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Interval-valued neutrosophic competition graphs

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Abstract. We first introduce the concept of interval-valued neutrosophic competition graphs. We then discuss certain types, including k-competition interval-valued neutrosophic graphs, p-competition interval-valued neutrosophic graphs and m-step interval-valued neutrosophic competition graphs. Moreover, we present the concept of m-step interval-valued neutrosophic neighbourhood graphs.

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1. Introduction

In 1975, Zadeh [35] introduced the notion of interval-valued fuzzy sets as an extension of fuzzy sets [34] in which the values of the membership degrees are intervals of numbers instead of the numbers. Interval-valued fuzzy sets provide a more adequate description of uncertainty than traditional fuzzy sets. It is therefore important to use interval-valued fuzzy sets in applications, such as fuzzy control. One of the computationally most intensive part of fuzzy control is defuzzification [19]. Atanassov [12] proposed the extended form of fuzzy set theory by adding a new component, called, intuitionistic fuzzy sets. Smarandache [26, 27] introduced the concept of neutrosophic sets by combining the non-standard analysis. In neutrosophic set, the membership value is associated with three components: truth-membership (t), indeterminacy-membership (i) and falsity-membership (f), in which each membership value is a real standard or non-standard subset of the non-standard unit interval $]0^-, 1^+[$ and there is no restriction on their sum. Smarandache [28] and Wang et al. [29] presented the notion of single-valued neutrosophic sets to apply neutrosophic sets in real life problems more conveniently. In single-valued neutrosophic sets, three components are independent and their values are taken from the standard unit interval $[0, 1]$. Wang et al. [30] presented the concept of interval-valued neutrosophic sets, which is more precise and more flexible than the single-valued neutrosophic set. An interval-valued neutrosophic set is a generalization of the concept of single-valued neutrosophic set, in which three membership (t, i, f) functions are independent, and their values belong to the unit interval $[0, 1]$.

Kauffman [18] gave the definition of a fuzzy graph. Fuzzy graphs were narrated by Rosenfeld [22]. After that, some remarks on fuzzy graphs were represented by Bhattacharya [13]. He showed that all the concepts on crisp graph theory do not have similarities in fuzzy graphs. Wu [32] discussed fuzzy digraphs. The concept of fuzzy k-competition graphs and p-competition fuzzy graphs was first developed by Samanta and Pal in [23], it was further studied in [11, 21, 25]. Samanta et al. [24] introduced the generalization of fuzzy competition graphs, called m-step fuzzy competition graphs. Samanta et al. [24] also introduced the concepts of fuzzy m-step neighbourhood graphs, fuzzy economic competition graphs, and m-step economic competition graphs. The concepts of bipolar fuzzy competition graphs and intuitionistic fuzzy competition graphs are discussed in [21, 25]. Hongmei and Lianhua [16], gave definition of interval-valued fuzzy graphs. Akram et al. [1, 2, 3, 4] have introduced several concepts on interval-valued fuzzy graphs and interval-valued neutrosophic graphs. Akram and Shahzadi [6] introduced the notion of neutrosophic soft graphs with applications. Akram [7] introduced the notion of single-valued neutrosophic planar graphs. Akram and Shahzadi [8] studied properties of single-valued neutrosophic graphs by level graphs. Recently, Akram and Nasir [5] have discussed some concepts of interval-valued neutrosophic graphs. In this paper, we first introduce the concept of interval-valued neutrosophic competition graphs. We then discuss certain types, including k-competition interval-valued neutrosophic graphs, p-competition interval-valued neutrosophic graphs and m-step interval-valued neutrosophic competition graphs. Moreover, we present the concept of m-step interval-valued neutrosophic neighbourhood graphs.

We have used standard definitions and terminologies in this paper. For other notations, terminologies and applications not mentioned in the paper, the readers are referred to [6, 9, 10, 13, 14, 15, 17, 20, 26, 33, 36].

2. Interval-Valued Neutrosophic Competition Graphs

Definition 2.1 ([35])**.** The interval-valued fuzzy set A in X is defined by

$$A = \{(s, [t^l_A(s), t^u_A(s)]) : s \in X\},$$

where, $t^l_A(s)$ and $t^u_A(s)$ are fuzzy subsets of X such that $t^l_A(s) \leq t^u_A(s)$ for all $x \in X$. An interval-valued fuzzy relation on X is an interval-valued fuzzy set B in $X \times X$.

Definition 2.2 ([30, 31])**.** The interval-valued neutrosophic set (IVN-set) A in X is defined by

$$A = \{(s, [t^l_A(s), t^u_A(s)], [i^l_A(s), i^u_A(s)], [f^l_A(s), f^u_A(s)]) : s \in X\},$$

where, $t^l_A(s)$, $t^u_A(s)$, $i^l_A(s)$, $i^u_A(s)$, $f^l_A(s)$, and $f^u_A(s)$ are neutrosophic subsets of X such that $t^l_A(s) \leq t^u_A(s)$, $i^l_A(s) \leq i^u_A(s)$ and $f^l_A(s) \leq f^u_A(s)$ for all $s \in X$. An interval-valued neutrosophic relation (IVN-relation) on X is an interval-valued neutrosophic set B in $X \times X$.

Definition 2.3 ([5]). An interval-valued neutrosophic digraph (IVN-digraph) on a non-empty set X is a pair $G = (A, \overrightarrow{B})$, (in short, G), where $A = ([t_A^l, t_A^u], [i_A^l, i_A^u], [f_A^l, f_A^u])$ is an IVN-set on X and $B = ([t_B^l, t_B^u], [i_B^l, i_B^u], [f_B^l, f_B^u])$ is an IVN-relation on X, such that:

(i) $t_B^l\overrightarrow{(s,w)} \leq t_A^l(s) \wedge t_A^l(w)$, $\quad t_B^u\overrightarrow{(s,w)} \leq t_A^u(s) \wedge t_A^u(w)$,
(ii) $i_B^l\overrightarrow{(s,w)} \leq i_A^l(s) \wedge i_A^l(w)$, $\quad i_B^u\overrightarrow{(s,w)} \leq i_A^u(s) \wedge i_A^u(w)$,
(iii) $f_B^l\overrightarrow{(s,w)} \leq f_A^l(s) \wedge f_A^l(w)$, $\quad f_B^u\overrightarrow{(s,w)} \leq f_A^u(s) \wedge f_A^u(w)$, for all $s, w \in X$.

Example 2.4. We construct an IVN-digraph $G = (A, \overrightarrow{B})$ on $X = \{a, b, c\}$ as shown in Fig. 1.

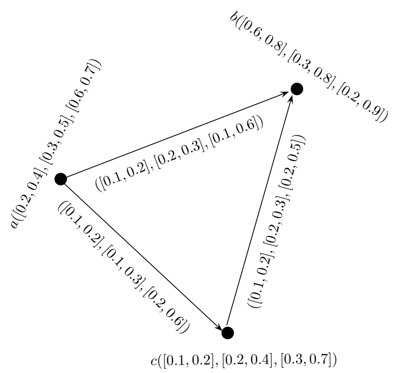

FIGURE 1. IVN-digraph

Definition 2.5. Let $\overrightarrow{G}$ be an IVN-digraph then interval-valued neutrosophic out-neighbourhoods (IVN-out-neighbourhoods) of a vertex s is an IVN-set

$$\mathbb{N}^+(s) = (X_s^+, [t_s^{(l)^+}, t_s^{(u)^+}], [i_s^{(l)^+}, i_s^{(u)^+}], [f_s^{(l)^+}, t_s^{(u)^+}]),$$

where

$$X_s^+ = \{w|[t_B^l\overrightarrow{(s,w)} > 0,\, t_B^u\overrightarrow{(s,w)} > 0], [i_B^l\overrightarrow{(s,w)} > 0, i_B^u\overrightarrow{(s,w)} > 0], [f_B^l\overrightarrow{(s,w)} > 0, f_B^u\overrightarrow{(s,w)} > 0]\},$$

such that $t_s^{(l)^+} : X_s^+ \to [0,1]$, defined by $t_s^{(l)^+}(w) = t_B^l\overrightarrow{(s,w)}$, $t_s^{(u)^+} : X_s^+ \to [0,1]$, defined by $t_s^{(u)^+}(w) = t_B^u\overrightarrow{(s,w)}$, $i_s^{(l)^+} : X_s^+ \to [0,1]$, defined by $i_s^{(l)^+}(w) = i_B^l\overrightarrow{(s,w)}$, $i_s^{(u)^+} : X_s^+ \to [0,1]$, defined by $i_s^{(u)^+}(w) = i_B^u\overrightarrow{(s,w)}$, $f_s^{(l)^+} : X_s^+ \to [0,1]$, defined by $f_s^{(l)^+}(w) = f_B^l\overrightarrow{(s,w)}$, $f_s^{(u)^+} : X_s^+ \to [0,1]$, defined by $f_s^{(u)^+}(w) = f_B^u\overrightarrow{(s,w)}$.

