THE EFFECT OF THE NEUTROSOPHIC LOGIC ON THE DECISION TREE

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ÖNSÖZ

Bu kitap Prof. Dr. Necati OLGUN danışmanlığında Gaziantep Üniversitesi Fen Bilimleri Enstitüsü' nde hazırlamış olduğum "NÖTROSOFİK MANTIĞIN KARAR AĞACI ÜZERINDEKI ETKİSİ" adlı doktora tezimden üretilmiştir.

Çalışmanın hazırlanması sırasında ayırdığı değerli zaman ve sağladığı destek için araştırmalarıma büyük katkıda bulunan Sayın Prof. Dr. Necati OLGUN' a sonsuz minnet ve saygılarımı sunarım.

Ahmed HATİP

Gaziantep, 2023

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LIST OF SYMBOLS

$\boldsymbol{I}$	Indeterminacy
$\boldsymbol{K}$	Field
$\boldsymbol{F}$	falsehood
$\boldsymbol{T}$	truth

LIST OF ABBREVIATIONS

EMV	Expected Monetary Value
NEMV	Neutrosophic Expected Monetary Value

CHAPTER I

INTRODUCTION

1.1 Motivation of Study

Ideas, like any living being, are born and nurtured and go through several stages. As soon as ideas are born, they begin to oscillate and sway between affirmation and denial, truth and lie, right and left; passing through many cases between the two opposite sides, including that moderate point which is equidistant between the two opposites where contradictions shake hands, embrace softly and complementarity. We may even reach a state of rapid fluctuation that turns issues upside down. What was a false case yesterday is proven to be true today. What one person confirms today with proof and conclusive evidence, another may deny it tomorrow. Thus, all constants are subject to change and all facts are subject to falsification.

Amid book contradictions and problems, you have to put things in perspective .You have to put the right thing in its rightful place, and the most effective way to do that is logic.

Historically, logic appeared in the writings of the Greek philosopher Aristotle (238 BC - 322 BC), and his discovery of the syllogism. Aristotle defined the analogy as “the inference in which the result follows the premises necessarily”. He presented the first formulation of the laws of syllogism in his book Analytics, "The First Analytics". Aristotelian logic was a formal abstraction that did not care about the content of the issue, but it was not symbolic/mathematical logic. At the end of the sixteenth century, the German philosopher and mathematician Gottfried Leibnitz (1646-1716) laid the foundations of modern mathematical logic as a deductive system that relies on signs - within special rules - in isolation from their meaning. Then the British scholar George Paul (1815-1864) invented the Boolean algebra system with commutative and associative properties, which would later allow its use in designing software circuits.

for computers, and in this respect, the work of Alfred Whitehood and Bertrand Russell in the development of mathematical logic should be commended.

After these mighty works, logic began to be crystallized in its modern form, where classical logic gives a result (0 or 1), yes or no, true or false, white or black for a case. Then, the fuzzy logic came out, which was first presented in the sixties of the last century by the Azerbaijani Professor Lotfi Zadeh [1]. This logic describes a case more broadly than (0 or 1), yes or no, true or false, white or black as it gives degrees of validity and invalidity. Thus, he filled a gap that existed in classical logic. After that, the poet and philosopher Professor Florentin Smarandache (Professor and Head of the Mathematics Department at the University of New Mexico, USA) inspired us with neutrosophic logic which is the focus of our research in this thesis. That logic is an extension of the fuzzy logic, but it includes an important factor which is indeterminacy (I) [2-4]. The indeterminacy is the gap that the fuzzy logic overlooked. Thus, the neutrosophic logic opened a wide door to understanding the relationship between the contradictions. According to him, things are not absolutely fixed or absolutely changing. There is only one absolute truth, which is God; everything else oscillates between relative stability and relative change. You are certainly able to describe the same thing as true and false, and yes and no together. This means that there is a situation that combines the two contradictions between lying and honesty, white and black, or two contracted points of view. This new neutral position between the contradicting sides is the neutrosophic logic and philosophy, which entered all aspects of life, introducing new concepts and interpretations that were always unexpected to us.

In the concept of neutrosophic logic and neutrosophic sets, there is a degree of membership T, a degree of uncertainty I and a degree of non-membership F. These are defined independently from each other. A neutrosophic value has the form (T, I, F). In other words, in neutrosophy, a situation is handled according to its trueness, its falsity, and its uncertainty. Therefore, neutrosophic logic and neutrosophic sets help us explain many uncertainties in our lives. Therefore, many researchers have made studies on this subject [5-7]. Recently, Şahin et al. obtain some operations for interval valued neutrosophic sets [8]; Uluçay et al. studied neutrosophic multigroups and Applications [9]; Hassan et al. introduced Q-neutrosophic soft expert set and its application [10]; Sahin et al. obtained neutrosophic soft expert sets [11]; Uluçay studied interval-valued refined neutrosophic sets and their applications [12]; Khalifa et al. obtained neutrosophic set significance on deep transfer learning models [13]; Kargın et al. studied generalized Hamming similarity measure aased on neutrosophic quadruple numbers and its applications [14]; Şahin et al. obtain Hausdorff Measures on generalized set valued neutrosophic quadruple numbers and decision making applications for adequacy of online education [15].

In 2019, Florentin Smarandache introduced new research areas in neutrosophy, which he called neutro-structures and anti-structures [16-17]. When evaluating <A> as an element (concept, attribute, idea, proposition, theory, etc.), during the neutrosification process, he worked on three regions; two opposites corresponding to <A> and <antiA> and also a neutral (indeterminate) <neutA> (also called <neutralA>). A neutro-algebra consists of at least one neutro-operation (indeterminate for other items and false for other items) or it is an algebra well-defined for some items (also called internally defined), indeterminate for others, and externally defined for others. Therefore, the subject attracted the attention of many researchers [18–24]. Recently, Smarandache et al. studied neutro-BCK algebras [25]; İbrahim et al. obtained neutro -vector spaces [26]; Al-Tahan et al. studied NeutroOrdered Algebra and applications [27]; Smarandache studied generalizations and alternatives of classical algebraic structures to neutroalgebraic structures and antialgebraic structures [28].

For example: In Physics, Florentin Smarandache presented a series of topics in "Quantum Mechanics". One of them is that there is no maximum speed limit in the universe, and this contradicts Einstein's relativity [29]. He also put forward the possibility of a third form between a matter and its opposite, which he called "netro-matter". The same case is in economics, medicine, politics, and artificial intelligence. There is no doubt about mathematics, as the idea of indeterminacy (I) has created for us a new mathematical logic that has produced new algebra and new concepts of probability, statistics, measurement, integration, and derivation [30-45].

In this book, we will present the neutrosophic decision-making mechanism which is an extension of the classical decision-making process by extending the data to include the indefinite cases that are ignored by classical logic and which, in fact, support the decision-making problem. This book consists of eight chapters. In the introductory part of the thesis, the historical development process of the neutrosophic structure theory is given. In the second part, the effect of the neutrosophic logic on the decision tree has been compiled. In the third chapter, the Prospector Neutro Function with their applications were studied. In the fourth chapter, the subject of Neutro ordered R-module and their properties is examined in detail. In the fifth chapter, the Fundamental Theorem in neutrosophic Euclidean Geometry is given. In the sixth chapter, the solutions of some Kandasamy-Smarandache problems about neutrosophic complex numbers and group of units' problem are given. In the seventh chapter, the algebraic creativity in the neutrosophic square matrices and the results are given with examples. Finally, in the eighth chapter, the results and suggestions obtained in the thesis are given.

CHAPTER II

NEUTROSOPHIC DECISION TREE

In this chapter, we present neutrosophic decision-making, which is an extension of the classical decisionmaking process by expanding the data to cover the non-specific cases ignored by the classical logic which, in fact, supports the decision-making problem.

Definition 2.1.1: [1] Let U be a universe of discourse, and let $A \subset U$ be a subset. Then: $A_{\mathrm{FS}} = \{ \left(x, T_A(x)\right): x \in U \}$, where $T_A: U \longrightarrow [0,1]$ is the membership degree of the generic element x with respect to the set A, is called a Fuzzy Set.

Definition 2.1.2: [3] Let $\mathcal{U}$ be a universe of discourse, and let $A \subset U$ be a subset. Then: $A_{IFS} = \{\left(x, T_A(x), F_A(x)\right): x \in U \}$, where $T_A(x), F_A(x): U \longrightarrow [0,1]$ are the membership degree respectively the nonmembership of the generic element x with respect to the set A, and sup $T_A(x) + supF_A(x) \leq 1$ for all $x \in U$, is called an Intuitionistic Fuzzy Set.

Definition 2.1.3: [2] Let U be a universe of discourse, and a set $A_{NS} \subset U$ then: $A_{NS} = \{(x, T_A(x), I_A(x), F_A(x)): x \in U \}$, where $T_A(x), I_A(x), F_A(x): U \rightarrow [0,1]$ represent the represent the degree of membership, the degree of indeterminacy, and the degree of non-membership respectively of each element $x \in U$ to the set A.

Definition 2.1.4: [47] Let K be a field, the neutrosophic field generated by $\langle K \cup I \rangle$ which is denoted by $K(I) = \langle K \cup I \rangle$.

Definition 2.1.5: [48] Classical neutrosophic number has the form $a + bI$ where a, b are real or complex numbers and I is the indeterminacy such that $0 \cdot I = 0$ and $I^2 = I$ which results that $I^n = I$ for all positive integers n.

Definition 2.1.6: [49] The neutrosophic probability of event A occurrence is $NP(A) = \big(ch(A), ch(neutA), ch(antiA)\big) = (T, I, F)$ where T, I, F standard or nonstandard subsets of the nonstandard unitary interval are $]^{-}0,1^{+}[$.

2.2 CLASSICAL DECISION TREE

We know from the definition of the Classical Decision Tree that it is a graphic in the form of a tree gives options and is used in choosing options in the case of one scale. Its root starts from the left and its branches spreads into the right showing the options and the possibilities of the natural causes (events). It is considered to be a suitable method to make a decision if one is not sure, and it is one of the strongest mathematical methods that is used to analyze many problems [50].

To build the classical decision tree we can follow this algorithm:

1- Expected Monetary Value (EMV)
2- Calculate the future monetary value for each option
3- Choose the option with the highest EMV.

Example 2.2.1: [50] You need to travel from one city to another to attend an important business meeting. Failure to attend the meeting will cost you 4000$

You can take either airline X or airline$Y$.

Knowing the following information, which airline would you choose?

Airline X costs 900$ and the give you a 90 % chance of arriving on time

Airline Y costs 300$ and the give you a 70 % chance of arriving on time

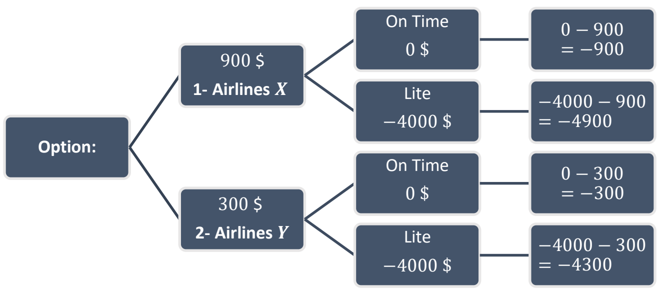

Figure 2.1 Classical decision tree

$EMV_1 = [0.1(-4900)] + [0.9(-900)] = -1300\$$

$EMV_2 = [0.3(-4300)] + [0.7(-300)] = -1500\$$

According to the graphic above, traveling in Airlines X is the best option because it includes the highest Expected Monetary Value (EMV).

The Neutrosophic Decisions Tree is the Classical Decision Tree with adding some indetermination to the data or by exchanging the classical probabilities with neutrosophical probabilities.

2.3 NEUTROSOPHIC DECISION TREE

Building the neutrosophic tree of decisions without including the probabilities is considered to be a suitable option when the decision makers don't have enough information that can make them estimate the probability of the events that built up the tree of decisions. It is also suitable at analyzing the best or the worst options away from probabilities. This theory agrees with the concept of the classical tree of decision. However, what the neutrosophic logic adds to the tree of decision without probabilities is that the expected benefits that matches each option, which is usually evaluated by the decision makers, according to their expertise or by related skills, will be evaluated more accurately and generally with less possible mistakes.

From another side, we may see that the expected values of the benefits whether good or bad are agreed on by some experts but others disagree. Therefore, the best solution to face this problem that absolutely affects the quality of the taken decision is to take the expected benefits with adding and reducing a value interval between (0) and another determinate value, for example (a). (0) which represents the minimum value in this interval means that there is no disagreement on the expected values among the experts or with the decision makers. (a) Which represents the maximum value in this interval means that there is a disagreement among the experts or between them and the decision makers about the expected values of benefits and (a) is the highest estimated value.

Therefore, we will present the expected value of benefits with adding and reducing the interval $[0, a]$ not forgetting that all the various opinions about the expected values will be contained in the $[0, a]$ interval. So that, the expected value of benefits will become an interval of values containing all the opinions.

By doing this, we move from the classical form that gives a determinant value of benefits in the neutrosophic form that doesn't do that, but gives an interval of expected values of benefits [51].

For example, we can consider three options d_1, d_2 and d_3 by the best and the worst expectations as it is clarified in the following:

Table 1.2 Classical and Neutrosophic Part of the Expected Values

	High turnout	Low turnout
d_1	$A \pm i_1$	$B \pm i_2$
d_2	$C \pm i_3$	$D \pm i_4$
d_3	$E \pm i_5$	$F \pm i_6$

A, B, C, D, E, F Represents the determinate part of the expected values.

$i_1, i_2, i_3, i_4, i_5, i_6$ Represents the indeterminate part of the expected values.

$i_k \in [0, i_k] : k = 1,2,3,4,5,6$

Numerical Example 2.3.1: [51] If the decision maker faces three options to invest in Education. These options are Science Institute, Languages Institute and Kindergarten. And for each option we have two natural causes (High turnout) and (Low turnout) depending on the following data, the benefits will change according to two variables (the options and the natural causes).

The experts evaluated the benefits saying that the Science Institute in case of (High turnout) will give the benefits of (55000) with an indeterminate value of estimation interval between [0,4000], and in case of (Low turnout) it will give the benefits of (8000) with an indeterminate value of estimation interval between [0,2000].

They also say that the Languages Institute in case of (High turnout) will give the benefits of (50000) with an indeterminate value of estimation interval between

[0,18000], and in case of (Low turnout) it will give the benefits of (20000) with an indeterminate value of estimation interval between [0,1000].

In addition, the Kindergarten in case of (High turnout) will give the benefits of (40000) with an indeterminate value of estimation interval between [0, 3000], and in case of (Low turnout) it will give the benefits of (18000) with an indeterminate value of estimation interval between [0, 4000].

Table 2.2 Example (Neutrosophic Case)

	High turnout	**Low turnout**
Science Institute	$\mathbf{55000 \mp [0, 400]}$	$\mathbf{8000 \mp [0, 2000]}$
Languages Institute	$\mathbf{50000 \mp [0, 18000]}$	$\mathbf{20000 \mp [0, 1000]}$
Kindergarten	$\mathbf{40000 \mp [0, 3000]}$	$\mathbf{18000 \mp [0, 4000]}$

Table 3.2 Example (Neutrosophic Case)

	High turnout	**Low turnout**
Science Institute	$[41000, 61000]$	$[6000, 10000]$
Languages Institute	$[32000, 68000]$	$[19000, 21000]$
Kindergarten	$[37000, 43000]$	$[14000, 22000]$

2.4 The Studying Of Approaches:

2.4.1 The Optimistic Approach

We know that this approach depends on evaluating the options paving the way to choose the option that guarantee the best possible benefits under the optimistic natural cases without taking the pessimistic cases for this option into consideration. This case is referred to as (**High** Max) as the first (Max) refers to the highest monetary value and the second (Max) the optimistic natural case [51].

