.0	239.71	241.08	239.71	236.71	101.70	102.28	101.70	100.08
	1.5	416.35	418.73	416.35	411.13	101.70	102.28	101.70	100.08
1.0	0.5	111.07	111.08	111.07	111.08	100.60	100.80	100.60	100.15
	1.0	148.77	149.06	148.77	148.11	100.60	100.80	100.60	100.15
	1.5	210.75	211.17	210.75	209.82	100.60	100.80	100.60	100.15
1.5	0.5	103.94	104.01	103.94	103.78	100.20	100.27	100.20	100.05
	1.0	116.57	116.65	116.57	116.40	100.20	100.27	100.20	100.05
	1.5	137.64	137.73	137.64	137.42	100.20	100.27	100.20	100.05
2.0	0.5	101.58	101.60	101.58	101.52	100.07	100.09	100.07	100.01
	1.0	105.75	105.77	105.75	105.70	100.07	100.09	100.07	100.01
	1.5	112.71	112.73	112.71	112.65	100.07	100.09	100.07	100.01



**Table 3: Relative efficiencies of  $\bar{y}_a$  with respect to  $\bar{y}$  and  $\bar{y}_r$**

$\alpha = 1.5$											
g	$\beta$	n = 10									
		E <sub>1</sub>					E <sub>2</sub>				
		A					A				
		0.60	1.20	1.80	2.40	2.90	0.60	1.20	1.80	2.40	2.90
0.0	0.5	183.82	192.25	192.25		183.82	108.77	113.76	113.76	108.77	
					171.79						101.65
	1.0	459.55	480.62	480.62		459.55	108.77	113.76	113.76	108.77	
					429.47						101.65
	1.5	919.10	961.25	961.25		919.10	108.77	113.76	113.76	108.77	
					858.94						101.65
0.5	0.5	130.34	132.52	132.52		130.34	103.29	105.01	105.01	103.29	
					127.01						100.64
	1.0	233.64	237.55	237.55		233.65	103.29	105.01	105.01	103.29	
					227.67						100.64
	1.5	405.82	412.60	412.60		405.82	103.29	105.01	105.01	103.29	
					395.44						100.64
1.0	0.5	110.36	111.01	111.01		110.36	101.17	101.77	101.77	101.17	
					109.34						100.23
	1.0	147.15	148.02	148.02		147.15	101.17	101.77	101.77	101.17	
					147.79						100.23
	1.5	208.46	209.69	209.69		208.46	101.17	101.77	101.77	101.17	
					206.53						100.23
1.5	0.5	103.77	103.98	103.98		103.77	100.40	100.60	100.60	100.40	
					103.44						100.08
	1.0	116.39	116.62	116.62		116.39	100.40	100.60	100.60	100.40	

				116.01					100.08
	1.5	137.41	137.69	137.69	137.41	100.40	100.60	100.60	100.40
				139.68					100.08
	0.5	102.15	102.22	102.22	102.15	100.13	100.20	100.20	100.13
				102.04					100.03
2.0	1.0	106.35	106.42	106.42	106.35	100.13	100.20	100.20	100.13
				106.24					100.03
	1.5	113.35	113.42	113.42	113.35	100.13	100.20	100.20	100.13
				113.23					100.03

$\alpha = 1.5$											
G	$\beta$	n = 20									
		E <sub>1</sub>					E <sub>2</sub>				
		A									
		0.60	1.20	1.80	2.40	2.90	0.60	1.20	1.80	2.40	2.90
0.0	0.5	187.47	196.97	195.97		187.47	108.67	113.59	113.59	108.67	
				175.33							101.63
	1.0	468.68	489.91	489.91		468.68	108.67	113.59	113.59	108.67	
				438.34							101.63
	1.5	937.36	979.83	979.83		937.36	108.67	113.59	113.59	108.67	
				876.67							101.63
0.5	0.5	132.05	134.21	134.21		132.05	103.23	104.92	104.92	103.23	
				128.73							100.63
	1.0	236.70	240.58	240.58		236.70	103.23	104.92	104.92	103.23	
				230.76							100.63
	1.5	411.13	417.87	417.87		411.13	103.23	104.92	104.92	103.23	
				400.80							100.63

1.0	0.5	111.08	111.72	111.72	111.08	101.14	101.72	101.72	101.14
				110.08					100.23
	1.0	148.11	148.96	148.96	148.11	101.14	101.72	101.72	101.14
			146.77						100.23
	1.5	209.82	211.02	211.02	209.82	101.14	101.72	101.72	101.14
			207.92						100.23
1.5	0.5	103.78	103.98	103.98	103.78	100.39	100.58	100.58	100.39
				103.46					100.08
	1.0	116.40	116.62	116.62	116.40	100.39	100.58	100.58	100.39
			116.40						100.08
	1.5	137.43	137.70	137.70	137.43	100.39	100.58	100.58	100.39
			137.00						100.08
2.0	0.5	101.53	101.59	101.59	101.53	100.13	100.19	100.19	100.03
				101.42					100.03
	1.0	105.70	105.77	105.77	105.70	100.13	100.19	100.19	100.03
			105.59						100.03
	1.5	112.65	112.72	112.72	112.65	100.13	100.19	100.19	100.03
			112.54						100.03