Definition 2.6. Let $\overrightarrow{G}$ be an IVN-digraph then interval-valued neutrosophic in-neighbourhoods (IVN-in-neighbourhoods) of a vertex s is an IVN-set

$$\mathbb{N}^-(s) = (X_s^-, [t_s^{(l)^-}, t_s^{(u)^-}], [i_s^{(l)^-}, i_s^{(u)^-}], [f_s^{(l)^-}, t_s^{(u)^-}]),$$

where

$$X_s^- = \{w|[t_B^l\overrightarrow{(w,s)} > 0,\, t_B^u\overrightarrow{(w,s)} > 0], [i_B^l\overrightarrow{(w,s)} > 0, i_B^u\overrightarrow{(w,s)} > 0], [f_B^l\overrightarrow{(w,s)} > 0, f_B^u\overrightarrow{(w,s)} > 0]\},$$

such that $t_s^{(l)^-} : X_s^- \to [0,1]$, defined by $t_s^{(l)^-}(w) = t_B^l\overrightarrow{(w,s)}$, $t_s^{(u)^-} : X_s^- \to [0,1]$, defined by $t_s^{(u)^-}(w) = t_B^u\overrightarrow{(w,s)}$, $i_s^{(l)^-} : X_s^- \to [0,1]$, defined by $i_s^{(l)^-}(w) = i_B^l\overrightarrow{(w,s)}$, $i_s^{(u)^-} : X_s^- \to [0,1]$, defined by $i_s^{(u)^-}(w) = i_B^u\overrightarrow{(w,s)}$, $f_s^{(l)^-} : X_s^- \to [0,1]$, defined by $f_s^{(l)^-}(w) = f_B^l\overrightarrow{(w,s)}$, $f_s^{(u)^-} : X_s^- \to [0,1]$, defined by $f_s^{(u)^-}(w) = f_B^u\overrightarrow{(w,s)}$.

Example 2.7. Consider an IVN-digraph $G = (A, \overrightarrow{B})$ on $X = \{a, b, c\}$ as shown in Fig. 2.

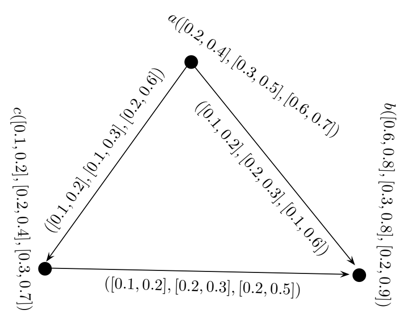

Figure 2. IVN-digraph

We have Table 1 and Table 2 representing interval-valued neutrosophic out and in-neighbourhoods, respectively.

Table 1. IVN-out-neighbourhoods

s	$\mathbb{N}^+(s)$
a	{(b, [0.1,0.2],[0.2,0.3],[0.1,0.6]), (c, [0.1,0.2],[0.1,0.3],[0.2,0.6])}
b	∅
c	{(b, [0.1,0.2],[0.2,0.3],[0.2,0.5])}

Table 2. IVN-in-neighbourhoods

s	$\mathbb{N}^-(s)$
a	∅
b	{(a, [0.1,0.2],[0.2,0.3],[0.1,0.6]), (c, [0.1,0.2],[0.2,0.3],[0.2,0.5])}
c	{(a, [0.1,0.2],[0.1,0.3],[0.2,0.6])}

Definition 2.8. The height of IVN-set $A = (s, [t_A^l, t_A^u], [i_A^l, i_A^u], [f_A^l, f_A^u])$ in universe of discourse X is defined as: for all $s \in X$,

$$h(A) = ([h_1^l(A), h_1^u(A)], [h_2^l(A), h_2^u(A)], [h_3^l(A), h_3^u(A)]),$$
$$= ([\sup_{s\in X} t_A^l(s), \sup_{s\in X} t_A^u(s)], [\sup_{s\in X} i_A^l(s), \sup_{s\in X} i_A^u(s)], [\inf_{s\in X} f_A^l(s), \inf_{s\in X} f_A^u(s)]).$$

Definition 2.9. An interval-valued neutrosophic competition graph (IVNC-graph) of an interval-valued neutrosophic graph (IVN-graph) $\overrightarrow{G} = (A, \overrightarrow{B})$ is an undirected IVN-graph $\mathbb{C}(\overrightarrow{G}) = (A, W)$ which has the same vertex set as in $\overrightarrow{G}$ and there is an edge between two vertices s and w if and only if $\mathbb{N}^+(s) \cap \mathbb{N}^+(w) \neq \varnothing$. The truth-membership, indeterminacy-membership and falsity-membership values of the edge (s, w) are defined as: for all $s, w \in X$,

(i) $t^l_W(s, w) = (t^l_A(s) \wedge t^l_A(w))h^l_1(\mathbb{N}^+(s) \cap \mathbb{N}^+(w)$,
$t^u_W(s, w) = (t^u_A(s) \wedge t^u_A(w))h^u_1(\mathbb{N}^+(s) \cap \mathbb{N}^+(w)$,
(ii) $i^l_W(s, w) = (i^l_A(s) \wedge i^l_A(w))h^l_2(\mathbb{N}^+(s) \cap \mathbb{N}^+(w)$,
$i^u_W(s, w) = (i^u_A(s) \wedge i^u_A(w))h^u_2(\mathbb{N}^+(s) \cap \mathbb{N}^+(w)$,
(iii) $f^l_W(s, w) = (f^l_A(s) \wedge f^l_A(w))h^l_3(\mathbb{N}^+(s) \cap \mathbb{N}^+(w)$,
$f^u_W(s, w) = (f^u_A(s) \wedge f^u_A(w))h^u_3(\mathbb{N}^+(s) \cap \mathbb{N}^+(w)$.

Example 2.10. Consider an IVN-digraph $G = (A, \overrightarrow{B})$ on $X = \{a, b, c\}$ as shown in Fig. 3.

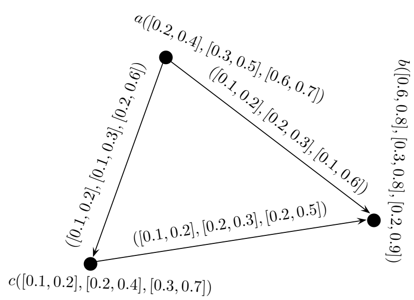

Figure 3. IVN-digraph

We have Table 3 and Table 4 representing interval-valued neutrosophic out and in-neighbourhoods, respectively.

Table 3. IVN-out-neighbourhoods

s	$\mathbb{N}^+(s)$
a	{(b, [0.1,0.2],[0.2,0.3],[0.1,0.6]), (c, [0.1,0.2],[0.1,0.3],[0.2,0.6])}
b	∅
c	{(b, [0.1,0.2],[0.2,0.3],[0.2,0.5])}

Table 4. IVN-in-neighbourhoods

s	$\mathbb{N}^-(s)$
a	∅
b	{(a, [0.1,0.2],[0.2,0.3],[0.1,0.6]), (c, [0.1,0.2],[0.2,0.3],[0.2,0.5])}
c	{(a, [0.1,0.2],[0.1,0.3],[0.2,0.6])}

Then IVNC-graph of Fig. 3 is shown in Fig. 4.

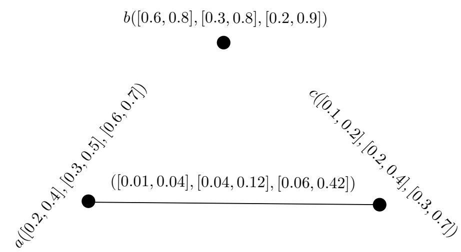

FIGURE 4. IVNC-graph

Definition 2.11. Consider an IVN-graph $G = (A,\, B)$, where $A = ([A_1^l,\, A_1^u],\, [A_2^l,\, A_2^u],\, [A_3^l,\, A_3^u)]$ and $B = ([B_1^l,\, B_1^u],\, [B_2^l,\, B_2^u],\, [B_3^l,\, B_3^u)]$. then, an edge $(s,\, w)$, $s,\, w \in X$ is called independent strong, if

$$\frac{1}{2}[A_1^l(s)\wedge A_1^l(w)] < B_1^l(s,w), \quad \frac{1}{2}[A_1^u(s)\wedge A_1^u(w)] < B_1^u(s,w),$$
$$\frac{1}{2}[A_2^l(s)\wedge A_2^l(w)] < B_2^l(s,w), \quad \frac{1}{2}[A_2^u(s)\wedge A_2^u(w)] < B_2^u(s,w),$$
$$\frac{1}{2}[A_3^l(s)\wedge A_3^l(w)] > B_3^l(s,w), \quad \frac{1}{2}[A_3^u(s)\wedge A_3^u(w)] > B_3^u(s,w).$$

Otherwise, it is called weak.

We state the following theorems without their proofs.

Theorem 2.12. *Suppose $\overrightarrow{G}$ is an IVN-digraph. If $\mathbb{N}^+(s)\cap\mathbb{N}^+(w)$ contains only one element of $\overrightarrow{G}$, then the edge $(s,\, w)$ of $\mathbb{C}(\overrightarrow{G})$ is independent strong if and only if*

$$|[\mathbb{N}^+(s)\cap\mathbb{N}^+(w)]|_{t^l} > 0.5, \quad |[\mathbb{N}^+(s)\cap\mathbb{N}^+(w)]|_{t^u} > 0.5,$$
$$|[\mathbb{N}^+(s)\cap\mathbb{N}^+(w)]|_{i^l} > 0.5, \quad |[\mathbb{N}^+(s)\cap\mathbb{N}^+(w)]|_{i^u} > 0.5,$$
$$|[\mathbb{N}^+(s)\cap\mathbb{N}^+(w)]|_{f^l} < 0.5, \quad |[\mathbb{N}^+(s)\cap\mathbb{N}^+(w)]|_{f^u} < 0.5.$$

Theorem 2.13. *If all the edges of an IVN-digraph $\overrightarrow{G}$ are independent strong, then*

$$\frac{B_1^l(s,w)}{(A_1^l(s)\wedge A_1^l(w))^2} > 0.5, \quad \frac{B_1^u(s,w)}{(A_1^u(s)\wedge A_1^u(w))^2} > 0.5,$$
$$\frac{B_2^l(s,w)}{(A_2^l(s)\wedge A_2^l(w))^2} > 0.5, \quad \frac{B_2^u(s,w)}{(A_2^u(s)\wedge A_2^u(w))^2} > 0.5,$$
$$\frac{B_3^l(s,w)}{(A_3^l(s)\wedge A_3^l(w))^2} < 0.5, \quad \frac{B_3^u(s,w)}{(A_3^u(s)\wedge A_3^u(w))^2} < 0.5,$$

for all edges $(s,\, w)$ in $\mathbb{C}(\overrightarrow{G})$.