Table 4.2 The Optimistic Approach (Neutrosophic Case)

	High Max
Science Institute	$\max[41000, 61000] = 61000$
Languages Institute	$\max[32000, 68000] = 68000$
Kindergarten	$\max[37000, 43000] = 43000$

According to The Optimistic Approach, investing in Languages Institute is the best option because it includes the most possible benefit (680000).

We notice that if we put $i_1 = i_3 = i_5 = 0$ (in the **Table (2)**) we returns to the classical case of the tree of decisions according to The Optimistic Approach and we notice the following:

Table 5.2 The Optimistic Approach (Classical Case)

	High turnout
Science Institute	**55000**
Languages Institute	**50000**
Kindergarten	**40000**

We notice that the highest monetary value in the classical optimistic case with high turnout is (55000). This leads us to take a decision that the investment in the Science Institute is the best option.

Consequently, we notice that there is a differentiation in the taken decisions when we widen the data (that represents the expected values of benefits) netrosophically. Moreover, it is normal to see that the resulted decision that comes from the neutrosophic form is better for investing than the classical one, because it is built upon more data including all the opinions and then the resulted decision is highly agreed on.

2.4.2 The Pessimistic Approach

Know that this approach depends on adjusting the options paving the way to choose the option that guarantee the best possible benefits under the pessimistic normal cases without taking the optimistic cases for this option into consideration. This case is referred to as (**Low, Max**) as the first (Max) refers to the highest monetary value, but it is related to the second part (Min) which is the pessimistic natural case [51].

Table 6.2 The pessimistic Approach (Neutrosophic Case)

	Low Max
Science Institute	$\max[6000,10000] = 10000$

Table 6.2 Continued

Languages Institute	$\max[19000, 21000] = 21000$
Kindergarten	$\max[14000, 22000] = 22000$

According to The Pessimistic Approach, investing in the Kindergarten is the best option because it includes the most possible benefit (22000).

We notice that if we put $i_2 = i_4 = i_6 = 0$ (in the **Table (2)**) we returns to the classical case of the tree of decisions according to The Pessimistic Approach and we notice the following:

Table 7.2 The pessimistic Approach (Classical Case)

	Low turnout
Science Institute	8000
Languages Institute	20000
Kindergarten	18000

We notice that the highest monetary value in the natural pessimistic case with low turnout is (20000). This leads us to take a decision that the investment in the Languages Institute is the best option.

By comparing this classical form with neutrosophic form, we find that the decision of choosing an option is changed. According to the neutrosophic form, this approach leads us to invest in The Kindergarten, but according to the classical form, it leads us to invest in The Languages Institute. However, when the data are defined accurately, it will absolutely lead us to the correct and best option.

2.4.3 The Caution Approach

This approach is not an optimistic nor a pessimistic one. It is a moderate approach that depends on adjusting the options too in order to choose the best option without losing any possible opportunity [51].

And choosing the most suitable option according to this approach demands to build a new matrix as the following by exchanging the option that makes the highest monetary value of zero (after taking the high value of the interval) taking into consideration that there is no lost opportunities for this option:

Table 8.2 The Caution Approach (Neutrosophic Case)

	High turnout	Low turnout
Science Institute	[32000,68000]-[41000,61000]	[14000,22000]-[6000,10000]
Languages Institute	[32000,68000]-[32000,68000]	[14000,22000]-[19000,21000]
Kindergarten	[32000,68000]-[37000,43000]	[14000,22000]-[14000,22000]

Table 9.2 The Caution Approach (Neutrosophic Case)

	High turnout	Low turnout
Science Institute	$[-9000, 7000]$	$[8000, 12000]$
Languages Institute	$[0, 0]$	$[-3000, 1000]$
Kindergarten	$[-5000, 25000]$	$[0, 0]$

We reduced the highest monetary value in the High turnout case from the other available monetary values in this natural case. Also, we reduced the highest monetary value in the Low turnout case from the other available monetary values in this case.

Now we make a concise matrix that includes the highest values of the lost opportunities for each option as the following:

Table 10.2 The Caution Approach (Neutrosophic Case)

	Lost opportunities
Science Institute	$[\mathbf{800}, \mathbf{1200}]$
Languages Institute	$[\mathbf{-3000}, \mathbf{1000}]$
Kindergarten	$[\mathbf{-5000}, \mathbf{25000}]$

Consequently, according to this approach, The Languages Institute is the best option because it leads to less lost opportunities.

When working according to this approach in the case of the classic logic, we will come to the same decision that The Languages Institute is the best option, but this does not happen always.

When $i_1 = i_2 = i_3 = i_4 = i_5 = i_6 = 0$ we get the following table:

Table 11.2 The Caution Approach (Classical Case)

	High turnout	Lowturnout
Science Institute	55000	8000
Languages Institute	50000	20000
Kindergarten	40000	18000

We built up the Caution matrix:

Table 12.2 The Caution Approach (Classical Case)

	High turnout	Lowturnout
Science Institute	0	12000
Languages Institute	5000	0
Kindergarten	15000	2000

We take the (Max) and get:

Table 13.2 The Caution Approach (Classical Case)

	Lost opportunities
Science Institute	12000
Languages Institute	5000

Kindergarten	15000

Taking into consideration that this approach has, the less lost opportunities, the most suitable option is The Languages Institute.

We notice that the classical form may agree with the neutrosophic form in the taken decision, but this does not happen always. However, it is better to depend on the method that has accurate data that leads us to choose the best option.

By studying the three approaches according the neutrosophic form we find that in most cases we get different options from the classical logic form.

In addition, we get different options according to the approaches. We look to this positively because it enriches the decision-making process and it reflects the circumstances of the decision maker and the opinions that affects him.

Dealing with the samples of the decision-making process according to the Neutrosophic logic provides us with a comprehensive and complete study for the problem that we are studying. So that, we don't miss any data just because it is clearly indeterminate. This makes us to choose the best option . The existence of indeterminacy in the problem actually affects the process of taking the suitable decision. Therefore, the indeterminate values can't be ignored while studying in order to get more accurate results that leads us to the best options. Nowadays, the classical logic is not sufficient to deal with all the data that we study. Therefore, we had to expand the data of the study and name it accurately to get more real possibilities and, therefore, make decision more accurate. And here appears the role of the Neutrosophic logic that generalizes the classical logic and gives us a wider horizon in interpreting the data in the study and expand it and then make correct decisions with the least possible mistakes.

2.5 The Neutrosophic Decisions Tree in View of the Neutrosophic Probabilities

In the case of decision trees in view of the classical probabilities, the decision maker has the opportunity to evaluate the possibility of each event of the normal cases.

Therefore, the monetary value approach *EMV* is used in order to choose the best options.

However, it is not logical to see that the possibility of the High Turnout of three options the same. For example, it is not possible to see that the probability of the Science Institute, the Languages Institute and the Kindergarten in the High Turnout to be 0.4 because that doesn't agree with the logic that says that each option has conditions and cases that differs from the other options.

We will discuss another method through the Neutrosophic Logic to discuss the decision tree in the review of probabilities depending on the Neutrosophic probabilities, and we will define another form of indeterminate data through this method [51].

We will clarify it as the following:

First, we will define the Neutrosophic expected monetary value and refer to it as (*NEMV*) depending on the expected Neutrosophic value as:

In the natural case (n) and the indeterminate case (m) we write:

$$NEMV\ (d_i) = \sum_{j=1}^{n} p\left(s_j\right) \cdot v\ \left(d_i, s_j\right) + \sum_{I=1}^{m} p\left(s_I\right) \cdot v\ \left(d_i, s_I\right)$$

$P\left(S_j\right)$ Refers to the probability of getting a high or low turnout

(S represents the natural cases)

$P\left(S_I\right)$ Refers to the probability of getting the indeterminate case. (I represent the indeterminacy)

$v\left(d_i, S_j\right)$ Represents the expected monetary value of the option d_i in the S_j case.

$v\left(d_i, S_I\right)$ Represents the expected monetary value of the option d_i in the S_I case.

And in our dealt example, it becomes:

$$NEMV\ (d_i) = p(s_{j=1}) \cdot v\ (d_i, s_{j=1}) + p(s_{j=2}) \cdot v\ (d_i, s_{j=2}) + p(s_{I=1}) \cdot v\ (d_i, s_{I=1})$$

$P(s_{j=1})$ The probability of high turnout

$P(s_{j=2})$ The probability of low turnout

Assuming that the neutrosophic probability in case of the high turnout for the Educational Science Institute is $NP(0.65,\ 0.05,\ 0.30)$ that means that there are three probabilities:

$P(S_{j=1}) = 0.65$ The probability of high turnout for the Science Institute

$P(S_{j=2}) = 0.30$ The probability of low turnout for the Science Institute

$P(S_{I=1}) = 0.05$ The probability of indeterminacy, which means that turnout for the Science Institute not high and not low but between the both. (We get these probabilities from research and expertise centers).

The matrix will be:

Table 14.2 Example (depending on the Neutrosophic probabilities)

	High turnout	Low turnout	Indeterminate turnout
Science Institute	55000	8000	25000
Languages Institute	50000	20000	27000
Kindergarten	40000	18000	22000

The values in the matrix are expected values of options by the experts. In this case, we recognized another form of indeterminacy, which is the turnout, is neither high nor low, but between the two possibilities and we called it indeterminate turnout (and the indeterminate turnout may be gradual).

Now let us calculate the Neutrosophic expected monetary value of the first option d_1 the Science Institute as:

$n = 2 \quad , m = 1$

$$NEMV\ (d_1) = p\left(s_{j=1}\right)\cdot v\ \left(d_1, s_{j=1}\right) + p\left(s_{j=2}\right)\cdot v\ \left(d_1, s_{j=2}\right) + p\left(s_{I=1}\right)\cdot v\ \left(d_1, s_{I=1}\right)$$
$$= (0.65)(55000) + (0.30)(8000) + (0.05)(25000) = 39400$$

Now let us calculate the neutrosophic expected monetary value of the Languages Institute d_2

If we know that the neutrosophic probability of the high turnout of the Languages Institute are: $NP\left(0.46,\ 0.09,\ 0.45\right)$

$P\left(S_{j=1}\right)\ =\ 0.46$ the probability of high turnout for the Languages Institute
$P\left(S_{j=2}\right)\ =\ 0.45$ the probability of low turnout for the Languages Institute
$P\left(S_{I=1}\right)\ =\ 0.09$ the probability of indeterminacy which means that turnout for the Languages Institute not high and not low but between the both.

$$NEMV\ \left(d_2\right) = \left(0.46\right)\left(50000\right) + \left(0.45\right)\left(20000\right) + \left(0.09\right)\left(27000\right) = 34430$$

Now let us calculate the neutrosophic expected monetary value of the Kindergarten d_3

If we know that the neutrosophic probability of the high turnout of the Kindergarten are: $NP\left(0.50,\ 0.08,\ 0.42\right)$

$P\left(S_{j=1}\right)\ =\ 0.50$ The probability of high turnout for the Kindergarten

$P\left(S_{j=2}\right)\ =\ 0.42$ The probability of low turnout for the Kindergarten

$P\left(S_{I=1}\right)\ =\ 0.08$ The probability of indeterminacy, which means that turnout for the Kindergarten not high and not low but between the both.

$$NEMV\ \left(d_3\right) = \left(0.50\right)\left(40000\right) + \left(0.42\right)\left(18000\right) + \left(0.08\right)\left(22000\right) = 29320$$

By calculating the neutrosophic expected monetary value, we see that the first option d_1 (the Science Institute) is the suitable opt. On because it presents. Highest monetary value (39400).

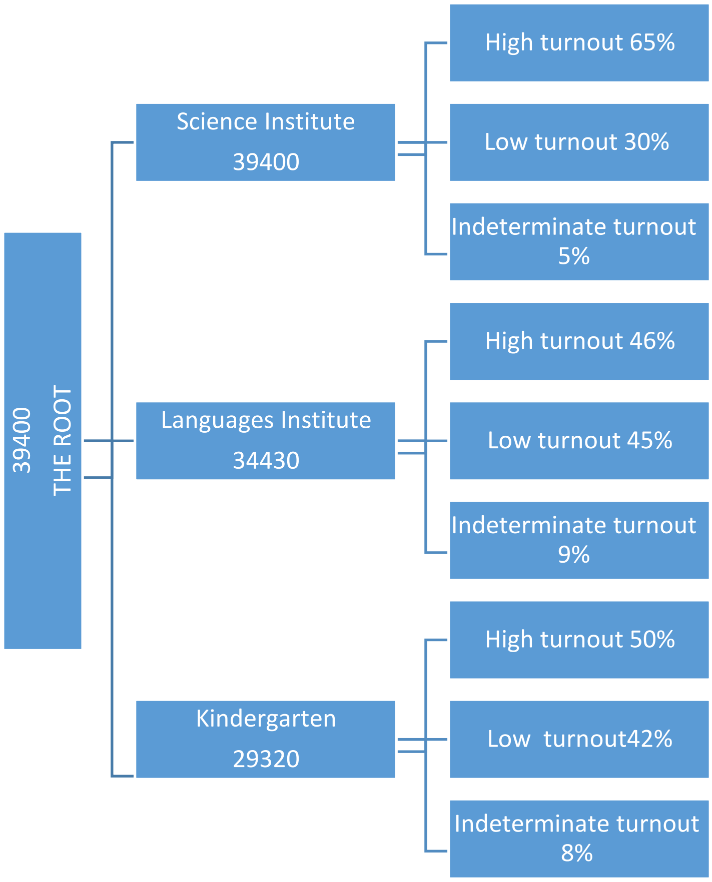

Figure 2.2 Neutrosophic decisions tree

CHAPTER III

PROSPECTOR NEUTRO-FUNCTION

In this section, we define the Prospector Neutro-Function, the method we will use to achieve its applications.

Definition 3.1.1: [52] Prospector Function is defined in the following way;

$$f: [-1, 1] \times [-1, 1] \ \rightarrow \ \ [-1, 1]$$

with formula:

$$f(x, y) \ = \ \frac{x + y}{1 + x \cdot y}$$

This function with $f(-1,1)$ and $f(1, -1)$ are undefined.

Definition 3.1.2: [53] The Extended Prospector Function we can extend $f(x, y)$ to $g(x, y)$ such that:

$$g(x, y) = \begin{cases} f(x, y), & if \ (x, y) \in [-1, 1] \times [-1, 1] \setminus \ \{(-1,1), (1, -1)\} \\ \text{undefined}, & if \ \ (x, y) = (-1,1) \text{ or } (1, -1) \end{cases}$$

$$g(-1,1) \ = \ g(1, -1) \ = undefined,$$

$$g(undefined, undefined) \ = \ undefined.$$

$$g(undefined, x) \ = \ g(x, undefined) \ = \begin{cases} undefined, & if x > \ 0 \\ x, & if \ x \leq 0 \end{cases}$$

Definition 3.1.3: [53] Let A be a finite set defined as $A = \{(x, y): \quad x, y \in \{k, undefined\}$. The Binary Law Θ is defined for every

1- If $g(x, y)$ is not undefined, then $x\Theta y = \frac{round(g(x,y)*10)}{10}$, where round is the function that output the integer nearest to the argument.