Definition 2.14. The interval-valued neutrosophic open-neighbourhood (IVN-open-neighbourhood) of a vertex s of an IVN-graph $G = (A, B)$ is IVN-set $\mathbb{N}(s) = (X_s,\, [t_s^l,\, t_s^u],\, [i_s^l,\, i_s^u],\, [f_s^l,\, f_s^u])$, where

$$X_s = \{w|[B_1^l(s,w) > 0,\ B_1^u(s,w) > 0],\ [B_2^l(s,w) > 0,\ B_2^u(s,w) > 0],\ [B_3^l(s,w) > 0,\ B_3^u(s,w) > 0]\},$$

and $t_s^l : X_s \to [0,1]$ defined by $t_s^l(w) = B_1^l(s,\ w)$, $t_s^u : X_s \to [0,1]$ defined by $t_s^u(w) = B_1^u(s,\ w)$, $i_s^l : X_s \to [0,1]$ defined by $i_s^l(w) = B_2^l(s,\ w)$, $i_s^u : X_s \to [0,1]$ defined by $i_s^u(w) = B_2^u(s,\ w)$, $f_s^l : X_s \to [0,1]$ defined by $f_s^l(w) = B_3^l(s,\ w)$, $f_s^u : X_s \to [0,1]$ defined by $f_s^u(w) = B_3^u(s,\ w)$. For every vertex $s \in X$, the interval-valued neutrosophic singleton set, $\breve{A}_s = (s, [A_1^{l\prime}, A_1^{u\prime}], [A_2^{l\prime}, A_2^{u\prime}], [A_3^{l\prime}, A_3^{u\prime})$ such that: $A_1^{l\prime} : \{s\} \to [0,1]$, $A_1^{u\prime} : \{s\} \to [0,1]$, $A_2^{l\prime} : \{s\} \to [0,1]$, $A_2^{u\prime} : \{s\} \to [0,1]$, $A_3^{l\prime} : \{s\} \to [0,1]$, $A_3^{u\prime} : \{s\} \to [0,1]$, defined by $A_1^{l\prime}(s) = A_1^l(s)$, $A_1^{u\prime}(s) = A_1^u(s)$, $A_2^{l\prime}(s) = A_2^l(s)$, $A_2^{u\prime}(s) = A_2^u(s)$, $A_3^{l\prime}(s) = A_3^l(s)$ and $A_3^{u\prime}(s) = A_3^u(s)$, respectively. The interval-valued neutrosophic closed-neighbourhood (IVN-closed-neighbourhood) of a vertex s is $\mathbb{N}[s] = \mathbb{N}(s) \cup A_s$.

Definition 2.15. Suppose $G = (A,\ B)$ is an IVN-graph. Interval-valued neutrosophic open-neighbourhood graph (IVN-open-neighbourhood-graph) of G is an IVN-graph $\mathbb{N}(G) = (A,\ B')$ which has the same IVN-set of vertices in G and has an interval-valued neutrosophic edge between two vertices $s,\ w \in X$ in $\mathbb{N}(G)$ if and only if $\mathbb{N}(s) \cap \mathbb{N}(w)$ is a non-empty IVN-set in G. The truth-membership, indeterminacy-membership, falsity-membership values of the edge $(s,\ w)$ are given by:

$$\begin{aligned}
B_1^{l\prime}(s,w) &= [A_1^l(s) \wedge A_1^l(w)]h_1^l(\mathbb{N}(s) \cap \mathbb{N}(w)),\\
B_2^{l\prime}(s,w) &= [A_2^l(s) \wedge A_2^l(w)]h_2^l(\mathbb{N}(s) \cap \mathbb{N}(w)),\\
B_3^{l\prime}(s,w) &= [A_3^l(s) \wedge A_3^l(w)]h_3^l(\mathbb{N}(s) \cap \mathbb{N}(w)),\\
B_1^{u\prime}(s,w) &= [A_1^u(s) \wedge A_1^u(w)]h_1^u(\mathbb{N}(s) \cap \mathbb{N}(w)),\\
B_2^{u\prime}(s,w) &= [A_2^u(s) \wedge A_2^u(w)]h_2^u(\mathbb{N}(s) \cap \mathbb{N}(w)),\\
B_3^{u\prime}(s,w) &= [A_3^u(s) \wedge A_3^u(w)]h_3^u(\mathbb{N}(s) \cap \mathbb{N}(w)),\ \text{respectively.}
\end{aligned}$$

Definition 2.16. Suppose $G = (A,\ B)$ is an IVN-graph. Interval-valued neutrosophic closed-neighbourhood graph (IVN-closed-neighbourhood-graph) of G is an IVN-graph $\mathbb{N}(G) = (A,\ B')$ which has the same IVN-set of vertices in G and has an interval-valued neutrosophic edge between two vertices $s,\ w \in X$ in $\mathbb{N}[G]$ if and only if $\mathbb{N}[s] \cap \mathbb{N}[w]$ is a non-empty IVN-set in G. The truth-membership, indeterminacy-membership, falsity-membership values of the edge $(s,\ w)$ are given by:

$$\begin{aligned}
B_1^{l\prime}(s,w) &= [A_1^l(s) \wedge A_1^l(w)]h_1^l(\mathbb{N}[s] \cap \mathbb{N}[w]),\\
B_2^{l\prime}(s,w) &= [A_2^l(s) \wedge A_2^l(w)]h_2^l(\mathbb{N}[s] \cap \mathbb{N}[w]),\\
B_3^{l\prime}(s,w) &= [A_3^l(s) \wedge A_3^l(w)]h_3^l(\mathbb{N}[s] \cap \mathbb{N}[w]),\\
B_1^{u\prime}(s,w) &= [A_1^u(s) \wedge A_1^u(w)]h_1^u(\mathbb{N}[s] \cap \mathbb{N}[w]),\\
B_2^{u\prime}(s,w) &= [A_2^u(s) \wedge A_2^u(w)]h_2^u(\mathbb{N}[s] \cap \mathbb{N}[w]),\\
B_3^{u\prime}(s,w) &= [A_3^u(s) \wedge A_3^u(w)]h_3^u(\mathbb{N}[s] \cap \mathbb{N}[w]),\ \text{respectively.}
\end{aligned}$$

We now discuss the method of construction of interval-valued neutrospohic competition graph of the Cartesian product of IVN-digraph in following theorem which can be proof using similar method as used in [21], hence we omit its proof.

Theorem 2.17. *Let $\mathbb{C}(\overrightarrow{G_1}) = (A_1,\, B_1)$ and $\mathbb{C}(\overrightarrow{G_2}) = (A_2,\, B_2)$ be two IVNC-graphs of IVN-digraphs $\overrightarrow{G_1} = (A_1,\, \overrightarrow{L_1})$ and $\overrightarrow{G_2} = (A_2,\, \overrightarrow{L_2})$, respectively. Then $\mathbb{C}(\overrightarrow{G_1} \square \overrightarrow{G_2}) = G_{\mathbb{C}(\overrightarrow{G_1})^* \square \mathbb{C}(\overrightarrow{G_2})^*} \cup G^{\square}$, where $G_{\mathbb{C}(\overrightarrow{G_1})^* \square \mathbb{C}(\overrightarrow{G_2})^*}$ is an IVN-graph on the crisp graph $(X_1 \times X_2, E_{\mathbb{C}(\overrightarrow{G_1})^*} \square E_{\mathbb{C}(\overrightarrow{G_2})^*})$, $\mathbb{C}(\overrightarrow{G_1})^*$ and $\mathbb{C}(\overrightarrow{G_2})^*$ are the crisp competition graphs of $\overrightarrow{G_1}$ and $\overrightarrow{G_2}$, respectively. $G^{\square}$ is an IVN-graph on $(X_1 \times X_2, E^{\square})$ such that:*

(1) $E^{\square} = \{(s_1, s_2)(w_1, w_2) : w_1 \in \mathbb{N}^-(s_1)^*, w_2 \in \mathbb{N}^+(s_2)^*\}$
$E_{\mathbb{C}(\overrightarrow{G_1})^*} \square E_{\mathbb{C}(\overrightarrow{G_2})^*} = \{(s_1, s_2)(s_1, w_2) : s_1 \in X_1, s_2 w_2 \in E_{\mathbb{C}(\overrightarrow{G_2})^*}\}$
$\cup \{(s_1, s_2)(w_1, s_2) : s_2 \in X_2, s_1 w_1 \in E_{\mathbb{C}(\overrightarrow{G_1})^*}\}$.

(2) $t^l_{A_1 \square A_2} = t^l_{A_1}(s_1) \wedge t^l_{A_2}(s_2)$, $\quad i^l_{A_1 \square A_2} = i^l_{A_1}(s_1) \wedge i^l_{A_2}(s_2)$, $\quad f^l_{A_1 \square A_2} = f^l_{A_1}(s_1) \wedge f^l_{A_2}(s_2)$,
$t^u_{A_1 \square A_2} = t^u_{A_1}(s_1) \wedge t^u_{A_2}(s_2)$, $\quad i^u_{A_1 \square A_2} = i^u_{A_1}(s_1) \wedge i^u_{A_2}(s_2)$, $\quad f^u_{A_1 \square A_2} = f^u_{A_1}(s_1) \wedge f^u_{A_2}(s_2)$.

(3) $t^l_B((s_1, s_2)(s_1, w_2)) = [t^l_{A_1}(s_1) \wedge t^l_{A_2}(s_2) \wedge t^l_{A_2}(w_2)] \times \vee_{a_2} \{t^l_{A_1}(s_1) \wedge t^l_{\overrightarrow{L_2}}(s_2 a_2) \wedge t^l_{\overrightarrow{L_2}}(w_2 a_2)\}$,
$(s_1, s_2)(s_1, w_2) \in E_{\mathbb{C}(\overrightarrow{G_1})^*} \square E_{\mathbb{C}(\overrightarrow{G_2})^*}, \quad a_2 \in (\mathbb{N}^+(s_2) \cap \mathbb{N}^+(w_2))^*$.

(4) $i^l_B((s_1, s_2)(s_1, w_2)) = [i^l_{A_1}(s_1) \wedge i^l_{A_2}(s_2) \wedge i^l_{A_2}(w_2)] \times \vee_{a_2} \{i^l_{A_1}(s_1) \wedge i^l_{\overrightarrow{L_2}}(s_2 a_2) \wedge i^l_{\overrightarrow{L_2}}(w_2 a_2)\}$,
$(s_1, s_2)(s_1, w_2) \in E_{\mathbb{C}(\overrightarrow{G_1})^*} \square E_{\mathbb{C}(\overrightarrow{G_2})^*}, \quad a_2 \in (\mathbb{N}^+(s_2) \cap \mathbb{N}^+(w_2))^*$.

(5) $f^l_B((s_1, s_2)(s_1, w_2)) = [f^l_{A_1}(s_1) \wedge f^l_{A_2}(s_2) \wedge f^l_{A_2}(w_2)] \times \vee_{a_2} \{f^l_{A_1}(s_1) \wedge f^l_{\overrightarrow{L_2}}(s_2 a_2) \wedge f^l_{\overrightarrow{L_2}}(w_2 a_2)\}$,
$(s_1, s_2)(s_1, w_2) \in E_{\mathbb{C}(\overrightarrow{G_1})^*} \square E_{\mathbb{C}(\overrightarrow{G_2})^*}, \quad a_2 \in (\mathbb{N}^+(s_2) \cap \mathbb{N}^+(w_2))^*$.

(6) $t^u_B((s_1, s_2)(s_1, w_2)) = [t^u_{A_1}(s_1) \wedge t^u_{A_2}(s_2) \wedge t^u_{A_2}(w_2)] \times \vee_{a_2} \{t^u_{A_1}(s_1) \wedge t^u_{\overrightarrow{L_2}}(s_2 a_2) \wedge t^u_{\overrightarrow{L_2}}(w_2 a_2)\}$,
$(s_1, s_2)(s_1, w_2) \in E_{\mathbb{C}(\overrightarrow{G_1})^*} \square E_{\mathbb{C}(\overrightarrow{G_2})^*}, \quad a_2 \in (\mathbb{N}^+(s_2) \cap \mathbb{N}^+(w_2))^*$.

(7) $i^u_B((s_1, s_2)(s_1, w_2)) = [i^u_{A_1}(s_1) \wedge i^u_{A_2}(s_2) \wedge i^u_{A_2}(w_2)] \times \vee_{a_2} \{i^u_{A_1}(s_1) \wedge i^u_{\overrightarrow{L_2}}(s_2 a_2) \wedge i^u_{\overrightarrow{L_2}}(w_2 a_2)\}$,
$(s_1, s_2)(s_1, w_2) \in E_{\mathbb{C}(\overrightarrow{G_1})^*} \square E_{\mathbb{C}(\overrightarrow{G_2})^*}, \quad a_2 \in (\mathbb{N}^+(s_2) \cap \mathbb{N}^+(w_2))^*$.

(8) $f^u_B((s_1, s_2)(s_1, w_2)) = [f^u_{A_1}(s_1) \wedge f^u_{A_2}(s_2) \wedge f^u_{A_2}(w_2)] \times \vee_{a_2} \{f^u_{A_1}(s_1) \wedge f^u_{\overrightarrow{L_2}}(s_2 a_2) \wedge f^u_{\overrightarrow{L_2}}(w_2 a_2)\}$,
$(s_1, s_2)(s_1, w_2) \in E_{\mathbb{C}(\overrightarrow{G_1})^*} \square E_{\mathbb{C}(\overrightarrow{G_2})^*}, \quad a_2 \in (\mathbb{N}^+(s_2) \cap \mathbb{N}^+(w_2))^*$.

(9) $t^l_B((s_1, s_2)(w_1, s_2)) = [t^l_{A_1}(s_1) \wedge t^l_{A_1}(w_1) \wedge t^l_{A_2}(s_2)] \times \vee_{a_1} \{t^l_{A_2}(s_2) \wedge t^l_{\overrightarrow{L_1}}(s_1 a_1) \wedge t^l_{\overrightarrow{L_1}}(w_1 a_1)\}$,
$(s_1, s_2)(w_1, s_2) \in E_{\mathbb{C}(\overrightarrow{G_1})^*} \square E_{\mathbb{C}(\overrightarrow{G_2})^*}, \quad a_1 \in (\mathbb{N}^+(s_1) \cap \mathbb{N}^+(w_1))^*$.

(10) $i^l_B((s_1, s_2)(w_1, s_2)) = [i^l_{A_1}(s_1) \wedge i^l_{A_1}(w_1) \wedge i^l_{A_2}(s_2)] \times \vee_{a_1} \{i^l_{A_2}(s_2) \wedge i^l_{\overrightarrow{L_1}}(s_1 a_1) \wedge i^l_{\overrightarrow{L_1}}(w_1 a_1)\}$,
$(s_1, s_2)(w_1, s_2) \in E_{\mathbb{C}(\overrightarrow{G_1})^*} \square E_{\mathbb{C}(\overrightarrow{G_2})^*}, \quad a_1 \in (\mathbb{N}^+(s_1) \cap \mathbb{N}^+(w_1))^*$.

(11) $f^l_B((s_1,s_2)(w_1,s_2)) = [f^l_{A_1}(s_1)\wedge f^l_{A_1}(w_1)\wedge f^l_{A_2}(s_2)]\times\vee_{a_1}\{t^l_{A_2}(s_2)\wedge f^l_{\overrightarrow{L_1}}(s_1a_1)\wedge f^l_{\overrightarrow{L_1}}(w_1a_1)\}$,
$(s_1,s_2)(w_1,s_2)\in E_{\mathbb{C}(\overrightarrow{G_1})^*}\Box E_{\mathbb{C}(\overrightarrow{G_2})^*},\quad a_1\in(\mathbb{N}^+(s_1)\cap\mathbb{N}^+(w_1))^*$.

(12) $t^u_B((s_1,s_2)(w_1,s_2)) = [t^u_{A_1}(s_1)\wedge t^u_{A_1}(w_1)\wedge t^u_{A_2}(s_2)]\times\vee_{a_1}\{t^u_{A_2}(s_2)\wedge t^u_{\overrightarrow{L_1}}(s_1a_1)\wedge t^u_{\overrightarrow{L_1}}(w_1a_1)\}$,
$(s_1,s_2)(w_1,s_2)\in E_{\mathbb{C}(\overrightarrow{G_1})^*}\Box E_{\mathbb{C}(\overrightarrow{G_2})^*},\quad a_1\in(\mathbb{N}^+(s_1)\cap\mathbb{N}^+(w_1))^*$.

(13) $i^u_B((s_1,s_2)(w_1,s_2)) = [i^u_{A_1}(s_1)\wedge i^u_{A_1}(w_1)\wedge i^u_{A_2}(s_2)]\times\vee_{a_1}\{i^u_{A_2}(s_2)\wedge i^u_{\overrightarrow{L_1}}(s_1a_1)\wedge i^u_{\overrightarrow{L_1}}(w_1a_1)\}$,
$(s_1,s_2)(w_1,s_2)\in E_{\mathbb{C}(\overrightarrow{G_1})^*}\Box E_{\mathbb{C}(\overrightarrow{G_2})^*},\quad a_1\in(\mathbb{N}^+(s_1)\cap\mathbb{N}^+(w_1))^*$.

(14) $f^u_B((s_1,s_2)(w_1,s_2)) = [f^u_{A_1}(s_1)\wedge f^u_{A_1}(w_1)\wedge f^u_{A_2}(s_2)]\times\vee_{a_1}\{t^u_{A_2}(s_2)\wedge f^u_{\overrightarrow{L_1}}(s_1a_1)\wedge f^u_{\overrightarrow{L_1}}(w_1a_1)\}$,
$(s_1,s_2)(w_1,s_2)\in E_{\mathbb{C}(\overrightarrow{G_1})^*}\Box E_{\mathbb{C}(\overrightarrow{G_2})^*},\quad a_1\in(\mathbb{N}^+(s_1)\cap\mathbb{N}^+(w_1))^*$.

(15) $t^l_B((s_1,s_2)(w_1,w_2)) = [t^l_{A_1}(s_1)\wedge t^l_{A_1}(w_1)\wedge t^l_{A_2}(s_2)\wedge t^l_{A_2}(w_2)]\times[t^l_{A_1}(s_1)\wedge t^l_{\overrightarrow{L_1}}(w_1s_1)\wedge t^l_{A_2}(w_2)\wedge t^l_{\overrightarrow{L_2}}(s_2w_2)]$,
$(s_1,w_1)(s_2,w_2)\in E^{\Box}$.

(16) $i^l_B((s_1,s_2)(w_1,w_2)) = [i^l_{A_1}(s_1)\wedge i^l_{A_1}(w_1)\wedge i^l_{A_2}(s_2)\wedge i^l_{A_2}(w_2)]\times[i^l_{A_1}(s_1)\wedge i^l_{\overrightarrow{L_1}}(w_1s_1)\wedge i^l_{A_2}(w_2)\wedge i^l_{\overrightarrow{L_2}}(s_2w_2)]$,
$(s_1,w_1)(s_2,w_2)\in E^{\Box}$.