2- If $g(x, y)$ is undefined then $x\ \Theta y = undefined$.

Then Θ is a finite Neutro Binary Law. This is because Θ is commutative and associative for the subset of elements of A without any undefined component, but it is not associative otherwise.

$E.g\ if\ a = -0.9, b = 0.8, c = undefined, then\ a\ \Theta\ (b\ \Theta\ c) = a$

$and\ (a\ \Theta\ b)\ \Theta\ c = -0.4 \neq a$ therefore, associativity is a Neutro Binary Law.

The following tables summarize the Cayley table of the Neutro Binary Law Θ which is not associative when we included the undefined value and it generates a Neutro Binary Law Θ. We preferred to maintain the undefinition of the Prospector function because this indicates there is contradiction.

Table 15.3 Example

$x\Theta y$	-1	-0.9	-0.8	-0.7	-0.6	-0.5	-0.4	-0.3	-0.2	-0.1	0
-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1
-0.9	-1	-1	-1	-1	-1	-1	-1	-0.9	-0.9	-0.9	-0.9
-0.8	-1	-1	-1	-1	-0.9	-0.9	-0.9	-0.9	-0.9	-0.8	-0.8
-0.7	-1	-1	-1	-0.9	-0.9	-0.9	-0.9	-0.8	-0.8	-0.7	-0.7
-0.6	-1	-1	-0.9	-0.9	-0.9	-0.8	-0.8	-0.8	-0.7	-0.7	-0.6
-0.5	-1	-1	-0.9	-0.9	-0.8	-0.8	-0.8	-0.7	-0.6	-0.6	-0.5
-0.4	-1	-1	-0.9	-0.9	-0.8	-0.8	-0.7	-0.6	-0.5	-0.5	-0.4

Table 15.3 Continued

-0.3	-1	-0.9	-0.9	-0.8	-0.8	-0.7	-0.6	-0.6	-0.4	-0.4	-0.3
-0.4	-1	-1	-0.9	-0.9	-0.8	-0.8	-0.7	-0.6	-0.5	-0.5	-0.4
-0.3	-1	-0.9	-0.9	-0.8	-0.8	-0.7	-0.6	-0.6	-0.4	-0.4	-0.3
-0.2	-1	-0.9	-0.9	-0.8	-0.7	-0.6	-0.6	-0.5	-0.3	-0.3	-0.2
-0.1	-1	-0.9	-0.8	-0.7	-0.7	-0.6	-0.5	-0.4	-0.2	-0.2	-0.1
undef	-1	-0.9	-0.8	-0.7	-0.6	-0.5	-0.4	-0.3	-0.2	-0.1	0
0	-1	-0.9	-0.8	-0.7	-0.6	-0.5	-0.4	-0.3	-0.1	-0.1	0
0.1	-1	-0.9	-0.8	-0.6	-0.5	-0.4	-0.3	-0.2	0	0	0.1
0.2	-1	-0.9	-0.7	-0.6	-0.5	-0.3	-0.2	-0.1	0.1	0.1	0.2
0.3	-1	-0.8	-0.7	-0.5	-0.4	-0.2	-0.1	0	0.2	0.2	0.3
0.4	-1	-0.8	-0.6	-0.4	-0.3	-0.1	0	0.1	0.3	0.3	0.4
0.5	-1	-0.7	-0.5	-0.3	-0.1	0	0.1	0.2	0.4	0.4	0.5
0.6	-1	-0.7	-0.4	-0.2	0	0.1	0.3	0.4	0.5	0.5	0.6
0.7	-1	-0.5	-0.2	0	0.2	0.3	0.4	0.5	0.6	0.6	0.7
0.8	-1	-0.4	0	0.2	0.4	0.5	0.6	0.7	0.7	0.8	0.8
0.9	-1	0	0.4	0.5	0.7	0.7	0.8	0.8	0.9	0.9	0.9
1	undef	1	1	1	1	1	1	1	1	1	1

Table 16.3 Example

$x\Theta y$	undef	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1
-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	undef
-0.9	-0.9	-0.9	-0.9	-0.8	-0.8	-0.7	-0.7	-0.5	-0.4	0	1
-0.8	-0.8	-0.8	-0.7	-0.7	-0.6	-0.5	-0.4	-0.2	0	0.4	1
-0.7	-0.7	-0.6	-0.6	-0.5	-0.4	-0.3	-0.2	0	0.2	0.5	1
-0.6	-0.6	-0.5	-0.5	-0.4	-0.3	-0.1	0	0.2	0.4	0.7	1
-0.5	-0.5	-0.4	-0.3	-0.2	-0.1	0	0.1	0.3	0.5	0.7	1
-0.4	-0.4	-0.3	-0.2	-0.1	0	0.1	0.3	0.4	0.6	0.8	1
-0.3	-0.3	-0.2	-0.1	0	0.1	0.2	0.4	0.5	0.7	0.8	1
-0.2	-0.2	-0.1	0	0.1	0.2	0.3	0.5	0.6	0.7	0.9	1
-0.1	-0.1	0	0.1	0.2	0.3	0.4	0.5	0.6	0.8	0.9	1
undef	undef	undef	undef	undef	undef	undef	undef	undef	undef	undef	undef
0	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1
0.1	undef	0.2	0.3	0.4	0.5	0.6	0.7	0.7	0.8	0.9	1
0.2	undef	0.3	0.4	0.5	0.6	0.6	0.7	0.8	0.9	0.9	1
0.3	undef	0.4	0.5	0.6	0.6	0.7	0.8	0.8	0.9	0.9	1
0.4	undef	0.5	0.6	0.6	0.7	0.8	0.8	0.9	0.9	1	1
0.5	undef	0.6	0.6	0.7	0.8	0.8	0.8	0.9	0.9	1	1

Table 16.3 Continued

0.6	undef	0.7	0.7	0.8	0.8	0.8	0.9	0.9	0.9	1	1
0.7	undef	0.7	0.8	0.8	0.9	0.9	0.9	0.9	1	1	1
0.8	undef	0.8	0.9	0.9	0.9	0.9	0.9	1	1	1	1
0.9	undef	0.9	0.9	0.9	1	1	1	1	1	1	1
1	undef	1	1	1	1	1	1	1	1	1	1

A group of Syrian migrants was surveyed on an evaluation scale between -5 to 5 in various aspects of access to health [54].

The World Health Organization (WHO) states that a health system brings together all the institutions and organizations whose primary objective is to maintain and improve the health of the population. Most health systems are made up of different sectors, public, private, traditional and informal, and must provide good treatments and services that respond to the needs of the population and are fair from a financial point of view.

Access to health services is the ability to get care when it is needed. This can be determined by various factors and variables such as the location of health centers and the availability of medical or health providers (geographical or physical barriers), up to health insurance and health care costs, also can be influenced by cultural barriers or language.

This research aims to evaluating the access barriers to health that the international migrant population faces in primary health care in Turkey. To achieve this objective, a group of 20 Syrian migrants of different sexes are surveyed. Respondents evaluated different relevant aspects in health care on a numerical scale with a maximum of 5 for approval and a minimum of -5 for disapproval.

Variables that have been used are the following:

1. Location access barriers

2. language access barriers

3. Financial access barriers

4. Legal access barrier.

The assessments provided by the interviewed on the four barriers were as follows:

Table 17.3 Example

Assessments	Location access barriers	Language access barriers	Financial access barriers	Legal access barrier
$v_{ij} = -5, \bar{v}_{ij} = -1$	0	1	0	0
$v_{ij} = -4, \bar{v}_{ij} = -0.8$	0	4	0	0
$v_{ij} = -3, \bar{v}_{ij} = -0.6$	0	2	0	2
$v_{ij} = -2, \bar{v}_{ij} = -0.4$	0	6	0	7
$v_{ij} = -1, \bar{v}_{ij} = -0.2$	0	3	0	5
$v_{ij} = 0, \bar{v}_{ij} = 0$	0	0	1	1
$v_{ij} = 1, \bar{v}_{ij} = 0.2$	0	2	2	2
$v_{ij} = 2, \bar{v}_{ij} = 0.4$	3	2	5	2
$v_{ij} = 3, \bar{v}_{ij} = 0.6$	6	0	8	0
$v_{ij} = 4, \bar{v}_{ij} = 0.8$	10	0	3	0
$v_{ij} = 5, \bar{v}_{ij} = 1$	1	0	1	1

1- The value obtained in the evaluation of each aspect for each migrant is rescaled to the interval $[-1,1]$, dividing by 5. That is, $\bar{v}_{ij} = \frac{v_{ij}}{5}$, we denote by $v_{ij}, (i = 1,2, \ldots , 20; \quad j = 1,2,3,4)$ the evaluation of the ith migrant on the jth aspect.
2- It is decided on two different situations:

If less than 33.333% of the respondents show contradictory results for each fixed *j*, that is, if there are 4 pairs or less of values $(-1,1)$ or $(1, -1)$, these values are eliminated for aggregating.

Otherwise, the *jth* aspect is evaluated as "undefined" and it should be reviewed in more detail because there is such a contradiction.

When we have the case (1) the aggregation of the remaining values is calculated by using Θ.The results obtained from applying this method were as follows:

Aggregating the data of **Table 17.3** using Θ we have the following results based on **Table 15.3** and **Table 16.3**:

$\Theta_{i=1}^{20} \bar{v}_{i1} = \bar{v}_{11} \Theta \bar{v}_{21} \Theta \bar{v}_{31} \Theta \ldots \Theta \bar{v}_{201} = 1$

which means there is sufficient evidence that "Location access barriers" is good.

$\Theta_{i=1}^{20} \bar{v}_{i1} = \bar{v}_{11} \Theta \bar{v}_{21} \Theta \bar{v}_{31} \Theta \ldots \Theta \bar{v}_{201} = -1$

which means there is sufficient evidence that "language access barriers" is bad.

$\Theta_{i=1}^{20} \bar{v}_{i1} = \bar{v}_{11} \Theta \bar{v}_{21} \Theta \bar{v}_{31} \Theta \ldots \Theta \bar{v}_{201} = 1$

which means there is sufficient evidence that "Financial access barriers" is good.

$\Theta_{i=1}^{20} \bar{v}_{i1} = \bar{v}_{11} \Theta \bar{v}_{21} \Theta \bar{v}_{31} \Theta \ldots \Theta \bar{v}_{201} = \text{undefined}$

which means there is no sufficient evidence that means "Legal access barrier" it should be reviewed in more detail why there is such a contradiction.

CHAPTER IV

NEUTRO ORDERED R-Module

In this section, we use the defined notion of NeutroOrdered Algebra and apply it to NeutroOrdered R-Module. As a result, we define NeutroOrdered R-Module and other related concepts. Moreover, we study some properties of NeutroOrdered R-Module and, NeutroOrdered R-Module homomorphism.

Definition 4.1.1: [55] Let R be a Neutro-Ring and let $({}_RM,+)$ be a Neutro abelian group and $"\cdot"$ be a binary operation such that $\cdot : R \times M \to M$. Then $({}_RM,+,\cdot)$ is called a Neutro left R-Module on Neutro-Ring $(R,+,.)$ if the following conditions are satisfied:

$"+"$ is left Neutro-Distributive over $"\cdot"$ that is there exists at least some $r \in R$ and $m,n \in {}_RM$ such that $r \cdot (m + n) = r \cdot m + r \cdot n$,there exists at least $q \in R$ and $t,v \in {}_RM$ such that $q \cdot (t+v)$ or $q \cdot t + q \cdot v$ are indeterminate and there exists at least $s \in R$, $x,y \in {}_RM$ such that $s \cdot (x+y) \neq s \cdot x + s \cdot y$.

$"+"$ is right Neutro-Distributive over $"\cdot"$ that is there exists at least some $r,s \in R$ and $m \in {}_RM$ such that $(r+s) \cdot m = r \cdot m + s \cdot m$,there exists at least $x,y \in R$ and $z \in {}_RM$ such that $(x+y) \cdot z$ or $x \cdot z + y \cdot z$ are indeterminate and there exists at least some $t,q \in R$, $n \in {}_RM$ such that

$(t+q) \cdot n \neq t \cdot n + q \cdot n.$

$"\cdot"$ is Neutro-Associative that is there exists at least some $r,s \in R$ and $m \in M$ such that $(rs) \cdot m = r \cdot (s \cdot m)$, there exists at least some $x,y \in R$, $z \in {}_RM$ such that $(x \cdot y) \cdot z$ or $x \cdot (y \cdot z)$ are indeterminate and there exists at least some $t,q \in R$, $n \in {}_RM$ such that $(tq) \cdot n \neq t \cdot (q \cdot n)$.

There is an element e (Neutro-Neutral element in R) that is there exists at least some $m \epsilon M$ such that

$e \cdot m = m$ there exists at least some $x \in {}_{R}M$ such that $e \cdot x$ is indeterminate and there exists at least some $n \in {}_{R}M$ such that $e \cdot n \neq n$.

Similarly, the form $(M_R, +, \cdot)$ is known as Neutro right $R - Module$ over a Neutro-Ring.

Notes 4.1.2:

If we have R as a commutative Neutro-Ring, then every Neutro left R-Module is a Neutro right $R - Module$.

M is called a finite Neutro $R - Module$ of order n if the number of elements in M is n that is $o(M) = n$. If no such n exists, then M is called an infinite Neutro $R - Module$ and we write $o(M) = \infty$.

An element $x \in M$ is called a NeutroIdempotent element if $x^2 = x$.

An element $x \in M$ is called a NeutroINilpotent element if for the least positive integer n, we have

$x^n = e$ where e is Neutro-Neutral element in M.

Example 4.1.3: Let R be a commutative Neutro-Ring. A very important example of an Neutro R-Module is R Neutro-Ring itself:

Example 4.1.4: Let $X = \{m, n, p, q, t\}$ be a universe of discourse and let $M = \{m, n, p\}$ be a subset of X. let ■ and $*$ be binary operation defined on M as shown in the Cayely tables below:

Table 18.4 Example

■	m	n	p
m	m	n	n or p
n	p or n	m or n	p
p	n	p	n

Table 19.4 Example

$*$	m	n	p
m	m	m	m
n	m or n	p	m
p	m	p	n

It is clear from the table that it $(R, \blacksquare, *)$ is a Neutro-Commutative Ring with Neutro-Unity and:

$m * (n \blacksquare p) = m * p = m$

$(m * n) \blacksquare (m * p) = m \blacksquare m = m$ [degree of truth (T)],

$p * (n \blacksquare m) = p \text{ or } n$

$(p * n) \blacksquare (p * m) = n \text{ or } m$ are indeterminacy [degree of indeterminacy (I)]

and $n * (p \blacksquare m) = n * n = p$

$(n * p) \blacksquare (n * m) = m \blacksquare m = m$ [degree of falsehood (F)].

This shows that "$\blacksquare$"is both left Neutro-Distributive over "$*$".