(17) $f^l_B((s_1,s_2)(w_1,w_2)) = [f^l_{A_1}(s_1)\wedge f^l_{A_1}(w_1)\wedge f^l_{A_2}(s_2)\wedge f^l_{A_2}(w_2)]\times[f^l_{A_1}(s_1)\wedge f^l_{\overrightarrow{L_1}}(w_1s_1)\wedge f^l_{A_2}(w_2)\wedge f^l_{\overrightarrow{L_2}}(s_2w_2)]$,
$(s_1,w_1)(s_2,w_2)\in E^{\Box}$.

(18) $t^u_B((s_1,s_2)(w_1,w_2)) = [t^u_{A_1}(s_1)\wedge t^u_{A_1}(w_1)\wedge t^u_{A_2}(s_2)\wedge t^u_{A_2}(w_2)]\times[t^u_{A_1}(s_1)\wedge t^u_{\overrightarrow{L_1}}(w_1s_1)\wedge t^u_{A_2}(w_2)\wedge t^u_{\overrightarrow{L_2}}(s_2w_2)]$,
$(s_1,w_1)(s_2,w_2)\in E^{\Box}$.

(19) $i^u_B((s_1,s_2)(w_1,w_2)) = [i^u_{A_1}(s_1)\wedge i^u_{A_1}(w_1)\wedge i^u_{A_2}(s_2)\wedge i^u_{A_2}(w_2)]\times[i^u_{A_1}(s_1)\wedge i^u_{\overrightarrow{L_1}}(w_1s_1)\wedge i^u_{A_2}(w_2)\wedge i^u_{\overrightarrow{L_2}}(s_2w_2)]$,
$(s_1,w_1)(s_2,w_2)\in E^{\Box}$.

(20) $f^u_B((s_1,s_2)(w_1,w_2)) = [f^u_{A_1}(s_1)\wedge f^u_{A_1}(w_1)\wedge f^u_{A_2}(s_2)\wedge f^u_{A_2}(w_2)]\times[f^u_{A_1}(s_1)\wedge f^u_{\overrightarrow{L_1}}(w_1s_1)\wedge f^u_{A_2}(w_2)\wedge f^u_{\overrightarrow{L_2}}(s_2w_2)]$,
$(s_1,w_1)(s_2,w_2)\in E^{\Box}$.

A. k-competition interval-valued neutrosophic graphs

We now discuss an extension of IVNC-graphs, called k-competition IVN-graphs.

Definition 2.18. The cardinality of an IVN-set A is denoted by

$$|A| = \big([|A|_{t^l},\, |A|_{t^u}],\ [|A|_{i^l},\, |A|_{i^u}],\ [|A|_{f^l},\, |A|_{f^u}]\big).$$

Where $[|A|_{t^l},\, |A|_{t^u}]$, $[|A|_{i^l},\, |A|_{i^u}]$ and $[|A|_{f^l},\, |A|_{f^u}]$ represent the sum of truth-membership values, indeterminacy-membership values and falsity-membership values, respectively, of all the elements of A.

Example 2.19. The cardinality of an IVN-set $A = \{(a, [0.5, 0.7], [0.2, 0.8], [0.1, 0.3]), (b, [0.1, 0.2], [0.1, 0.5], [0.7, 0.9]), (c, [0.3, 0.5], [0.3, 0.8], [0.6, 0.9])\}$ in $X = \{a, b, c\}$ is

$$|A| = \big([|A|_{t^l}, |A|_{t^u}], [|A|_{i^l}, |A|_{i^u}], [|A|_{f^l}, |A|_{f^u}]\big)$$
$$= ([0.9, 1.4], [0.6, 2.1], [1.4, 2.1]).$$

We now discuss k-competition IVN-graphs.

Definition 2.20. Let k be a non-negative number. Then k-competition IVN-graph $\mathbb{C}_k(\overrightarrow{G})$ of an IVN-digraph $\overrightarrow{G} = (A, \overrightarrow{B})$ is an undirected IVN-graph $G = (A, B)$ which has same IVN-set of vertices as in $\overrightarrow{G}$ and has an interval-valued neutrosophic edge between two vertices $s, w \in X$ in $\mathbb{C}_k(\overrightarrow{G})$ if and only if $|(\mathbb{N}^+(s) \cap \mathbb{N}^+(w))|_{t^l} > k$, $|(\mathbb{N}^+(s) \cap \mathbb{N}^+(w))|_{t^u} > k$, $|(\mathbb{N}^+(s) \cap \mathbb{N}^+(w))|_{i^l} > k$, $|(\mathbb{N}^+(s) \cap \mathbb{N}^+(w))|_{i^u} > k$, $|(\mathbb{N}^+(s) \cap \mathbb{N}^+(w))|_{f^l} > k$ and $|(\mathbb{N}^+(s) \cap \mathbb{N}^+(w))|_{f^u} > k$. The interval-valued truth-membership value of edge (s, w) in $\mathbb{C}_k(\overrightarrow{G})$ is $t^l_B(s, w) = \frac{k^l_1-k}{k^l_1}[t^l_A(s) \wedge t^l_A(w)]h^l_1(\mathbb{N}^+(s) \cap \mathbb{N}^+(w))$, where $k^l_1 = |(\mathbb{N}^+(s) \cap \mathbb{N}^+(w))|_{t^l}$ and $t^u_B(s, w) = \frac{k^u_1-k}{k^u_1}[t^u_A(s) \wedge t^u_A(w)]h^u_1(\mathbb{N}^+(s) \cap \mathbb{N}^+(w))$, where $k^u_1 = |(\mathbb{N}^+(s) \cap \mathbb{N}^+(w))|_{t^u}$, the interval-valued indeterminacy-membership value of edge (s, w) in $\mathbb{C}_k(\overrightarrow{G})$ is $i^l_B(s, w) = \frac{k^l_2-k}{k^l_2}[i^l_A(s) \wedge i^l_A(w)]h^l_2(\mathbb{N}^+(s) \cap \mathbb{N}^+(w))$, where $k^l_2 = |(\mathbb{N}^+(s) \cap \mathbb{N}^+(w))|_{i^l}$, and $i^u_B(s, w) = \frac{k^u_2-k}{k^u_2}[i^u_A(s) \wedge i^u_A(w)]h^u_2(\mathbb{N}^+(s) \cap \mathbb{N}^+(w))$, where $k^u_2 = |(\mathbb{N}^+(s) \cap \mathbb{N}^+(w))|_{i^u}$, the interval-valued falsity-membership value of edge (s, w) in $\mathbb{C}_k(\overrightarrow{G})$ is $f^l_B(s, w) = \frac{k^l_3-k}{k^l_3}[f^l_A(s) \wedge f^l_A(w)]h^l_3(\mathbb{N}^+(s) \cap \mathbb{N}^+(w))$, where $k^l_3 = |(\mathbb{N}^+(s) \cap \mathbb{N}^+(w))|_{f^l}$, and $f^u_B(s, w) = \frac{k^u_3-k}{k^u_3}[f^u_A(s) \wedge f^u_A(w)]h^u_3(\mathbb{N}^+(s) \cap \mathbb{N}^+(w))$, where $k^u_3 = |(\mathbb{N}^+(s) \cap \mathbb{N}^+(w))|_{f^u}$.

Example 2.21. Consider an IVN-digraph $G = (A, \overrightarrow{B})$ on $X = \{s, w, a, b, c\}$, such that $A = \{(s, [0.4, 0.5], [0.5, 0.7], [0.8, 0.9]), (w, [0.6, 0.7], [0.4, 0.6], [0.2, 0.3]), (a, [0.2, 0.6], [0.3, 0.6], [0.2, 0.6]), (b, [0.2, 0.6], [0.1, 0.6], [0.2, 0.6]), (c, [0.2, 0.7], [0.3, 0.5], [0.2, 0.6])\}$, and $B = \{(\overrightarrow{(s, a)}, [0.1, 0.4], [0.3, 0.6], [0.2, 0.6]), (\overrightarrow{(s, b)}, [0.2, 0.4], [0.1, 0.5], [0.2, 0.6]), (\overrightarrow{(s, c)}, [0.2, 0.5], [0.3, 0.5], [0.2, 0.6]), (\overrightarrow{(w, a)}, [0.2, 0.5], [0.2, 0.5], [0.2, 0.3]), (\overrightarrow{(w, b)}, [0.2, 0.6], [0.1, 0.6], [0.2, 0.3]), (\overrightarrow{(w, c)}, [0.2, 0.7], [0.3, 0.5], [0.2, 0.3])\}$, as shown in Fig. 5.

We calculate $\mathbb{N}^+(s) = \{(a, [0.1, 0.4], [0.3, 0.6], [0.2, 0.6]), (b, [0.2, 0.4], [0.1, 0.5], [0.2, 0.6]), (c, [0.2, 0.5], [0.3, 0.5], [0.2, 0.6])\}$ and $\mathbb{N}^+(w) = \{(a, [0.2, 0.5], [0.2, 0.5], [0.2, 0.3]), (b, [0.2, 0.6], [0.1, 0.6], [0.2, 0.3]), (c, [0.2, 0.7], [0.3, 0.5], [0.2, 0.3])\}$. Therefore, $\mathbb{N}^+(s) \cap \mathbb{N}^+(w) = \{(a, [0.1, 0.4], [0.2, 0.5], [0.2, 0.3]), (b, [0.2, 0.4], [0.1, 0.5], [0.2, 0.3]), (c, [0.2, 0.5], [0.3, 0.5], [0.2, 0.3)\}$. So, $k^l_1 = 0.5$, $k^u_1 = 1.3$, $k^l_2 = 0.6$, $k^u_2 = 1.5$, $k^l_3 = 0.6$ and $k^u_3 = 0.9$. Let $k = 0.4$, then, $t^l_B(s, w) = 0.02$, $t^u_B(s, w) = 0.56$, $i^l_B(s, w) = 0.06$, $i^u_B(s, w) = 0.82$, $f^l_B(s, w) = 0.02$ and $f^u_B(s, w) = 0.11$. This graph is depicted in Fig. 6.

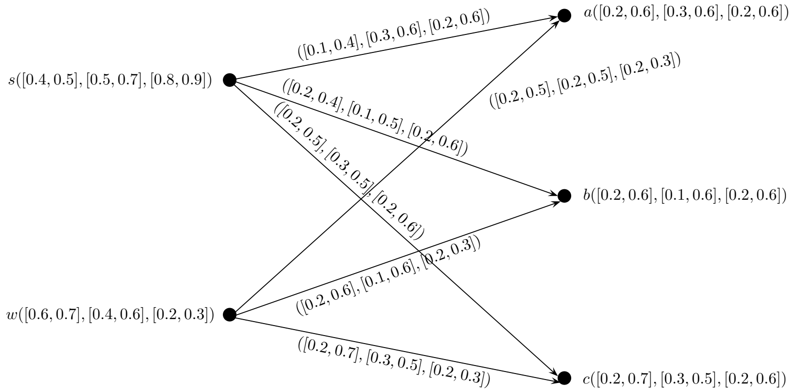

Figure 5. IVN-digraph

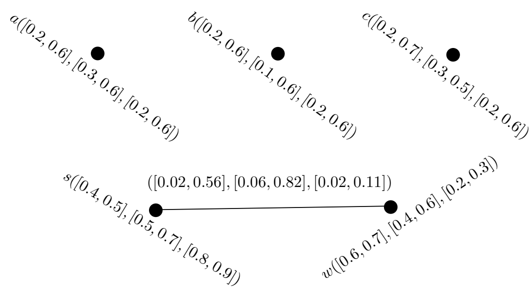

Figure 6. 0.4-Competition IVN-graph

Theorem 2.22. *Let* $\overrightarrow{G} = (A,\ \overrightarrow{B})$ *be an IVN-digraph. If*

$$h_1^l(\mathbb{N}^+(s) \cap \mathbb{N}^+(w)) = 1, \quad h_2^l(\mathbb{N}^+(s) \cap \mathbb{N}^+(w)) = 1, \quad h_3^l(\mathbb{N}^+(s) \cap \mathbb{N}^+(w)) = 1,$$
$$h_1^u(\mathbb{N}^+(s) \cap \mathbb{N}^+(w)) = 1, \quad h_2^u(\mathbb{N}^+(s) \cap \mathbb{N}^+(w)) = 1, \quad h_3^u(\mathbb{N}^+(s) \cap \mathbb{N}^+(w)) = 1,$$

and

$$|(\mathbb{N}^+(s) \cap \mathbb{N}^+(w))|_{t^l} > 2k, \quad |(\mathbb{N}^+(s) \cap \mathbb{N}^+(w))|_{i^l} > 2k, \quad |(\mathbb{N}^+(s) \cap \mathbb{N}^+(w))|_{f^l} < 2k,$$
$$|(\mathbb{N}^+(s) \cap \mathbb{N}^+(w))|_{t^u} > 2k, \quad |(\mathbb{N}^+(s) \cap \mathbb{N}^+(w))|_{i^u} > 2k, \quad |(\mathbb{N}^+(s) \cap \mathbb{N}^+(w))|_{f^u} < 2k,$$

Then the edge (s, w) *is independent strong in* $\mathbb{C}_k(\overrightarrow{G})$*.*

Proof. Let $\overrightarrow{G} = (A,\ \overrightarrow{B})$ be an IVN-digraph. Let $\mathbb{C}_k(\overrightarrow{G})$ be the corresponding k-competition IVN-graph.

If $h_1^l(\mathbb{N}^+(s)\cap\mathbb{N}^+(w))=1$ and $|(\mathbb{N}^+(s)\cap\mathbb{N}^+(w))|_{t^l}>2k$, then $k_1^l>2k$. Thus,

$$\begin{aligned}
t_B^l(s,w) &= \frac{k_1^l-k}{k_1^l}[t_A^l(s)\wedge t_A^l(w)]h_1^l(\mathbb{N}^+(s)\cap\mathbb{N}^+(w))\\
\text{or,}\quad t_B^l(s,w) &= \frac{k_1^l-k}{k_1^l}[t_A^l(s)\wedge t_A^l(w)]\\
\frac{t_B^l(s,w)}{[t_A^l(s)\wedge t_A^l(w)]} &= \frac{k_1^l-k}{k_1^l}>0.5.
\end{aligned}$$

If $h_1^u(\mathbb{N}^+(s)\cap\mathbb{N}^+(w))=1$ and $|(\mathbb{N}^+(s)\cap\mathbb{N}^+(w))|_{t^u}>2k$, then $k_1^u>2k$. Thus,

$$\begin{aligned}
t_B^u(s,w) &= \frac{k_1^u-k}{k_1^u}[t_A^u(s)\wedge t_A^u(w)]h_1^u(\mathbb{N}^+(s)\cap\mathbb{N}^+(w))\\
\text{or,}\quad t_B^u(s,w) &= \frac{k_1^u-k}{k_1^u}[t_A^u(s)\wedge t_A^u(w)]\\
\frac{t_B^u(s,w)}{[t_A^u(s)\wedge t_A^u(w)]} &= \frac{k_1^u-k}{k_1^u}>0.5.
\end{aligned}$$

If $h_2^l(\mathbb{N}^+(s)\cap\mathbb{N}^+(w))=1$ and $|(\mathbb{N}^+(s)\cap\mathbb{N}^+(w))|_{i^l}>2k$, then $k_2^l>2k$. Thus,

$$\begin{aligned}
i_B^l(s,w) &= \frac{k_2^l-k}{k_2^l}[i_A^l(s)\wedge i_A^l(w)]h_2^l(\mathbb{N}^+(s)\cap\mathbb{N}^+(w))\\
\text{or,}\quad i_B^l(s,w) &= \frac{k_2^l-k}{k_2^l}[i_A^l(s)\wedge i_A^l(w)]\\
\frac{i_B^l(s,w)}{[i_A^l(s)\wedge i_A^l(w)]} &= \frac{k_2^l-k}{k_2^l}>0.5.
\end{aligned}$$

If $h_2^u(\mathbb{N}^+(s)\cap\mathbb{N}^+(w))=1$ and $|(\mathbb{N}^+(s)\cap\mathbb{N}^+(w))|_{i^u}>2k$, then $k_2^u>2k$. Thus,

$$\begin{aligned}
i_B^u(s,w) &= \frac{k_2^u-k}{k_2^u}[i_A^u(s)\wedge i_A^u(w)]h_2^u(\mathbb{N}^+(s)\cap\mathbb{N}^+(w))\\
\text{or,}\quad i_B^u(s,w) &= \frac{k_2^u-k}{k_2^u}[i_A^u(s)\wedge i_A^u(w)]\\
\frac{i_B^u(s,w)}{[i_A^u(s)\wedge i_A^u(w)]} &= \frac{k_2^u-k}{k_2^u}>0.5.
\end{aligned}$$

If $h_3^l(\mathbb{N}^+(s)\cap\mathbb{N}^+(w))=1$ and $|(\mathbb{N}^+(s)\cap\mathbb{N}^+(w))|_{f^l}<2k$, then $k_3^l<2k$. Thus,

$$\begin{aligned}
f_B^l(s,w) &= \frac{k_3^l-k}{k_3^l}[f_A^l(s)\wedge f_A^l(w)]h_3^l(\mathbb{N}^+(s)\cap\mathbb{N}^+(w))\\
\text{or,}\quad f_B^l(s,w) &= \frac{k_3^l-k}{k_3^l}[f_A^l(s)\wedge f_A^l(w)]\\
\frac{f_B^l(s,w)}{[f_A^l(s)\wedge f_A^l(w)]} &= \frac{k_3^l-k}{k_3^l}<0.5.
\end{aligned}$$

If $h_3^u(\mathbb{N}^+(s)\cap\mathbb{N}^+(w))=1$ and $|(\mathbb{N}^+(s)\cap\mathbb{N}^+(w))|_{f^u}<2k$, then $k_3^u<2k$. Thus,

$$f_B^u(s,w)=\frac{k_3^u-k}{k_3^u}[f_A^u(s)\wedge f_A^u(w)]h_3^u(\mathbb{N}^+(s)\cap\mathbb{N}^+(w))$$

$$\text{or,}\quad f_B^u(s,w)=\frac{k_3^u-k}{k_3^u}[f_A^u(s)\wedge f_A^u(w)]$$

$$\frac{f_B^u(s,w)}{[f_A^u(s)\wedge f_A^u(w)]}=\frac{k_3^u-k}{k_3^u}<0.5.$$

So, the edge (s,w) is independent strong in $\mathbb{C}_k(\overrightarrow{G})$. □

B. p-competition interval-valued neutrosophic graphs

We now define another extension of IVNC-graphs, called p-competition IVN-graphs.

Definition 2.23. The support of an IVN-set $A=(s,\,[t_A^l,\,t_A^u],\,[i_A^l,\,i_A^u],\,[f_A^l,\,f_A^u])$ in X is the subset of X defined by

$$supp(A)=\{s\in X:\,[t_A^l(s)\neq 0,\,t_A^u(s)\neq 0],\,[i_A^l(s)\neq 0,\,i_A^u(s)\neq 0],\,[f_A^l(s)\neq 1,\, f_A^u(s)\neq 1]\}$$

and $|supp(A)|$ is the number of elements in the set.

Example 2.24. The support of an IVN-set $A=\{(a,\,[0.5,\,0.7],\,[0.2,\,0.8],[0.1,\,0.3]),\,(b,\,[0.1,\,0.2],\,[0.1,\,0.5],\,[0.7,\,0.9]),\,(c,\,[0.3,\,0.5],\,[0.3,\,0.8],\,[0.6,\,0.9]),\,(d,\,[0,\,0],\,[0,\,0],\,[1,\,1])\}$ in $X=\{a,\,b,\,c,\,d\}$ is $supp(A)=\{a,b,c\}$ and $|supp(A)|=3$.