$(m \blacksquare n) * p = n * p = m$

$(m * p) \blacksquare (n * p) = m \blacksquare m = m$ [degree of truth (T)],

$(n \blacksquare m) * p = n \text{ or } m$

$(n * p) \blacksquare (m * p) = m$ [degree of indeterminacy (I)]

and $(p \blacksquare m) * n = n * n = p$

$(p * n) \blacksquare (m * n) = p \blacksquare m = n$ [degree of falsehood (F)].

This shows that "$\blacksquare$"is both right Neutro-Distributive over "$*$".

$m * (n * p) = m * m = m$

$(m * n) * p = m * p = m$

$(n * m) * p = m$[degree of truth (T)],

$n * (m * p) = n \text{ or } m$[degree of indeterminacy (I)]

and $p * (n * n) = p * p = n$

$(p * n) * n = p * n = p$ [degree of falsehood (F)].

This shows that " $*$ " is a Neutro-Associative.

$p * n = p$, $m * n = m$ [degree of truth (T)],

$m * n = m$, $n * m = m \text{ or } n$ [degree of indeterminacy (I)]

$n * n = p \neq n$ [degree of falsehood (F)].

It follows that $(M, \blacksquare, *)$ Neutro R-Module over Neutro-Ring $(R, \blacksquare, *)$.

4.2 Neutro-Sub R-Module

Definition 4.2.1: [55] Let M be a Neutro R-Module. A nonempty subset N of M is called a Neutro-Sub R-Module of M if N is also a Neutro R-Module.

Example 4.2.2: Let M be a Neutro R-Module. M is a Neutro-Sub R-Module called a trivial Neutro-Sub R-Module.

Theorem 4.2.3: [55] Let M be a Neutro R-Module over a Neutro-Ring R and let N be a nonempty subset of M.

N is a Neutro-Sub R-Module of M if the following conditions hold:

(1) That is there exists at least some $m, n \in N$ such that $m + n \in N$.

(2) That is there exists at least some $m \in N$, $r \in R$ such that $rm \in N$.

Corollary 4.2.4: Let M be a Neutro R-Module over a Neutro-Ring R and let N be a nonempty subset of M.

N is a Neutro-Sub R-Module of M if the following conditions hold:

That is there exists at least some $m, n \in N$, $r, s \in R$ such that $rm + sn \in N$.

Example 4.2.5: Let $(M, ■,*)$ be a the Neutro R-Module of **Example 4.1.4** and let $N = \{p, n\}$:

$p, n \in N$ $, p■n = p \in N$ but $n■n = m$

$p, n \in N$ $, p \in R, p * n = p \in N$ but $n * p = m$

It follows that N is Neutro-Sub R-Module of M.

Theorem 4.2.6: Let M be a Neutro R-Module over a Neutro-Ring R and let $\{N_n\}_{n\in\lambda}$ be a family of Neutro-Sub $R - Module$ of M .Then $\cap N_n$ is a Neutro-Sub R-Module.

4.3 Neutro $R - Module$ Homomorphism

Definition 4.3.1: [55] Let $(M, +,\cdot)$ and $(N, ■,*)$ be any two Neutro R-Modules. The mapping $\varphi : M \to N$ is called a Neutro R-Module Homomorphism if the following conditions hold:

for at least a pair $(x, y) \in M$, we have:

$$\varphi(x + y) = \varphi(x)■\varphi(y)$$

$$\varphi(x \cdot y) = \varphi(x) * \varphi(y)$$

If in addition φ is a Neutro-Bijection, then φ is called a Neutro R-Module Isomorphism and we write $M \cong N$. Neutro $R - Module$ Epimorphism, Neutro R-Module Monomorphism, Neutro $R - Module$ Endomorphism and Neutro R-Module Automorphism are defined similarly.

Definition 4.3.2:[37] The kernel of φ denoted by $Ker\varphi$ is defined as:

$Ker\varphi = \{x : \varphi(x) = e_N\}$ where $e_N \in N$ is Neutro-Neutral element in N.

The image of φ denoted by $Im\varphi$ is defined as:

$Im\varphi = \{y \in N : y = \varphi(x)\, for\ at\ least\ one\ y \in N\}$.

Example 4.3.3: Let $(M, ■,*)$ be a the Neutro $R - Module$ of **Example 4.1.4** and let $\varphi: (M, ■,*) \to (M, ■,*)$ be a mapping defined by:

$$\varphi(m) = m * m$$

It can be shown that φ is a Neutro R-Module Homomorphism such that

for $m, n, p \in M$, we have:

$$\varphi(m \blacksquare m) = \varphi(m) = m * m = m$$

$\varphi(m) \blacksquare \varphi(m) = (m * m) \blacksquare (m * m) = m \blacksquare m = m$ but

$$\varphi(m \blacksquare n) = \varphi(n) = n * n = p$$

$\varphi(m) \blacksquare \varphi(n) = (m * m) \blacksquare (n * n) = m \blacksquare p = n$

$$\varphi(m * n) = \varphi(m) = m * m = m$$

$\varphi(m) * \varphi(n) = m * p = m$ but

$$\varphi(p * n) = \varphi(p) = p * p = n$$

$\varphi(p) * \varphi(n) = n * p = m$

The kernel of φ is $Ker\varphi = \{x : \varphi(x) = e_M\} = \{m, p\}$ where $e_M \in M$ is Neutro-Neutral element in M.

The image of φ is $Im\varphi = \{y \in N : y = \varphi(x) for\ at\ least\ one\ y \in N\} = \{m, n, p\}$

Theorem 4.3.4: Let $(M, \cdot, +)$ and $(N, \blacksquare, *)$ be any two Neutro R-Modules. Suppose that $\varphi : M \to N$ is a Neutro R-Module Homomorphism. Then:

$\varphi(e_M)$ is not necessarily equals e_N.

$Ker\varphi$ is a Neutro-Sub R-Module of M.

$Im\varphi$ is not necessarily a Neutro-Sub R-Module of N.

φ is NeutroInjective if and only if $Ker\varphi = \{e_M\}$ for at least one $e_M \in M$.

Definition 4.3.5: [55] Let K, M and N be Neutro R-Module over a Neutro-Ring R and let

$$\phi : K \to M, \psi : M \to N$$

be Neutro $R-Module$ homomorphisms. The composition $\psi\,\phi : K \to N$ is defined by

$$\psi\phi\,(k) \;=\; \psi(\phi(k))\; for\ all\ k\in K.$$

Theorem 4.3.6: Let K, M and N be Neutro R-Module over a Neutro-Ring R and let

$$\phi : K \to M, \psi : M \to N$$

be Neutro R-Module homomorphisms. Then the composition $\psi\phi : K \to N$ is a Neutro R-Module homomorphisms.

Theorem 4.3.7: Let K, M and N be Neutro R-Modules over a Neutro-Ring R and let

$$\phi : K \to M, \psi : M \to N$$

be Neutro R-Module homomorphisms. Then

If $\psi\phi$ is Monomorphism Neutro R-Module, then ϕ Monomorphism Neutro R-Module.

If $\psi\phi$ is Neutro R-Module Epimorphism, then ψ is Neutro R-Module Epimorphism.

If ψ and ϕ are Monomorphism Neutro R-Module, then $\psi\phi$ is Monomorphism Neutro R-Module.

4.4 Neutro Ordered $R-Module$

Definition 4.4.1: [56] Let M be a Neutro R-Module with n (Neutro) operations "i" and "$\leq$" be a partial order (reflexive, anti-symmetric, and transitive) on M. Then $(M, *_1, *_2, \leq)$ is a NeutroOrdered R-Module if the following conditions hold.

(1) There exist $x \leq y \in M$ with $x \neq y$ such that $z *_i x \leq z *_i y\ and\ x *_i z \leq y *_i z$ for some $i = 1, 2$ and $z \in M$ (This condition is called degree of truth, "T".)

(2) There exist $x \leq y \in M$ and $z \in A$ such that $z *_i x \not\leq z *_i y\ and\ x *_i z \not\leq y *_i z$ for some $i = 1,2$. (This condition is called degree of falsity, "F".)

(3) There exist $x \leq y \in M$ and $z \in A$ such that $z *_i x \ or \ z *_i y \ or \ x *_i z \ or \ y *_i z$ are indeterminate, or the relation between that $z \ *_i x$ and $z *_i y$, or the relation between $x *_i z$ and $y *_i z$ are indeterminate for some $i = 1, 2$. (This condition is called degree of indeterminacy, "I".)

Where (T, I, F) is different from $(1, 0, 0)$ that represents the classical Ordered R-Module as well from $(0, 0, 1)$ that represents the AntiOrdered R-Module.

Definition 4.4.2: [56] Let $(M, *_1, *_2, \leq)$ be a NeutroOrdered R-Module. If " $\leq$ " is a total order on A then M is called NeutroTotal Ordered R-Module.

Definition 4.4.3: [56] Let $(M, *_1, *_2, \leq)$ be a Neutro Ordered R-Module and $\emptyset \neq S \subseteq M$. Then S is a Neutro Ordered Sub R-Module of S if $(S, *_1, *_2, \leq)$ is a Neutro Ordered R-Module and there exist.

Example 4.4.4: Let $M = \{m, n, p\}$ and $(M, ■, *,)$ be defined by the following table.

Table 20.4 Example

■	m	n	p
m	m	n	n
n	$p \ or \ n$	$m \ or \ n$	p
p	n	p	n

Table 21.4 Example

$*$	m	n	p
m	m	m	m
n	$m \ or \ n$	p	m
p	m	p	n

As showed $(M, ■, *,)$ in **Example 4.1.3** is a Neutro R-Module.

By defining the total order

$$\leq = \{(m, m), (n, n), (p, p), (m, n), (m, p), (n, p)\}$$

on M, we get that $(M, ■,*, \leq)$ is a NeutroTotalOrdered R-Module. This is easily seen as:

$m \leq p$ implies that $m * x \leq p * x$ and $x * m \leq x * p$ for all $x \in M$.

And having $n \leq p$ but $■n.p■p \nleq$

$m \leq n$ implies that $m * x \leq n * x$ and $x * m \leq x * n$ for all $x \in M$.

And having $n \leq p$ but $* n \nleq p * p$.

Example 4.4.5: Let $(M, ■,*, \leq)$ be a the Neutro $R - Module$ of **Example 4.1.3** and let $N = \{p, n\}$:

$p, n \in N$, $p■n = p \in N$ but $n■n = m$

$p, n \in N$, $p \in R$, $p * n = p \in N$ but $n * p = m$

By defining the total order

$$\leq = \{(m, m), (n, n), (p, p), (m, n), (m, p), (n, p)\}$$

It follows that $(N, ■,*, \leq)$ is Neutro-Sub $R - Module$ of M.

Definition 4.4.6: [56] Let $(M, *_1, *_2, \leq_1)$ and $(N, ■_1, ■_2, \leq_2)$ be any two Neutro Ordered $R - Modules$. The mapping $\varphi : M \to N$ is called a Neutro Ordered R-Module Homomorphism if the following conditions hold:

for some $(x, y) \in M$, we have:

$$\varphi(x *_1 y) = \varphi(x)■_1 \varphi(y)$$

$$\varphi(x *_2 y) = \varphi(x)■_2 \varphi(y)$$

and there exist $a \leq_1 b$, $a \neq b$, $\varphi(a) \leq_2 \varphi(b)$

φ is called Neutro Ordered R-Module Isomorphism if φ is a bijective NeutroOrdered R-Module Homomorphism.

i)There exists a double $(p, q) \in M$ such that $\varphi(p *_1 q) = \varphi(p)■_1\varphi(q)$ (degree of truth T) and there exist two doubles $(s, t), (k, m)$ (F, V) such that $[\varphi(s *_1 t) \neq$

$\varphi(s) \blacksquare_1 \varphi(t)$ (degree of falsehood F) or $\varphi(k \ *_1 \ m) =_{\text{indeterminacy}} \ \varphi(k) \blacksquare_1 \varphi(m)$ (degree of indeterminacy I)]; where (T, I, F) is different from $(1, 0, 0)$ and $(0, 0, 1)$.

ii)There exists a double $(p, q) \in M$ such that $\varphi(p *_2 \ q) \ = \ \varphi(p) \blacksquare_2 \varphi(q)$ (degree of truth T) and there exist two doubles $(s, t), (k, m)$ (F, V) such that $[\varphi(s \ *_2 \ t) \neq \varphi(s) \blacksquare_2 \varphi(t)$ (degree of falsehood F) or $\varphi(k \ *_2 \ m) =_{\text{indeterminacy}} \ \varphi(k) \blacksquare_2 \varphi(m)$ (degree of indeterminacy I)]; where (T, I, F) is different from $(1, 0, 0)$ and $(0, 0, 1)$.

Example 4.4.7: Let $\varphi: (M, *_1, *_2, \leq) \rightarrow (M, *_1, *_2, \leq)$ be a mapping defined by:

$$\varphi(m) = m *_2 m$$

It can be shown that φ is a Neutro Ordered $R - Module$ Homomorphism such that

for $m, n, p \in \ M$, we have:

1- and 2- it proved in **Example 4.3.2**

There exist $m \leq n$ such that $\varphi(m) \leq \varphi(n)$.

CHAPTER V

NEUTROSOPHIC EUCLIDEAN GEOMETRY

In the beginning, we will define the basic concepts in neutrosophic Euclidean geometry and then we will study their relations with classical geometry.

Definition 5.1.1: [57] Let $R(I) = \{a + bI;\ a, b \in R\}$ be the real neutrosophic field, we say that $a + bI \leq c + dI$ if and only if $a \leq c\ and\ a + b \leq c + d$.

Theorem 5.1.2: [57] The relation defined in **Definition 5.1.1** is a partial order relation (reflexive, anti-symmetric, and transitive).

Proof:

Let $x = a + bI, y = c + dI, z = m + nI \in R(I)$, we have

$x \leq x$ that is because $a \leq a\ and\ a + b \leq a + b$.

Now, suppose that $x \leq y\ and\ y \leq x$, then $a \leq c, a + b \leq c + d, c \leq a,$

$c + d \leq a + b$, hence

$a = c, a + b = c + d$, which means that $d = b$ and $x = y$.

Assume that $x \leq y\ and\ y \leq z$, hence $a \leq c, a + b \leq c + d, c \leq m, c + d \leq m + n$, this implies that

$a \leq m, a + b \leq m + n$, so $x \leq z$. Thus $\leq$ is a partial order relation on $R(I)$.

Remark 5.1.3:

According to **Theorem 5.1.2**, we can define positive neutrosophic real numbers as follows:

$a + bI \geq 0 = 0 + 0.I$ implies that $a \geq 0, a + b \geq 0$.

Absolute value on $R(I)$ can be defined as follows:

$|a + bI| = |a| + I[|a + b| - |a|]$, we can see that $|a + bI| \geq 0$.

We can compute the square root of a neutrosophic positive real number as follows:

$$\sqrt{a + bI} = \sqrt{a} + I\left[\sqrt{a + b} - \sqrt{a}\right] \quad , it\ is\ clear\ that$$

$(\sqrt{a} + I\left[\sqrt{a + b} - \sqrt{a}\right])^2 = a + bI\ and\ \sqrt{a + bI} \geq 0$.