We now define p-competition IVN-graphs.

Definition 2.25. Let p be a positive integer. Then p-competition IVN-graph $\mathbb{C}^p(\overrightarrow{G})$ of the IVN-digraph $\overrightarrow{G}=(A,\,\overrightarrow{B})$ is an undirected IVN-graph $G=(A,\,B)$ which has same IVN-set of vertices as in $\overrightarrow{G}$ and has an interval-valued neutrosophic edge between two vertices $s,\,w\in X$ in $\mathbb{C}^p(\overrightarrow{G})$ if and only if $|supp(\mathbb{N}^+(s)\cap\mathbb{N}^+(w))|\geq p$. The interval-valued truth-membership value of edge $(s,\,w)$ in $\mathbb{C}^p(\overrightarrow{G})$ is $t_B^l(s,\,w)=\frac{(i-p)+1}{i}[t_A^l(s)\wedge t_A^l(w)]h_1^l(\mathbb{N}^+(s)\cap\mathbb{N}^+(w))$, and $t_B^u(s,\,w)=\frac{(i-p)+1}{i}[t_A^u(s)\wedge t_A^u(w)]h_1^u(\mathbb{N}^+(s)\cap\mathbb{N}^+(w))$, the interval-valued indeterminacy-membership value of edge $(s,\,w)$ in $\mathbb{C}^p(\overrightarrow{G})$ is $i_B^l(s,\,w)=\frac{(i-p)+1}{i}[i_A^l(s)\wedge i_A^l(w)]h_2^l(\mathbb{N}^+(s)\cap\mathbb{N}^+(w))$, and $i_B^u(s,\,w)=\frac{(i-p)+1}{i}[i_A^u(s)\wedge i_A^u(w)]h_2^u(\mathbb{N}^+(s)\cap\mathbb{N}^+(w))$, the interval-valued falsity-membership value of edge $(s,\,w)$ in $\mathbb{C}^p(\overrightarrow{G})$ is $f_B^l(s,\,w)=\frac{(i-p)+1}{i}[f_A^l(s)\wedge f_A^l(w)]h_3^l(\mathbb{N}^+(s)\cap\mathbb{N}^+(w))$, and $f_B^u(s,\,w)=\frac{(i-p)+1}{i}[f_A^u(s)\wedge f_A^u(w)]h_3^u(\mathbb{N}^+(s)\cap\mathbb{N}^+(w))$, where $i=|supp(\mathbb{N}^+(s)\cap\mathbb{N}^+(w))|$.

Example 2.26. Consider an IVN-digraph $G=(A,\,\overrightarrow{B})$ on $X=\{s,w,a,b,c\}$, such that $A=\{(s,\,[0.4,0.5],\,[0.5,0.7],\,[0.8,0.9]),\,(w,\,[0.6,0.7],\,[0.4,0.6],\,[0.2,0.3]),\,(a,\,[0.2,0.6],\,[0.3,0.6],\,[0.2,0.6]),\,(b,\,[0.2,0.6],\,[0.1,0.6],\,[0.2,0.6]),\,(c,\,[0.2,0.7],\,[0.3,0.5],\,[0.2,0.6])\}$, and $B=\{(\overrightarrow{(s,a)},\,[0.1,0.4],\,[0.3,0.6],\,[0.2,0.6]),\,(\overrightarrow{(s,b)},\,[0.2,0.4],\,[0.1,0.5],\,[0.2,0.6]),\,(\overrightarrow{(s,c)},\,[0.2,0.5],\,[0.3,0.5],\,[0.2,0.6]),\,(\overrightarrow{(w,a)},\,[0.2,0.5],\,[0.2,0.5],\,[0.2,0.3]),\,(\overrightarrow{(w,b)},\,[0.2,0.6],\,[0.1,0.6],\,[0.2,0.3]),\,(\overrightarrow{(w,c)},\,[0.2,0.7],\,[0.3,0.5],\,[0.2,0.3])\}$, as shown in Fig. 7.

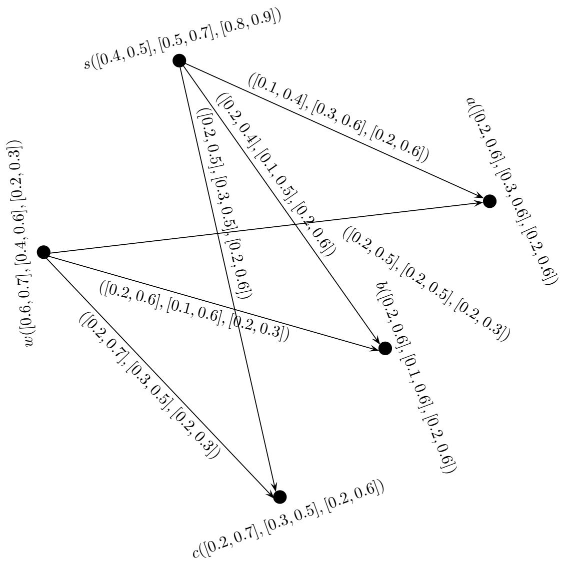

FIGURE 7. IVN-digraph

We calculate $\mathbb{N}^+(s) = \{(a,\ [0.1, 0.4],\ [0.3, 0.6],\ [0.2, 0.6]),\ (b,\ [0.2, 0.4],\ [0.1, 0.5],\ [0.2, 0.6]),\ (c,\ [0.2, 0.5],\ [0.3, 0.5],\ [0.2, 0.6])\}$ and $\mathbb{N}^+(w) = \{(a,\ [0.2, 0.5],\ [0.2, 0.5],\ [0.2, 0.3]),\ (b,\ [0.2, 0.6],\ [0.1, 0.6],\ [0.2, 0.3]),\ (c,\ [0.2, 0.7],\ [0.3, 0.5],\ [0.2, 0.3])\}$. Therefore, $\mathbb{N}^+(s) \cap \mathbb{N}^+(w) = \{(a,\ [0.1, 0.4],\ [0.2, 0.5],\ [0.2, 0.3]),\ (b,\ [0.2, 0.4],\ [0.1, 0.5],\ [0.2, 0.3]),\ (c,\ [0.2, 0.5],\ [0.3, 0.5],\ [0.2, 0.3)\}$. Now, $i = |supp(\mathbb{N}^+(s) \cap \mathbb{N}^+(w))| = 3$. For $p = 3$, we have, $t^l_B(s,\ w) = 0.02$, $t^u_B(s,\ w) = 0.08$, $i^l_B(s,\ w) = 0.04$, $i^u_B(s,\ w) = 0.1$, $f^l_B(s,\ w) = 0.01$ and $f^u_B(s,\ w) = 0.03$. This graph is depicted in Fig. 8.

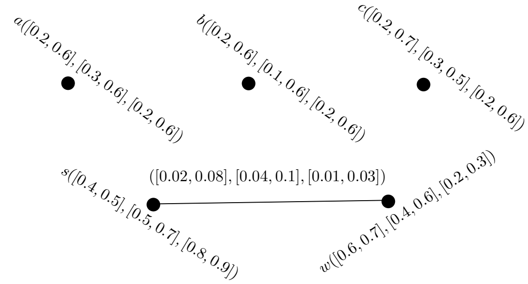

FIGURE 8. 3-Competition IVN-graph

We state the following theorem without its proof.

Theorem 2.27. *Let* $\overrightarrow{G} = (A,\ \overrightarrow{B})$ *be an IVN-digraph. If*

$$h^l_1(\mathbb{N}^+(s) \cap \mathbb{N}^+(w)) = 1, \quad h^l_2(\mathbb{N}^+(s) \cap \mathbb{N}^+(w)) = 1, \quad h^l_3(\mathbb{N}^+(s) \cap \mathbb{N}^+(w)) = 0,$$
$$h^u_1(\mathbb{N}^+(s) \cap \mathbb{N}^+(w)) = 1, \quad h^u_2(\mathbb{N}^+(s) \cap \mathbb{N}^+(w)) = 1, \quad h^u_3(\mathbb{N}^+(s) \cap \mathbb{N}^+(w)) = 0,$$

in $\mathbb{C}^{[\frac{i}{2}]}(\overrightarrow{G})$*, then the edge* (s,w) *is strong, where* $i = |supp(\mathbb{N}^+(s) \cap \mathbb{N}^+(w))|$*. (Note that for any real number* s*,* $[s]$*=greatest integer not esceeding* s*.)*

C. m**-step interval-valued neutrosophic competition graphs**

We now define another extension of IVNC-graph known as m-step IVNC-graph. We will use the following notations:
$P^m_{s,w}$: An interval-valued neutrosophic path of length m from s to w.
$\overrightarrow{P}^m_{s,w}$: A directed interval-valued neutrosophic path of length m from s to w.
$\mathbb{N}^+_m(s)$: m-step interval-valued neutrosophic out-neighbourhood of vertex s.
$\mathbb{N}^-_m(s)$: m-step interval-valued neutrosophic in-neighbourhood of vertex s.
$\mathbb{N}_m(s)$: m-step interval-valued neutrosophic neighbourhood of vertex s.
$\mathbb{N}_m(G)$: m-step interval-valued neutrosophic neighbourhood graph of the IVN-graph G.
$\mathbb{C}_m(\overrightarrow{G})$: m-step IVNC-graph of the IVN-digraph $\overrightarrow{G}$.

Definition 2.28. Suppose $\overrightarrow{G} = (A,\ \overrightarrow{B})$ is an IVN-digraph. The m-step IVN-digraph of $\overrightarrow{G}$ is denoted by $\overrightarrow{G}_m = (A,\ B)$, where IVN-set of vertices of $\overrightarrow{G}$ is same with IVN-set of vertices of $\overrightarrow{G}_m$ and has an edge between s and w in $\overrightarrow{G}_m$ if and only if there exists an interval-valued neutrosophic directed path $\overrightarrow{P}^m_{s,w}$ in $\overrightarrow{G}$.