Examples:

- $x = 2 - I$ is a neutrosophic positive real number, since $2 \geq 0$

$and\ (2-1)=1\geq 0.$

- $2+I\geq 2$, that is because $2\geq 2\ and\ (2+1)=3\geq (2+0)=2.$
- $|1+3I|=|1|+I[|1+3|-|1|]=1+3I.$
- $|-3+2I|=|-3|+I[|-3+2|-|-3|]=3-2I$ notice $0+0I\geq -3+2I$
- $\sqrt{9+4I}=\sqrt{9}+I\left[\sqrt{13}-\sqrt{9}\right]=3+\left(\sqrt{13}-3\right)I.$

Definition 5.1.4: [57] We define the neutrosophic plane with n-dimensional neutrosophic as follows:

$$R(I)\times R(I)\times R(I)\times \underbrace{\ldots\ldots}_{n-times}\times R(I)$$

Example 5.1.5: $R(I)=\{a+bI;a,b\in R\}$ is a neutrosophic plane with one n-dimension.

$R(I)^2=\{(a+bI,c+dI);a,b,c,d\in R\}$ is a neutrosophic plane with two n-dimensions.

In the following, we will focus on the two n-dimensional neutrosophic planes.

Definition 5.1.6: [57] Let $A(a+bI,c+dI),B(x+yI,z+tI)$ be two neutrosophic points from $R(I)^2$, we define:

$\overrightarrow{AB}=([x+yI]-[a+bI],[z+tI]-[c+dI])$, is called a neutrosophic vector with 2-dimensions neutrosophic.

Definition 5.1.7: [57] Let $\vec{u}=(a+bI,c+dI)$ be a neutrosophic vector, we define its norm as follows:

$\|\vec{u}\|=\sqrt{(a+bI)^2+(c+dI)^2}=\sqrt{a^2+c^2+I[(a+b)^2+(c+d)^2-a^2-c^2]}.$

It is easy to see that $\|\vec{u}\|\geq 0, according\ to$ **Remark 5.1.3**

Definition 5.1.8: [57] Let $A(a+bI,c+dI),B(x+yI,z+tI)$ be two neutrosophic points from $R(I)^2$, we define:

(a) The midpoint of $[AB]$ is $C\left(\frac{a+bI+x+yI}{2},\frac{c+dI+z+tI}{2}\right)$.

(b) The neutrosophic distance between A and B is equal to $\left\|\overrightarrow{AB}\right\|$.

Example 5.1.9: Consider the following neutrosophic points $A(1+I,2-3I),B(-I,-1+2I)$, The neutrosophic vector $\overrightarrow{AB} = (-1 - 2I, -3 + 5I)$, the square of the neutrosophic distance between A and B is $\|\overrightarrow{AB}\|^2 = 1 + 9 + I[9 + 4 - 1 - 9] = 10 + 3I$.

Hence the neutrosophic distance is equal to $\sqrt{10 + 3I} = \sqrt{10} + I[\sqrt{13} - \sqrt{10}]$. We can find easily that $(\sqrt{10 + 3I}\ )^2 = (\sqrt{10} + I[\sqrt{13} - \sqrt{10}])^2 = 10 + 3I$.

Let C be the neutrosophic midpoint of $[AB]$, then $C\left(\frac{1}{2}, \frac{1}{2} - \frac{1}{2}I\right)$.

Now, we list some geometrical and algebraic properties of the classical space $R^2 \times R^2$. We will need them in the forthcoming sections.

Remark 5.1.10: Let $V = R^2 \times R^2$ be the Cartesian product of the classical Euclidean plane with itself, we have

(a) V has a module structure over the ring $R \times R$, with respect to the following operations:

Addition: $\big((a, b), (c, d)\big) + \big((x, y), (z, t)\big) = \big((a + x, b + y), (c + z, d + t)\big)$,

Multiplication by a duplet scalar from $\times R$: $(m, n).\big((a, b), (c, d)\big) = \big((m.a, n.b), (m.c, n.d)\big)$.

(b) The norm of any vector in V can be defined as a duplet number from $\times R$, as follows:

$$\|((a, b), (c, d))\| = \left(\sqrt{a^2 + c^2}, \sqrt{b^2 + d^2}\right).$$

Example 5.1.11: Consider the following two points from the space V, $A((1,2), (2,5)), B((-1,4), (3, -2))$, we have:

(a) $\overrightarrow{AB} = \big((-2,2), (1, -7)\big)$.

(b) $\|\overrightarrow{AB}\| = \left(\sqrt{(-2)^2 + (1)^2}, \sqrt{(2)^2 + (-7)^2}\right) = (\sqrt{5}, \sqrt{53})$.

(c) Let $r = (5,8) \in R \times R$ be a duplet scalar, we have:

$r.\overrightarrow{AB} = \big((-10,16), (5, -56)\big)$, it is clear that $\|r.\overrightarrow{AB}\| = r.\|\overrightarrow{AB}\|$.

5.2 The Connection Between Neutrosophic and Classical Geometry

This section is devoted to clarifying the relationships between neutrosophic coordinates defined above, and between classical geometrical coordinates.

Many important questions arise according to section 5.1 The first one is about famous relations in classical geometry: Does the midpoint of $[AB]$ have the same neutrosophic distance from A and B? If the answer is no, then our geometrical system is weak and has no importance because it contradicts logical statements.

The second, do the neutrosophic points have relationships with classical points. This question is the most important one, that is because if it has a positive answer, then we can study geometrical shapes in the neutrosophic plane.

The third is about how can we define neutrosophic lines, circles, elliptic curves… etc.

We try to answer these important questions by using algebra since the neutrosophic plane with two N-dimensions is a module over the ring $R(I)$.

Definition 5.2.1: [57] Let $M = R(I)^2 = R(I) \times R(I), V = R^2 \times R^2$be the neutrosophic plane with two N-dimensions and the Cartesian product of the classical Euclidean space R^2 with itself, we define the AH-isometry map as follows:

$f: M \to V; f(a + bI, c + dI) = ((a, a + b), (c, c + d))$.

We can define the one-dimensional isometry between $R(I)$ and the space $R \times R$ as follows:

$g: R(I) \to R \times R; g(a + bI) = (a, a + b)$.

Remark 5.2.2: The one-dimensional isometry is an algebraic isomorphism between $R(I)$ and $R \times R$.

Proof: Let $a + bI, c + dI$ be two neutrosophic real numbers, then

$$f(a + bI + c + dI) = f([a + c] + [b + d]I) = (a + c, a + c + b + d)$$
$$= (a, a + b) + (c, c + d)$$
$$= f(a + bI) + f(c + dI).$$

$$f([a+bI].[c+dI]) = f(ac + I[ad + bc + bd]) = (ac, ac + ad + bc + bd)$$
$$= (a, a + b).(c, c + d)$$
$$= f(a + bI).f(c + dI).$$

f is a correspondence one-to-one, that is because $Ker(f) = \{0\}$, and for every pair $(a, b) \in R \times R$, there exists $a + (b - a)I \in R(I)$ $such\ that$ $f(a + [b - a]I) = (a, b)$. Thus, f is an isomorphism.

Example 5.2.3: Consider the following neutrosophic point $A(1 + I, 3 - 6I)$, its isometric image is $\big((1,2), (3, -3)\big)$.

Consider the following neutrosophic vector $\vec{u} = (2 - I, 4 + I)$, its isometric vector is

$$\vec{v} = \big((2,1), (4,5)\big).$$

The idea behind the AH-isometry is to deal with neutrosophic points as classical points and to explore their properties using classical Euclidean geometry.

The following theorem is considered as the fundamental theorem in neutrosophic Euclidean geometry since it describes the relation between neutrosophic space with two N-dimensions and the classical module generated by the Cartesian product of the classical Euclidean space by itself.

Theorem 5.2.4: [57] (Fundamental Theorem in neutrosophic Euclidean Geometry)

Let $f: M \to V; f(a + bI, c + dI) = ((a, a + b), (c, c + d))$ be the AH-isometry defined above, we have:

(a) f preserves addition operation between vectors.

(b) f preserves distances between points.

(c) f is a bijection one-to-one between M and V.

(d) Multiplying a neutrosophic vector by a neutrosophic real number is preserved up to isometry, i.e. The direct image of a neutrosophic vector multiplied by a neutrosophic real number is exactly equal to its AH-isometric image multiplied by the one-dimensional isometric image of the corresponding neutrosophic real number.

Proof:

(a) Let $\vec{u} = (a + bI, c + dI), \vec{v} = (x + yI, z + tI)$ be two neutrosophic vectors, we have

$$f(\vec{u} + \vec{v}) = f(a + x + I[b + y], c + z + I[d + t])$$
$$= ((a + x, a + x + b + y), (c + z, c + z + d + t))$$

$$= \big((a, a + b), (c, c + d)\big) + ((x, x + y), (z, z + t)) = f(\vec{u}) + f(\vec{v})$$

(b) We must prove that the norm of the classical vector $\overrightarrow{f(u)}$,

is exactly equal to the one-dimensional isometric image of the norm of the neutrosophic vector $\vec{u}$.

$\|f(\vec{u})\|^2 = (a^2 + c^2, (a + b)^2 + (c + d)^2)$, on the other hand, we have

$$g(\|\vec{u}\|^2) = g(a^2 + c^2 + I[(a + b)^2 + (c + d)^2 - a^2 - c^2]$$
$$= (a^2 + c^2, (a + b)^2 + (c + d)^2) = \|f(\vec{u})\|^2.$$

(c) Suppose that $f(a + bI, c + dI) = f(x + yI, z + tI)$, hence $\big((a, a + b), (c, c + d)\big)$

$= \big((x, x + y), (z, z + t)\big)$, thus $x = a, b = y, z = c, d = t$, so that f is injective.

It is clear that f is surjective, thus it is a bijection.

(d) Consider the following neutrosophic vector $\vec{u} = (a + bI, c + dI)$

With the following neutrosophic real number $m + nI, we\ have$

$(m + nI).\vec{u} = ((m + nI)(a + bI), (m + nI)(c + dI) = ((ma + I[mb + na + nb]), (mc + I[md + nc + nd]),$ on the other hand, we have

$$f\big((m + nI).\vec{u}\big) = \big((ma, (ma + mb + na + nb)), (mc, mc + md + nc + nd)\big)$$
$$= (m, m + n).\big((a, a + b), (c, c + d)\big)$$
$$= g(m + nI).f(a + bI, c + dI).$$

Example 5.2.5: Consider the following two neutrosophic points $A(1 + 2I, I), B(3I, -2 + I)$, we have:

(a) The isometric points of A, B are $A' = \big((1,3),(0,1)\big), B' = \big((0,3),(-2,-1)\big)$.

(b) $\overrightarrow{AB} = (-1+I,-2), the\ corresponding\ isometric\ vector\ is\ \overrightarrow{A'B'} = \big((-1,0),(-2,-2)\big) = f(\overrightarrow{AB})$.

(c) The neutrosophic distance $[AB] = \sqrt{1+4+I[0+4-1-4]}$

$= \sqrt{5-I} = \sqrt{5} + I[4-\sqrt{5}]$. The classical distance between isometric images is

$$[A'B'] = \left(\sqrt{(-1)^2+(-2)^2}, \sqrt{(0)^2+(-2)^2}\right) = \left(\sqrt{5},4\right) = g([AB]).$$

Theorem 5.2.6: [57] introduces an algorithm to transform any neutrosophic point to a classical Cartesian product of two classical points. The following theorem describes the inverse relation between classical coordinates and neutrosophic coordinates, i.e. It clarifies how to go back from classical coordinates to neutrosophic coordinates.

Theorem 5.2.7: [57] Let $A((a,b),(c,d))$ be a Cartesian product of two classical points, then the inverse isometric image (the corresponding neutrosophic point) is

$B(a+(b-a)I, c+(d-c)I)$.

Proof:

It holds directly by taking the image of B with respect to AH-isometry, the point A is obtained.

5.2.8 Example: Consider the following classical point $A((1,2),(-1,4))$, its corresponding neutrosophic point is

$B(1+I,-1+5I)$.

As a result of Section 4, we can find that all geometrical famous properties are still true in neutrosophic Euclidean geometry, that is because we can transform any neutrosophic point to a corresponding classical point by preserving addition, distances, and multiplication by scalars.

5.3 Some Neutrosophic Geometrical Shapes With 2-dimensions

Definition 5.3.1: [57] (Neutrosophic circle)

Let $M(a+bI, c+dI)$ be a fixed neutrosophic point, we define the neutrosophic circle with center M and radius $R = r_1 + r_2 I \geq 0$ to be the set of all two n-dimensional points $N(X, Y) = N(x_0 + x_1 I, y_0 + y_1 I); dist(M, N) = R = const.$

Theorem 5.3.2: [57] Let $M(a+bI, c+dI)$ be a fixed neutrosophic point, $R = r_1 + r_2 I$ be a neutrosophic real positive number, we have:

(a) The equation of the circle with center M and radius R is

$$([(x_0 + x_1 I] - [a+bI])^2 + ([y_0 + Iy_1] - [c+dI])^2 = R^2 .$$

(b) The previous neutrosophic circle is equivalent to the following direct product of two classical circles

$C_1: (x_0 - a)^2 + (y_o - c)^2 = {r_1}^2, C_2: ([x_0 + x_1] - [a+b])^2 + ([y_0 + y_1] - [c + d])^2 = (r_1 + r_2)^2.$

Proof:

(a) By using the neutrosophic distance form defined in **Definition 5.1.7** and **Definition 5.1.8** we get

is $([(x_0 + x_1 I] - [a+bI])^2 + ([y_0 + Iy_1] - [c+dI])^2 = R^2$.

(b) To obtain the classical equivalent geometrical system of the neutrosophic circle, it is sufficient to take its isometric image as follows:

$f(([(x_0 + x_1 I] - [a+bI])^2) + f(([y_0 + Iy_1] - [c+dI])^2) = f(R^2)$, hence

$((x_0 - a)^2, (x_0 + x_1 - [a+b])^2) + ((y_0 - c)^2, (y_0 + y_1 - [c+d])^2) = ({r_1}^2, (r_1 + r_2)^2)$, thus

$$\big(((x_0 - a)^2 + (y_0 - c)^2), ((x_0 + x_1 - [a+b])^2 + (y_0 + y_1 - [c+d])^2)\big) = ({r_1}^2, (r_1 + r_2)^2),$$

Thus, we get $(x_0 - a)^2 + (y_o - c)^2 = {r_1}^2$ and

$$([x_0 + x_1] - [a+b])^2 + ([y_0 + y_1] - [c+d])^2 = (r_1 + r_2)^2.$$

Example 5.3.3: Consider the following neutrosophic circle: $C: (X - I)^2 + (Y - (2 - 3I))^2 = (2 + I)^2$

It is equivalent to the direct product of the following two classical circles:

$C_1: (x_0 - 0)^2 + (y_0 - 2)^2 = 2^2, C_2: ([x_0 + x_1] - [1])^2 + ([y_0 + y_1] - [-1])^2 = (2+1)^2.$

Definition 5.3.4: [57] (Neutrosophic line)

We define the neutrosophic line by the set of all two n-dimensional points (X, Y) with the property

$$AX + BY + C = 0; X = x_0 + x_1 I, Y = y_0 + y_1 I, A = a_0 + a_1 I, B = b_0 + b_1 I, C = c_0 + c_1 I.$$

Theorem 5.3.5: [57] Let $AX + BY + C = 0$ be an equation of a neutrosophic line d, this line is equivalent to the direct product of the following two classical lines:

$$d_1: a_0 x_0 + b_0 y_0 + c_0 = 0, d_2: (a_0 + a_1)(x_0 + x_1) + (b_0 + b_1)(y_0 + y_1) + c_0 + c_1 = 0.$$

Proof:

By taking the isometric image to the equation $+BY + C = 0$, we get the proof.

Example 5.3.6:

Consider the following neutrosophic line $(1 + I)X + (2 - 4I)Y + 1 - 3I = 0$, it is equivalent to the following two classical lines

$$d_1: x_0 + 2y_0 + 1 = 0, d_2: 2\,(x_0 + x_1) - 2(y_0 + y_1) - 2 = 0.$$

Remark 5.3.7:

(a) If we have two classical circles $C_1: (x_0 - a)^2 + (y_0 - c)^2 = (r_1)^2, C_2: (x_1 - b)^2 + (y_1 - d)^2 = (r_2)^2$, then we can transform the set of their direct product $C_1 \times C_2$, into one neutrosophic circle by using the inverse image of the AH-isometry as follows:

$C: (X - M)^2 + (Y - N)^2 = r^2; X = x_0 + (x_1 - x_0)I, Y = y_0 + (y_1 - y_0)I, M = a + (b - a)I, N = c + (d - c)I, r = r_1 + (r_2 - r_1)I.$

The proof holds easily by taking the inverse image with respect to AH-isometry.

(b) By the same argument, if we have two classical lines:

$a_0x_0 + b_0y_0 + c_0 = 0, a_1x_1 + b_1y_1 + c_1 = 0.$, we can transform the set of their direct product into one neutrosophic line as follows:

$$AX + BY + C = 0; A = a_0 + (a_1 - a_0)I, B = b_0 + (b_1 - b_0)I, X = x_0 + (x_1 - x_0)I, Y = y_0 + (y_1 - y_0)I, C = c_0 + (c_1 - c_0)I.$$

CHAPTER VI

SOME OF KANDASAMY-SMARANDACHE PROBLEMS

In this section, we present the basic definitions that are useful in this chapter. Here the solution for 18 problems of Kandasamy–Smarandache open problems.

Definition 6.1.1 : [58] Let the set $C(\langle Z \cup I \rangle) = \{a + bI + ci + idI \mid a, b, c, d \in Z\}$ then:

1- Is an integer complex neutrosophic group under addition

2- Is an integer neutrosophic complex monoid commutative monoid under multiplication.

3- Is an integer neutrosophic complex commutative ring with unit of infinite order under addition $+$ and multiplication $\times$.

Definition 6.1.2: [58] Let $S = \{(a_{ij}) \mid a_{ij} \in C(\langle Z \cup I \rangle);\ 1 \leq i, j \leq n\}$ be a collection of $n \times n$ complex neutrosophic integer matrices. S is a ring of $n \times n$ integer complex neutrosophic ring of infinite order and is non commutative. S has zero divisors, units, idempotents, subrings and ideals.

Definition 6.1.3: [58] Let the set $C(\langle Q \cup I \rangle) = \{a + bi + cI + idI \mid a, b, c, d \in Q\}$ then:

1- $C(\langle Q \cup I \rangle)$ is a rational complex neutrosophic ring has no zero divisors.

2- $C(\langle Q \cup I \rangle)$ is a rational complex neutrosophic field.

Definition 6.1.4: [58] Let $C(Z_n) = \{a + bi_F \mid a, b \in Z_n, i_F$ is the finite complex modulo number such that $i_F^2 = n - 1, n < \infty\}$ we define i_F as the finite complex modulo number. $C(Z_n)$ is the finite complex modulo integer numbers.

Definition 6.1.5: [59] A Smarandache ring (S-ring) is defined to be a ring A, such that a proper subset of A is a field with respect to the operations induced. By proper subset we understand a set included in A different from the empty set, from the unit element if any and from A.

6.2 Problem (3) and problem (4) of Kandasamy–Smarandache

Problem (3) 6.2.1: [59] Can $C(< Z \cup I >)$ be a Smarandache ring?

The answer is no. We give a proof.

We suppose that $C(< Z \cup I >)$ is a Smarandache ring, then there is a proper subset

$A \subseteq C(< Z \cup I >)$, such that A has a field structure with respect to multiplication on $C(< Z \cup I >)$. Consider an arbitrary element $n = a + bI + (c + dI)i; a, b, c, d \in Z$ in A, since A is an abelian group under addition we can see that $r \cdot x \in A$ for every $r \in Z$, thus A has infinite cardinality.

It is well known that the minimal field of infinite cardinality is the field of rationales Q, hence the field A has a characteristic zero (A contains an isomorphic image of Q).

The field A has only two principal ideals $\{0\}$ and A, hence $A =< a + bI + (c + dI)i >$. It is probable that A has a unity different from 1, we will prove that the identity of A must be 1.

For this goal, we suppose that $m = x + yI + (z + tI)i$ is the identity of A and different from 1,

We have $m \cdot n = n$, thus

$[a + bI + (c + dI)i][x + yI + (z + tI)i] = a + bI + (c + dI)i$, we get

$(a \cdot x - c \cdot z) + I(a \cdot y + b \cdot x + b \cdot y) + a \cdot zi + Ii(a \cdot t + b \cdot z + b \cdot t + c \cdot xi +$
$Ii(c \cdot y + d \cdot x + d \cdot y) + I(-c \cdot t - d \cdot z - d \cdot t) = a + bI + (c + dI)i$, this implies

$(a \cdot x - c \cdot z) + (a \cdot z + c \cdot x)i + I(a \cdot y + b \cdot x + b \cdot y - c \cdot t - d \cdot z - d \cdot t) +$
$Ii(a \cdot t + b \cdot z + b \cdot t + c \cdot y + d \cdot x + d \cdot y) = a + ci + bI + dIi$, hence

(1)$a + ci = (a.x - c.z) + (a.z + c.x)i$,

(2) $a \cdot y + b \cdot x + b \cdot y - c \cdot t - d \cdot t - d \cdot z = b$,

(3) $a \cdot t + b \cdot z + b \cdot t + c \cdot y + d \cdot x + d \cdot y = d$,

From equation (1) we get the following two equivalent equations

(I) $a = a \cdot x - c \cdot z$,

$(II) c = a \cdot z + c \cdot x$,

We multiply (*I*) by c and (*II*) by a to get

(I) $a \cdot c = a \cdot c \cdot x - c^2 z$,

$(II) a \cdot c = a^2 z + a \cdot c \cdot x$, we compute $(II) - (I)$,

$0 = a^2 z + c^2 z = z(a^2 + c^2)$, thus $z = 0$, that is because $a^2 + c^2 \neq 0$, so that $x = 1$.

By putting $z = 0$ and $x = 1$ in equations (2), (3), we get

$a \cdot y + b + b \cdot y - c \cdot t - d. t = b$, hence $y(a + b) - t(c + d) = 0$ (III).

$a \cdot t + b \cdot t + c \cdot y + d + d \cdot y = d$, hence $y(c + d) + t(a + b) = 0$ (IV).

We multiply equation (*III*) by $(a + b)$, and equation (*IV*) by $(c + d)$ and then we add them to get:

$y \cdot [(a + b)^2 + (c + d)^2] = 0$, hence $y = 0$ and then $t = 0$. So that $m = 1$.

The previous discussion implies that $Q \subseteq A$, thus $Q \subseteq C(< Z \cup I >)$ which is a contradiction, thus A cannot be a field and $C(< Z \cup I >)$ is not a Smarandache ring.

Problem (4) 6.2.2: [59] Is $M = \{(a_1, a_2);\ a_1, a_2 \in C(< Z \cup I >)\}$ under $(\times)$ a Smarandache semigroup?.

The answer is yes, we give a proof.

We have to search for a proper subset A of M, where A has a group structure.

It is easy to see that $Z(I) \subseteq C(< Z \cup I >)$, so if we take the group of units in the ring $Z(I)$, which is equal to $U\big(Z(I)\big) = \{1, -1, 1 - 2I, -1 + 2I\}$ it will be a subgroup of the semi group $C(< Z \cup I >)$, hence the direct product $U(Z(I)) \times U(Z(I))$ is a subgroup contained in the semigroup M, thus M is a Smarandache semi group.

6.3 Neutrosophic Complex Rings As Direct Products

This section is devoted to solve problems (1), (13), (15), (11), (14), (12), (17), (82), (83).

Problem (1) 6.3.1 : [59] Obtain some interesting results enjoyed by

(a) Neutrosophic complex reals.

(b) Neutrosophic complex modulo integers.

(c) Neutrosophic complex rationales.

The best answer we can obtain is to classify those rings as direct products of classical algebraic rings. This desired classification can help in many other problems.

Theorem 6.3.2: [59] Let $C(< R \cup I >)$be the neutrosophic complex ring of reals, then $C(< R \cup I >) = C(I)$.

Proof:

It is clear that the neutrosophic ring $C(I)$ is contained in $C(< R \cup I >)$. Conversely, suppose that

$x = a + bI + ci + diI \in C(< R \cup I >)$, then $x = (a + ci) + I(b + di) \in C(I)$.

Thus our proof is complete.

Theorem 6.3.3: [59] $C(< R \cup I >) \cong C \times C$.

Proof:

We shall prove that there is a ring isomorphism between $C(I) and\ C \times C$.

We define $f: C(I) \to C \times C; f(x + yI) = (x, x + y); x, y \in C$.

(a) f is well defined:

Suppose that $a + bI = x + yI; a, b, x, y \in C$, this implies $a = x, b = y$. Hence

$(a, a + b) = (x, x + y), i.e\ f(a + bI) = f(x + yI)$.

(b) f is a bijective:

It is clear that f is a surjective function. On the other hand we assume that

$f(x + yI) = f(a + bI), this\ means\ (a, a + b) = (x, x + y)$, thus $a = x\ and\ b = y$.

(c) f is a ring homomorphism:

We take $m = a + bI, n = c + dI \in C(I)$, then $m + n = (a + c) + (b + d)I$,

$m.n = a.c + I(a \cdot d + b \cdot c + b \cdot d)$, so that

$f(m + n) = (a + c, a + c + b + d) = (a, a + b) + (c, c + d) = f(m) + f(n)$, and

$f(m.n) = (a \cdot c, a \cdot c + a \cdot d + b \cdot c + b \cdot d) = \big(a \cdot c, (a + b)(c + d)\big) = (a, a + b) \cdot (c, c + d) = f(m) \cdot f(n)$.

Thus f is an isomorphism and the proof holds.

Problem (13) 6.3.4: [59] Can $C(< R \cup I >)$ have irreducible polynomials?

The answer is no, we give a proof.

Since $C(< R \cup I >) \cong C \times C$, then $C(< R \cup I >)[x] \cong C \times C[x]$, i.e for each polynomial $P(x)$ in $C(< R \cup I >)[x]$ there is a corresponding polynomial with form $(g(x), h(x))$ in $C \times C[x]$, if $p(x)$ is not irreducible in $C(< R \cup I >)[x]$, then one of $g(x), h(x)$ at least is irreducible over the field of complex numbers C, which is not possible, that is because C is algebraically closed field.

The following theorem classifies $C(< Q \cup I >)$.

Theorem 6.3.5: [59] Let $C(< Q \cup I >)$ be the neutrosophic complex ring of rationales, then $C(< Q \cup I >) \cong Q(i) \times Q(i)$, where $Q(i) = \{a + bi; a, b \in Q\}$ is the algebraic extension of the field Q by i.

Proof:

We define $f: C(< Q \cup I >) \to Q(i) \times Q(i); f(a + bi + (c + di)I) = (a + bi, (a + bi) + (c + di))$, where $a, c, b, d \in Q$. We have

(a) f is well defined:

Suppose that $a+bi+(c+di)I=x+yi+(z+ti)I$, then $a=x, b=y, c=z, d=t$.

(b) f is a bijective map:

It is similar to that of **Theorem 6.3.3**

(c) f is a ring homomorphism:

Consider two arbitrary elements $m=a+bi+(c+di)I, n=x+yi+(z+ti)I$, we have

$f(m+n)=([a+bi+a+yi],[(c+di)+(z+ti)+(a+bi)+(x+yi)])=f(m)+f(n)$.

$f(m.n)=f([(a+bi).(x+yi)]+I[(a+bi)(z+ti)+(c+di)(x+yi)+(c+di)(z+ti)]=$

$([(a+bi)(x+yi),[(a+bi).(x+yi)+(a+bi)(z+ti)+(c+di)(x+yi)+(c+di)(z+ti)])=f(m).f(n)$.

Remark 6.3.6: By same argument, we can write $C(<Z\cup I>)\cong Z(i)\times Z(i)$, i.e $C(<Z\cup I>)$ can be classified as a direct product of the ring $Z(i)=\{a+bi; a,b\in Z\}$ with itself.

By our classification results we can answer many other open problems.

Problem (11) 6.3.7: [59] Is $C(<Q\cup I>)$ a field? Is it a prime field?

The answer is no, that is because:

Since $C(<Q\cup I>)\cong Q(i)\times Q(i)$, we can find that it is not a field since the element

$x=(1,0)\in Q(i)\times Q(i)$ and it is not invertible, hence its inverse isomorphic image $1-I$ is not invertible in $C(<Q\cup I>)$. Thus $C(<Q\cup I>)$ cannot be a field.

Problem (14) 6.3.8 : [59] Determine the irreducible polynomials over $C(<Q\cup I>)$?.

It is really a hard problem, but by using the fact $C(< Q \cup I >) \cong Q(i) \times Q(i)$, we can find all irreducible polynomial over $C(< Q \cup I >)$.

Let $p(x)$ be any polynomial defined over $C(< Q \cup I >)$, then it has a corresponding polynomial $(g(x), h(x))$ in $(Q(i) \times Q(i))[x]$. It is sufficient to compute its isomorphic image $(g(x), h(x))$.

If one of $g(x), h(x)$ is irreducible at least over $Q(i)$, then $p(x)$ is irreducible over $C(< Q \cup I >)$.

Example 6.3.9: Let $p(x) = X^2 + (1 + (1 + i)I)X + 1 + (5 - i)I$, where $X = x_1 + x_2 I$; $x_1, x_2 \in Q(i)$ be a polynomial defined over $C(< Q \cup I >)$.

The corresponding isomorphic polynomial of $p(x)$ is

$$n(x_1, x_2) = f\big(p(x)\big) = f(X^2) + f(1 + (1 + i)I).f(X) + f(1 + (5 - i)I)$$

$$= (x_1{}^2, (x_1 + x_2)^2) + (1,2 + i).(x_1, x_1 + x_2) + (1,6 - i)$$

$$= (x_1{}^2 + x_1 + 1, (x_1 + x_2)^2 + (2 + i)(x_1 + x_2) + 6 - i),$$

We get the following two equivalent polynomials:

$g(x_1) = x_1{}^2 + x_1 + 1, h(x_1, x_2) = (x_1 + x_2)^2 + (2 + i)(x_1 + x_2) + 6 - i$, it is clear that $g(x)$ is irreducible over $Q(i)$, thus $p(x)$ is irreducible over $C(< Q \cup I >)$.

Problem (15) 6.3.10: [59] Find irreducible polynomials in $C(< Z \cup I >)[x]$? Is every ideal in $C(< Z \cup I >)$ is principal?

The first part of Problem (15) can be solved in a similar way of Problem (14), just by taking the isomorphic corresponding polynomial, since $C(< Z \cup I >) \cong Z(i) \times Z(i)$.

Example 6.3.11: Let $p(x) = X^2 + (1 + (1 + i)I)X + 1 + (5 - i)I$, where $X = x_1 + x_2 I$; $x_1, x_2 \in Z(i)$ be a polynomial defined over $C(< Z \cup I >)$.

The corresponding isomorphic polynomial of $p(x)$ is

$n(x_1, x_2) = f\big(p(x)\big) = f(X^2) + f(1 + (1 + i)I).f(X) + f(1 + (5 - i)I)=$

$\left(x_1{}^2, (x_1 + x_2)^2\right) + (1,2+i).(x_1, x_1 + x_2) + (1,6-i) = (x_1{}^2 + x_1 + 1, (x_1 + x_2)^2 + (2+i)(x_1 + x_2) + 6 - i),$

We get the following two equivalent polynomials:

$$g(x_1) = x_1{}^2 + x_1 + 1, h(x_1, x_2) = (x_1 + x_2)^2 + (2+i)(x_1 + x_2) + 6 - i$$

it is clear that g is irreducible over $Z(i)$, thus $p(x)$ is irreducible over $C(< Z \cup I >)$.

Problem (12) 6.3.12: [59] Can one say for all polynomials with complex neutrosophic coefficients C $(\langle R \ \cup \ I\rangle)$ is algebraically closed?.

The answer is yes. That is because $C(< R \cup I >) \cong C \times C$, and C is algebraically closed, thus $C(< R \cup I >)$ is an algebraically closed ring, i.e each root of any polynomial with coefficients from $C(< R \cup I >)$ is from $C(< R \cup I >)$.

Problem (17) 6.3 .12: [59] Is $G = \{\begin{pmatrix} a & b \\ c & d \end{pmatrix}; a.d - c.b \neq 0 \ and \ a, b, c, d \in C(< Z \cup I >)\}$ a group? Is G simple?

The answer in no. G is not even a group, we take $\begin{pmatrix} 1 & 2 \\ 3 & 7 \end{pmatrix} \in G$, its inverse is not in G, thus G is not a group.

Problem (82) 6.3.13: [59] Is $C(\langle Z \ \cup \ I\rangle)$ a unique factorization domain?

The answer is no. We clarify our claim by the following discussion.

We have $C(< Z \cup I >) \cong Z(i) \times Z(i)$, it is east to see that

$(1,0) = (1,2).(1,0) = (1,3).(1.0)$, i.e $Z(i) \times Z(i)$ is not a unique factorization domain, hence $C(\langle Z \ \cup \ I\rangle)$ is not a unique factorization domain.

Problem (83) 6.3.14: [59] Can $C(\langle R \ \cup \ I\rangle)$ be a principal ideal domain?

The answer is no. It is sufficient to prove that $C(\langle R \ \cup \ I\rangle)$has zero divisors.

We have $\mathrm{C}(\langle \mathrm{R} \ \cup \ \mathrm{I}\rangle) \cong C \times C$, and $(1,0).(0,1) = (0,0)$, so that $C \times C$ is not a principal ideal domain because it has zero divisors, thus $C(\langle R \ \cup \ I\rangle)$ is not a principal ideal domain.

6.4 Other open problems

This section is devoted to study Problems (5), (6), (7), (8), (19), (20), (2).

Problem (2) 6.4.1: [59] Can any geometrical interpretation be given to the field of neutrosophic complex numbers $C\ (\langle Q\ \cup\ I\rangle)$?.

The answer is no. But there is an algebraic interpretation of $C\ (\langle Q\ \cup\ I\rangle)$. We describe it by the following theorem.

Theorem 6.4.2: [59] Let $C\ (\langle Q\ \cup\ I\rangle)$ be the complex neutrosophic ring of rationales. Then it can be considered as an algebraic extension of the neutrosophic ring $Q(I)$ with degree two.

Proof:

We have $P(x) = x^2 + 1$ is a monic polynomial over $Q(I)$, we shall prove that it is irreducible over $Q(I)$.

Suppose that $p(x) = (x + a + bI)(x + c + dI); a, b, c, d \in Q$, then

$P(x) = x^2 + x(a + bI + c + dI) + a.c + I(a.d + b.c + b.d)$, hence

$a + bI + c + dI = 0\ and\ a.d + b.c + b.d = 0\ and\ a.c = 1$, so that $a + c = 0(*)\ and\ b + d = 0$.

We get from (*) $a = -c, thus - a^2 = 1$, which is a contradiction since $a \in Q$. Thus $p(x)$ is irreducible.

$P(x)$ has a root $m = i$, hence the ring $[Q(I)](i)$ is an algebraic extension of Q(I) with degree equal to $deg(P) = 2$. It is clear that $[Q(I)](i) = \{x + yi; x, y \in Q(I)\} = \{a + bI + ci + diI; a, b, c, d \in Q\} = \mathrm{C}\ (\langle \mathrm{Q}\ \cup\ \mathrm{I}\rangle)$.

Problem (5) 6.4.3: [59] Let $V = \{(a_1, a_2, a_3, a_4);\ a_i \in C(< Q \cup I >), +\}$ be a group

i) Define an automorphism $\eta : V \rightarrow V$ so that $ker\ \eta$ is a nontrivial subgroup.

ii) Is $V \cong C(\langle Q\ \cup\ I\rangle) \times C(\langle Q\ \cup\ I\rangle) \times C(\langle Q\ \cup\ I\rangle) \times C(\langle Q\ \cup\ I\ \rangle)$?

(i) is not possible, since every group automorphism needs to be a bijective map, hence its kernel will be trivial.

The question (ii) is easy and clear, that is because V is defined to be the direct product of $C(< Q \cup I >)$ with itself four times.

Problem (6) 6.4.4: [59] Let $M = \{\begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix}; \ a_i \in C(< Q \cup I >)\}$ be a semigroup under multiplication.

(i) Prove M is a S-semigroup.

(ii) Is M commutative?.

(iii) Find at least three zero divisors in M.

(iv) Does M have ideals?.

(v) Give subsemigroups in M which are not ideals.

(i) We define $A = \{\begin{pmatrix} x & 0 \\ 0 & x \end{pmatrix}; x \in C(< Q \cup I >)\}$**,** A is a proper subset of M, and it is an abelian group clearly. Thus M is a S-semigroup.

(ii) No it is not. Since matrices over Q do not commute, and Q is contained in $C(< Q \cup I >)$**.**

(iii) Take $x = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, y = \begin{pmatrix} 0 & 0 \\ 0 & 2 \end{pmatrix}, z = (\begin{matrix} 2 & 3 \\ 0 & 0 \end{matrix})$. It is easy to see that $x \cdot z = y \cdot z =$ 0, thus x, y, z are zero divisors.

(iv) Take $S = \{\begin{pmatrix} aI & bI \\ cI & dI \end{pmatrix}; a, b, c, d \in C(< Q \cup I >)\}$**,** S is a subgroup with respect to addition.

Let $m = (\begin{matrix} x & y \\ z & t \end{matrix}) \in M$ and $= \begin{pmatrix} aI & bI \\ cI & dI \end{pmatrix} \in S$, we have $m.n \in S$, thus S is an ideal and M has ideals.

(v) We define $S_1 = \left\{\begin{pmatrix} a & b \\ c & d \end{pmatrix}; a, b, c, d \in Q(i)\right\}, S_2 = \{\begin{pmatrix} a & b \\ c & d \end{pmatrix}; a, b, c, d \in Q\}$**.** These two sets are semi groups with respect to multiplication but they are not ideals clearly.

Problem (7) 6.4.5: [59] Let $S = \{\begin{pmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \\ a_{10} & a_{11} & a_{12} \end{pmatrix}; \ a_i \in C(< Q \cup I >), +\}$.

i. Find subgroups of S.
ii. Can S have ideals?
iii. Can S have idempotents?
iv. Can S have zero divisors?

We summarize the answer as follows:

(i) Consider that $(H_i, +)$ is a subgroup of $(C(< Q \cup I >), +)$, and $M = \{\begin{pmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \\ a_{10} & a_{11} & a_{12} \end{pmatrix}; a_i \in H_i; 1 \leq i \leq 12\}$, then S has the following property:

For every $x, y \in M, we\ have\ x - y \in M$, hence M is a subgroup of S. All subgroups will have the same form.

(ii) The answer in no. That is because the multiplication is not defined on S, thus it is not even a ring.

(iii) No, for the same reason.

(iv) No, for the same reason.

Problem (8) 6.4.6: [59] Let $V = \{\begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix}; a_i \in C(< Q \cup I >)\}$ be a semigroup under product.

i. Is V commutative?.
ii. Can V have idempotents?.
iii. Does V have a semigroup which is not an ideal?.
iv. Can V have zero divisors?.
v. Give an ideal in V.
vi. Is V a Smarandache semigroup?.
vii. Is V a Smarandache semigroup?.

The answer is:

(i) No it is not. Since matrices over Q do not commute, and Q is contained in $C(< Q \cup I >)$**.**

(ii) Yes, for example take $x = \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}$.

(iii) Yes, the set $S = \{\begin{pmatrix} a & b \\ c & d \end{pmatrix}; a, b, c, d \in Q\}$ is a semigroup which is not an ideal.

(iv) Yes, we have $\begin{pmatrix} I & I \\ I & I \end{pmatrix} . \begin{pmatrix} -I & -I \\ I & I \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$.

(v) Consider $S = \{\begin{pmatrix} aI & bI \\ cI & dI \end{pmatrix}; a, b, c, d \in C(< Q \cup I >)\}$, it is an ideal in V.

(vi) Yes, V contains the set $S = \{\begin{pmatrix} a & b \\ c & d \end{pmatrix}; a.d - c.b \neq 0 \; and \; a, b, c, d \in Q\}$, which is a group with respect to multiplication.

(vii) Yes, V contains the set $S = \{\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}; a \neq 0 \; and \; a \in Q\}$,which is an abelian group with respect to multiplication.**.**

Problem (19) 6.4.7: [59] What are the advantages of using the algebraic structure$C(\langle R \cup I \rangle)$?

Problem (20) 6.4.8: [59] Give some uses of this complete algebraic structure

$C(\langle R \cup I \rangle)$.

Problems (19) and (20) are solved partially, that is because $C(\langle R \cup I \rangle)$ is being classified as a direct product of the complex field C with itself according to **Theorem 6.3.5**.

6.5 Group of units problem

A well known problem in the theory of rings is to describe the group of units under multiplication for a ring R. We will solve this famous problem in the case of

$C(< R \cup I >), C(< Q \cup I >), C(< Z \cup I >)$, by using classification properties.

Theorem 6.5.1: [59] The group of units of the ring $C(< R \cup I >)$ is U= $C^{*} \times C^{*}$.

Proof:

Since $C(< R \cup I >) \cong C \times C$**,** then $U = U(C) \times U(C)$, but C is a field, hence $U(C) = C^{*}$. Thus the proof is complete.

Theorem 6.5.2: [59] The group of units of the ring $C(< Q \cup I >)$ is U= $(Q(i))^* \times (Q(i))^*$.

Proof:

Since $C(< Q \cup I >) \cong Q(i) \times Q(i)$, then $U = U(Q(i))) \times U(Q(i))$, but $Q(i)$ is a field, hence $U(Q(i)) = (Q(i))^*$. Thus the proof is complete.

Theorem 6.5.3: [59] The group of units of the ring $C(< Z \cup I >)$ is U= $Z_2 \times Z_2 \times Z_2 \times Z_2$.

Proof:

Since $C(< Z \cup I >) \cong Z(i) \times Z(i)$, then $U = U(Z(i))) \times U(Z(i))$, but $U\big(Z(i)\big) = \{1, -1, i, -i\} \cong Z_2 \times Z_2$, thus U= $Z_2 \times Z_2 \times Z_2 \times Z_2$.

CHAPTER VII

NEUTROSOPHIC SQUARE MATRICES

The objective of this chapter is to study algebraic properties of neutrosophic matrices, where a necessary and sufficient condition for the invertibility of a square neutrosophic matrix is presented by defining the neutrosophic determinant.

Definition7.1.1: [60] Classical neutrosophic number has the form $a + bI$ where a, b are real or complex numbers and I is the indeterminacy such that $0 \cdot I = 0$ and $I^2 = I$ which results that $I^n = I$ for all positive integers n.
Definition 7.1.2: [61] Let $M_{m\times n} = \{(a_{ij}) : a_{ij} \in K(I)\}$, where $K(I)$ is a neutrosophic field. We call to be the neutrosophic matrix.

7.2 Invertible Neutrosophic *n* Square Matrix

Definition 7.2.1: [62] Let $M = A + BI$ a neutrosophic n square matrix, where A and B are two n squares matrices, then M is called an invertible neutrosophic n square matrix, if and only if there exists an n square matrix $S = S_1 + S_2 I$, where S_1 and S_2 are two n square matrices such that

$S \cdot M = M \cdot S = U_{n\times n}$, where $U_{n\times n}$ denotes the $n \times n$ identity matrix.

Definition 7.2.2: [63] Let $M = A + BI$ be a neutrosophic n square matrix. The determinant of M is defined as:

$$detM = detA + I[det(A + B) - detA]$$

Theorem 7.2.3: [63] Let $M = A + BI$ a neutrosophic square $n \times n$ matrix, where , B are two squares $n \times n$ matrices, then M is invertible if and only if A and $A + B$ are invertible matrices and $M^{-1} = A^{-1} + I[(A + B)^{-1} - A^{-1}]$ **.**

Poof: If A and $A + B$ are invertible matrices, then $(A + B)^{-1}$, A^{-1} are existed, and $M^{-1} = A^{-1} + I[(A + B)^{-1} - A^{-1}]$ exists too. Now to prove $M^{-1}$is the inverse of M,

$$MM^{-1} = (A + BI) \cdot (A^{-1} + I[(A + B)^{-1} - A^{-1}])$$

$$= AA^{-1} + I[A(A + B)^{-1} - AA^{-1} + B \cdot A^{-1} + B(A + B)^{-1} - BA^{-1}]$$

$$= U_{n\times n} + I[(A + B)(A + B)^{-1} - U_{n\times n}]$$

$$= U_{n\times n} + I[U_{n\times n} - U_{n\times n}] = U_{n\times n} = M^{-1}M.$$

conversely, we suppose that M is invertible, thus there is a matrix $S = S_1 + S_2 I$, with the property $M \cdot S = S \cdot M = U_{n\times n}$.

$MS = (A + BI)(S_1 + S_2 I) = AS_1 + I[(A + B)(S_1 + S_2) - AS_1] = U_{n\times n} + 0_{n\times n}$=$SM$. Hence, we get:

(a) $S_1 A = AS_1 = U_{n\times n}, thus\ A\ is\ invertible\ and\ A^{-1} = S_1.$

(b) $(A + B)(S_1 + S_2) - AS_1 = (S_1 + S_2)(A + B) - S_1 A = O_{n\times n}$, thus,

$(S_1 + S_2)(A + B) = (A + B)(S_1 + S_2) = AS_1 = U_{n\times n}$. This implies that $(A + B)$ is invertible.

Theorem7.2.4: [63] M is invertible matrix if and only if $detM \neq 0$.

Proof:

From **Theorem 7.1.3** we find that M is invertible matrix if and only if $A + B, A$ are two invertible matrices, hence $det[A + B] \neq 0, detA \neq 0$ which means

$$detM = detA + I[\det(A + B) - detA] \neq 0.$$

Example 7.2.5 : Consider the following neutrosophic matrix

$M = A + BI = \begin{pmatrix} 1 & -1+I \\ I & 2+I \end{pmatrix}$. Where $A = \begin{pmatrix} 1 & -1 \\ 0 & 2 \end{pmatrix}$, $B = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$.

(a)$detA = 2, A + B = \begin{pmatrix} 1 & 0 \\ 1 & 3 \end{pmatrix}$, $\det(A + B) = 3, detM = 2 + I[3 - 2] = 2 + I \neq 0, hence\ M\ is\ invertible.$

(b) We have $A^{-1} = \begin{pmatrix} 1 & \frac{1}{2} \\ 0 & \frac{1}{2} \end{pmatrix}, (A + B)^{-1} = \begin{pmatrix} 1 & 0 \\ -\frac{1}{3} & \frac{1}{3} \end{pmatrix}, thus\ M^{-1} = (A^{-1}) + I[(A + B)^{-1} - A^{-1}]$

$$= \begin{pmatrix} 1 & \frac{1}{2} \\ 0 & \frac{1}{2} \end{pmatrix} + I \begin{pmatrix} 0 & -\frac{1}{2} \\ -\frac{1}{3} & -\frac{1}{6} \end{pmatrix} = \begin{pmatrix} 1 & \frac{1}{2} - \frac{1}{2} I \\ -\frac{1}{3} I & \frac{1}{2} - \frac{1}{6} I \end{pmatrix}.$$

(c) We can compute $MM^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = U_{2\times2}$.

Theorem 7.2.6: [63] Let $M = A + BI$ be a neutrosophic n square matrix, were A and B are two n square matrices, then

1) $M^r = A^r + I[(A + B)^r - A^r]$.

2)$M\ is\ nilpotent\ if\ and\ only\ if\ A, A + B\ are\ nilpotent.$

3)$M\ is\ idempotent\ if\ and\ only\ if\ A, A + B\ are\ idempotent.$

Proof:

1) By using mathematical induction, it easy to see $P(r = 1)$ is true.

Suppose $P(k)$, then we must prove $P(k + 1)$ is true like the following

$M^{k+1} = M^k \cdot M = (A^k + I[(A + B)^k - A^k]) \cdot (A + IB)$

$= A^{k+1} + I[(A\text{^}k \cdot B + (A + B)\text{^}k \cdot A + (A + B)\text{^}k \cdot B - A\text{^}k \cdot A - A\text{^}k \cdot B)]$

$= A^{k+1} + I[(A + B)^k \cdot (A + B) - A^{k+1}]$

$= A^{k+1} + I[(A + B)^{k+1} - A^{k+1}]$.

2) M is nilpotent $if\ and\ only\ if$ $\exists\ n \in N^+;\ M^n = 0$, this is equivalent to

$A^n + I[(A+B)^n - A^n] = 0$, thus

$A^n = (A+B)^n = 0$. Which is equivalent to

$A, A+B$ are nilpotent.

3) The proof is similar to (2).

Theorem 7.2.7: [63] Let $M = A + BI$ and $N = C + DI$ be two neutrosophic n square matrices, then

1) $det(M \cdot N) = detM \cdot detN$.

2) $\det(M^{-1}) = (detM)^{-1}$.

3) $detM$ =1 if and only if $detA = det(A+B) = 1$.

Proof:

1) $M \cdot N = A \cdot C + I[B \cdot C + B \cdot D + A \cdot D]$

$= A \cdot C + I[(A+B)(C+D) - A \cdot C]$.

$\det(M \cdot N) = \det(A \cdot C) + I\big[\det\big((A+B)(C+D)\big) - \det(A \cdot C)\big]$,

$= \det A \cdot \det C + I[\det(A+B) \cdot \det(C+D) - \det(A \cdot C)]$,

$= \det A \cdot \det C + I[\det(A+B) \cdot \det(C+D) - \det A \cdot \det C]$,

$= (\det A + I[\det(A+B) - \det A]) \cdot (\det C + I[\det(C+D) - \det C])$,

$= \det M \cdot \det N$.

2) We have

$\det(MM^{-1}) = \det(U_{n\times n}) = 1, thus\ detM. \det(M^{-1}) = 1, so\ that\ \det(M^{-1}) = (detM)^{-1}$.

3) $det\ M = 1$ is equivalent to $\det A + I[\det(A+B) - \det A] = 1$, thus it is equivalent to

$detA = det(A+B) = 1$.

Definition 7.2.8: [63] Let $M = A + BI$ be a neutrosophic n square matrix, where A and B are two n square matrices. M is satisfying the orthogonality property if and only if $M \cdot M^T = U_{n\times n}$.

Theorem 7.1.9: [63] Let $M = A + BI$ a neutrosophic n square matrix, then

(a) $M\ is$ orthogonal $if\ and\ only\ if\ A, B$ are two orthogonal matrices .

(b) $If\ M\ is$ orthogonal, then $detM = \pm 1$.

Proof:

(a) $M\ is$ orthogonal neutrosophic matrix $if\ and\ only\ if\ M^T = M^{-1}$, this is equivalent to

$A^T + B^T I = A^{-1} + I[(A + B)^{-1} - A^{-1}]$, thus

$A^{-1} = A^T, (A + B)^{-1} - A^{-1} = B^T$. This is equivalent to

$A^{-1} = A^T\ and\ (A + B)^{-1} = B^T + A^{-1} = B^T + A^T = (A + B)^T$. Thus the proof is complete.

(b) If M is orthogonal, we get that $det(M \cdot M^T) = det(U_{n\times n}) = 1$. This implies

$detM \cdot detM^T = 1$,

$(\det M)^2 = 1$, hence

$detM = \pm 1$.

Definition 7.2.10: [63] Let $M = A + BI$ be a square neutrosophic matrix, we say that M is diagonalizable if and only if there is an invertible neutrosophic matrix $S = C + DI$ such that $S^{-1}MS = D$. Where D is a diagonal neutrosophic matrix$\left(i.e.\ d_{ij} = 0\ \ \forall i \neq j, and\ d_{ij} \neq 0\ \ \forall i = j\right)$.

Theorem 7.2.11: [63] Let $M = A + BI$ be any square neutrosophic matrix. Then M is diagonalizable if and only if $A, A + B$ are diagonalizable.

Proof:

Consider a diagonalizable neutrosophic matrix M, then there exists S such that

$S^{-1}MS = K(k_{ij})$ $(*)$.

Now, to compute the entries elements k_{ij} , solve $(*)$ as follows:

$\left[C^{-1} + I[(C+D)^{-1} - C^{-1}]\right](A+BI)(C+DI) = \left[C^{-1} + I[(C+D)^{-1} - C^{-1}]\right]\left[AC + I[(A+B)(C+D) - AC]\right] = C^{-1}AC + I[(C+D)^{-1}(A+B)(C+D) - C^{-1}AC] = D_1 + (D_2 - D_1)I = K$. Where K is a diagonal matrix, thus D_1, D_2 are diagonal, and $A, A+B$ are diagonalizable. conversely, assume that $A, A+B$ are diagonalizable, then there are C, D where $C^{-1}AC = D_1, D^{-1}(A+B)D = D_2$. put $S = C + (D-C)I$.

Now we compute $S^{-1}MS = \left[C^{-1} + I[D^{-1} - C^{-1}]\right](A+BI)(C + (D-C)I)$

$$= \left[C^{-1} + I[D^{-1} - C^{-1}]\right]\left[AC + I[(A+B)(D) - AC]\right]$$
$$= C^{-1}AC + I[D^{-1}(A+B)D - C^{-1}AC]$$

$= D_1 + (D_2 - D_1)I = K$. Thus, M is diagonalizable, that is because D_1, D_2 are diagonal matrices.

Remark 7.2.12: If C is the diagonalization matrix of A, and D is the diagonalization matrix of $A+B$, then

$S = C + (D-C)I$ is the diagonalization matrix of $M = A + BI$.

Example 7.2.13: Consider the neutrosophic matrix defined in **Example 7.2.5**, we have:

(a) A is a diagonalizable matrix. Its diagonalization matrix is $= \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix}$, the corresponding diagonal matrix is $D_1 = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$, we can see that $C^{-1}AC = D_1$. Also, the diagonalization matrix of $A+B$ is $D = \begin{pmatrix} 1 & 0 \\ -\frac{1}{2} & 1 \end{pmatrix}$, the corresponding diagonal matrix is $D_2 = \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix}$. It is easy to check that

$D^{-1}(A+B)D = D_2$.

(b) Since $A, A+B$ are diagonalizable, then M is diagonalizable. The neutrosophic diagonalization matrix of M is $S = C + (D-C)I = \begin{pmatrix} 1 & 1-I \\ -\frac{1}{2}I & -1+2I \end{pmatrix}$. The corresponding diagonal matrix is

$$L = D_1 + I[D_2 - D_1] = \begin{pmatrix} 1 & 0 \\ 0 & 2+I \end{pmatrix}.$$

(c) It is easy to see that $S^{-1} = C^{-1} + I[D^{-1} - C^{-1}] = \begin{pmatrix} 1 & 1-I \\ \frac{1}{2}I & -1+2I \end{pmatrix}$.

(d) We can compute $S^{-1}MS = \begin{pmatrix} 1 & 1-I \\ \frac{1}{2}I & -1+2I \end{pmatrix} \begin{pmatrix} 1 & -1+I \\ I & 2+I \end{pmatrix} \begin{pmatrix} 1 & 1-I \\ -\frac{1}{2}I & -1+2I \end{pmatrix}$

$= \begin{pmatrix} 1 & 0 \\ 0 & 2+I \end{pmatrix} = L.$

Definition 7.2.14: [63] Let $M = A + BI$ be a n square neutrosophic matrix over the neutrosophic field $F(I)$, we say that $Z = X + YI$ is a neutrosophic Eigen vector if and only if $MZ = (a+bI)Z$. The neutrosophic number $a + bI$ is called the Eigen value of the eigen vector Z.

Theorem 7.1.15: [63] Let $M = A + BI$ be a n square neutrosophic matrix, then $a + bI$ is an Eigen value of M if and only if a is an eigen value of A, and $a + b$ is an eigen value of $A + B$. As well as, the eigen vector of M is $Z = X + YI$ if and only if X is the corresponding eigen vector of A, and $X + Y$ is the corresponding eigen vector of $A + B$.

Proof:

We suppose that $Z = X + YI$ is an eigen vector of M with the corresponding eigen value $a + bI$, hence $MZ = (a + bI)Z$, this implies

$(A + BI)(X + YI) = (a + bI)(X + YI), thus\ AX + I[(A + B)(X + Y) - AX] = aX + I[(a + b)(X + Y) - aX]$. We get:

$AX = aX, (A + B)(X + Y) = (a + b)(X + Y)$, so that X is an eigen vector of A, $X + Y$ is an eigen vector of $A + B$. The corresponding eigen value of X is a, and the corresponding eigen value of

$X + Y$ is $a + b$.

For the converse, we assume that X is an eigen vector of A with a as the corresponding eigen value, and $X+Y$ is an eigen vector of $A+B$ with $a+b$ as the corresponding eigen value, so that we get $AX = aX, (A+B)(X+Y) = (a+b)(X+Y)$.

Let us compute

$MZ = (A+BI)(X+YI) = AX + I[(A+B)(X+Y) - AX]$

$= aX + I[(a+b)(X+Y) - aX] = (a+bI)(X+YI) = (a+bI)Z.$ Thus $Z = X + YI$ is an eigen vector of M with $a+bI$ as a neutrosophic eigen value.

Theorem 7.2.16: [63] The eigen values of a neutrosophic matrix $M = A + BI$ can be computed by solving the neutrosophic equation $det(M - (a+bI)\ U_{n\times n}) = 0$.

Proof:

We have $det(M - (a+bI)U_{n\times n}) = \det([A - aU_{n\times n}] + I[B - bU_{n\times n}])$

$= \det\ ([A - aU_{n\times n}] + I[\det((A+B) - (a+b)U_{n\times n}) - \det[A - aU_{n\times n}]].$ Thus, the equation

$det(M - (a+bI)\ U_{n\times n}) = 0$ is equivalent to

$$\det\ ([A - aU_{n\times n}] = 0\ (i),$$

and $[\det\ ((A+B) - (a+b)U_{n\times n}) - \det[A - aU_{n\times n}] = 0\ \ (ii).$

From equation (i), we get a as eigen value of A, and from (ii) we get

$[\det((A+B) - (a+b)U_{n\times n}) = \det[A - aU_{n\times n}] = 0$, thus $a+b$ is an eigen value of $A+B$.

Example 7.2.17: Consider M the neutrosophic matrix defined in **Example 7.2.5**, we have

(a) The eigen values of the matrix A are $\{1,2\}, and$ $\{1,3\}$ for the matrix $A+B$. This implies that the eigen values of the neutrosophic matrix M are

$$\{1 + (3-1)I, 1 + (1-1)I, 2 + (3-2)I, 2 + (1-2)I\} = \{1 + 2I, 1, 2 + I, 2 - I\}.$$

(b) If we solved the equation $\det(M-(a+bI)U_{n\times n})=0$ has been solved, the same values will be gotten.

(c) The eigen vectors of A are $\{(1,0),(1,-1)\}$, the eigen vectors of $A+B$ are $\{(1,-1/2),(0,1)\}$. Thus, the neutrosophic eigen vectors of M are

$\left\{(1,0)+I[(0,1)-(1,0)],(1,0)+I\left[\left(1,-\frac{1}{2}\right)-(1,0)\right],(1,-1)+I[(0,1)-(1,-1)],(1,-1)+I\left[\left(1,-\frac{1}{2}\right)-(1,-1)\right]\right\}=\{(1,0)+I(-1,1),(1,0)+I(0,-1/2),(1,-1)+I(-1,2),(1,-1)+I(0,1/2)\}=\{(1-I,I),(1,-1/2\,I),(1-I,-1+2I),(1,-1+1/2\,I)\}$.

To determine the neutrosophic eigen vectors using **Theorem 7.1.15** let X be an eigen vector of $A, and\ Y$ be an eigen vector of $A+B$, hence $X+[(Y)-X]I=X+(Y-X)I$ is an Eigen vector of $M=A+BI$.

CHAPTER VII

CONCLUSION

This book, the historical development process of the neutrosophic structure theory is given. In the second part, the effect of the neutrosophic logic on the decision tree has been compiled. The existence of indeterminacy in the problem actually affects the process of taking the suitable decision. Therefore, the indeterminate values can't be ignored while studying in order to get more accurate results that leads us to the best options. In the third chapter, the prospector neutro function with their applications were studied. We provided the Cayley table of Θ, which is not associative when we included the undefined value and it generates a Neutro Binary Law. In the fourth chapter, the subject of Neutro ordered R-module and their properties is examined in detail. Several interesting results and examples on NeutroOrdered R-module, NeutroOrdered Sub R-module and NeutroOrdered R-module Homomorphisms are presented. In the fifth chapter, the Fundamental Theorem in neutrosophic Euclidean Geometry is given. In the sixth chapter, the solutions of some Kandasamy-Smarandache problems about neutrosophic complex numbers and group of units' problem are given. In the seventh chapter**,** the algebraic creativity in the neutrosophic square matrices and the results are given with examples, necessary and sufficient conditions for the invertibility and diagonalization of neutrosophic matrices are determined. Also, an easy algorithm to compute the inverse of a neutrosophic matrix and its Eigen values and vectors are found.

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