Definition 2.29. The m-step interval-valued neutrosophic out-neighbourhood (IVN-out-neighbourhood) of vertex s of an IVN-digraph $\overrightarrow{G} = (A,\ \overrightarrow{B})$ is IVN-set

$$\mathbb{N}^+_m(s) = (X^+_s,\ [t^{(l)^+}_s,\ t^{(u)^+}_s],\ [i^{(l)^+}_s,\ i^{(u)^+}_s],\ [f^{(l)^+}_s,\ f^{(u)^+}_s]), \quad \text{where}$$

$X^+_s = \{w|$ there exists a directed interval-valued neutrosophic path of length m from s to w, $\overrightarrow{P}^m_{s,w}\}$, $t^{(l)^+}_s : X^+_s \to [0,\ 1]$, $t^{(u)^+}_s : X^+_s \to [0,\ 1]$, $i^{(l)^+}_s : X^+_s \to [0,\ 1]$, $i^{(u)^+}_s : X^+_s \to [0,\ 1]$, $f^{(l)^+}_s : X^+_s \to [0,\ 1]$ $f^{(u)^+}_s : X^+_s \to [0,\ 1]$ are defined by $t^{(l)^+}_s = \min\{t^l\overrightarrow{(s_1,\ s_2)}$, $(s_1,\ s_2)$ is an edge of $\overrightarrow{P}^m_{s,w}\}$, $t^{(u)^+}_s = \min\{t^u\overrightarrow{(s_1,\ s_2)}$, $(s_1,\ s_2)$ is an edge of $\overrightarrow{P}^m_{s,w}\}$, $i^{(l)^+}_s = \min\{i^l\overrightarrow{(s_1,\ s_2)}$, $(s_1,\ s_2)$ is an edge of $\overrightarrow{P}^m_{s,w}\}$, $i^{(u)^+}_s = \min\{i^u\overrightarrow{(s_1,\ s_2)}$, $(s_1,\ s_2)$ is an edge of $\overrightarrow{P}^m_{s,w}\}$, $f^{(l)^+}_s = \min\{f^l\overrightarrow{(s_1,\ s_2)}$, $(s_1,\ s_2)$ is an edge of $\overrightarrow{P}^m_{s,w}\}$, $f^{(u)^+}_s = \min\{f^u\overrightarrow{(s_1,\ s_2)}$, $(s_1,\ s_2)$ is an edge of $\overrightarrow{P}^m_{s,w}\}$, respectively.

Example 2.30. Consider an IVN-digraph $G = (A,\ \overrightarrow{B})$ on $X = \{s, w, a, b, c, d\}$, such that $A = \{(s,\ [0.4, 0.5],\ [0.5, 0.7],\ [0.8, 0.9]),\ (w,\ [0.6, 0.7],\ [0.4, 0.6],\ [0.2, 0.3]),\ (a,\ [0.2, 0.6],\ [0.3, 0.6],\ [0.2, 0.6]),\ (b,\ [0.2, 0.6],\ [0.1, 0.6],\ [0.2, 0.6]),\ (c,\ [0.2, 0.7],\ [0.3, 0.5],\ [0.2, 0.6]),\ d([0.2, 0.6], [0.3, 0.6], [0.2, 0.6])\}$, and $B = \{(\overrightarrow{(s, a)},\ [0.1, 0.4],\ [0.3, 0.6],\ [0.2, 0.6]),\ (\overrightarrow{(a, c)},\ [0.2, 0.6],\ [0.3, 0.5],\ [0.2, 0.6]),\ (\overrightarrow{(a, d)},\ [0.2, 0.6],\ [0.3, 0.5],\ [0.2, 0.4]),\ (\overrightarrow{(w, b)},\ [0.2, 0.6],\ [0.1, 0.6],\ [0.2, 0.3]),\ (\overrightarrow{(b, c)},\ [0.2, 0.4],\ [0.1, 0.2],\ [0.1, 0.3]),\ (\overrightarrow{(b, d)},\ [0.1, 0.3],\ [0.1, 0.2],\ [0.2, 0.4])\}$, as shown in Fig. 9.

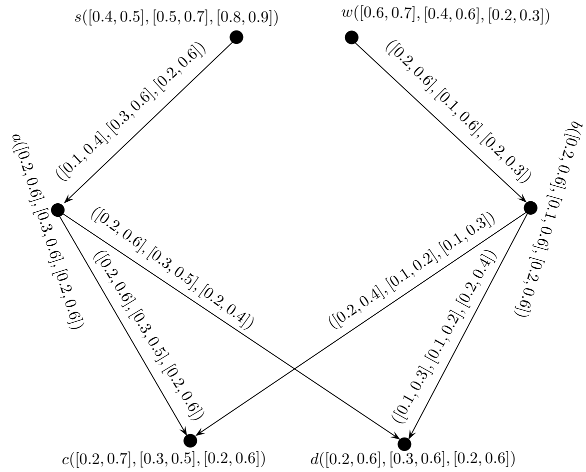

FIGURE 9. IVN-digraph

We calculate 2-step IVN-out-neighbourhoods as, $\mathbb{N}_2^+(s) = \{(c,\ [0.1, 0.4],\ [0.3, 0.5],\ [0.2, 0.6]),\ (d,\ [0.1, 0.4],\ [0.3, 0.5],\ [0.2, 0.4])\}$ and $\mathbb{N}_2^+(w) = \{(c,\ [0.2, 0.4],\ [0.1, 0.2],\ [0.1, 0.3]),\ (d,\ [0.1, 0.3],\ [0.1, 0.2],\ [0.2, 0.3])\}$.

Definition 2.31. The m-step interval-valued neutrosophic in-neighbourhood (IVN-in-neighbourhood) of vertex s of an IVN-digraph $\overrightarrow{G} = (A,\ \overrightarrow{B})$ is IVN-set

$$\mathbb{N}_m^-(s) = (X_s^-,\ [t_s^{(l)^-},\ t_s^{(u)^-}],\ [i_s^{(l)^-},\ i_s^{(u)^-}],\ [f_s^{(l)^-},\ f_s^{(u)^-}]), \quad \text{where}$$

$X_s^- = \{w|$ there exists a directed interval-valued neutrosophic path of length m from w to s, $\overrightarrow{P}^m_{w,s}\}$, $t_s^{(l)^-} : X_s^- \to [0,\ 1]$, $t_s^{(u)^-} : X_s^- \to [0,\ 1]$, $i_s^{(l)^-} : X_s^- \to [0,\ 1]$, $i_s^{(u)^-} : X_s^- \to [0,\ 1]$, $f_s^{(l)^-} : X_s^- \to [0,\ 1]$ $f_s^{(u)^-} : X_s^- \to [0,\ 1]$ are defined by $t_s^{(l)^-} = \min\{t^l\overrightarrow{(s_1,\ s_2)}$, $(s_1,\ s_2)$ is an edge of $\overrightarrow{P}^m_{w,s}\}$, $t_s^{(u)^-} = \min\{t^u\overrightarrow{(s_1,\ s_2)}$, $(s_1,\ s_2)$ is an edge of $\overrightarrow{P}^m_{w,s}\}$, $i_s^{(l)^-} = \min\{i^l\overrightarrow{(s_1,\ s_2)}$, $(s_1,\ s_2)$ is an edge of $\overrightarrow{P}^m_{w,s}\}$, $i_s^{(u)^-} = \min\{i^u\overrightarrow{(s_1,\ s_2)}$, $(s_1,\ s_2)$ is an edge of $\overrightarrow{P}^m_{w,s}\}$, $f_s^{(l)^-} = \min\{f^l\overrightarrow{(s_1,\ s_2)}$, $(s_1,\ s_2)$ is an edge of $\overrightarrow{P}^m_{w,s}\}$, $f_s^{(u)^-} = \min\{f^u\overrightarrow{(s_1,\ s_2)}$, $(s_1,\ s_2)$ is an edge of $\overrightarrow{P}^m_{w,s}\}$, respectively.

Example 2.32. Consider an IVN-digraph $G = (A,\ \overrightarrow{B})$ on $X = \{s, w, a, b, c, d\}$, such that $A = \{(s,\ [0.4, 0.5],\ [0.5, 0.7],\ [0.8, 0.9]),\ (w,\ [0.6, 0.7],\ [0.4, 0.6],\ [0.2, 0.3]),\ (a,\ [0.2, 0.6],\ [0.3, 0.6],\ [0.2, 0.6]),\ (b,\ [0.2, 0.6],\ [0.1, 0.6],\ [0.2, 0.6]),\ (c,\ [0.2, 0.7],\ [0.3, 0.5],\ [0.2, 0.6]),\ d([0.2, 0.6], [0.3, 0.6], [0.2, 0.6])\}$, and $B = \{(\overrightarrow{(s, a)},\ [0.1, 0.4],\ [0.3, 0.6],\ [0.2, 0.6]),\ (\overrightarrow{(a, c)},\ [0.2, 0.6],\ [0.3, 0.5],\ [0.2, 0.6]),\ (\overrightarrow{(a, d)},\ [0.2, 0.6],\ [0.3, 0.5],\ [0.2, 0.4]),\ (\overrightarrow{(w, b)},\ [0.2, 0.6],\ [0.1, 0.6],\ [0.2, 0.3]),\ (\overrightarrow{(b, c)},\ [0.2, 0.4],\ [0.1, 0.2],\ [0.1, 0.3]),\ (\overrightarrow{(b, d)},\ [0.1, 0.3],\ [0.1, 0.2],\ [0.2, 0.4])\}$, as shown in Fig. 10.

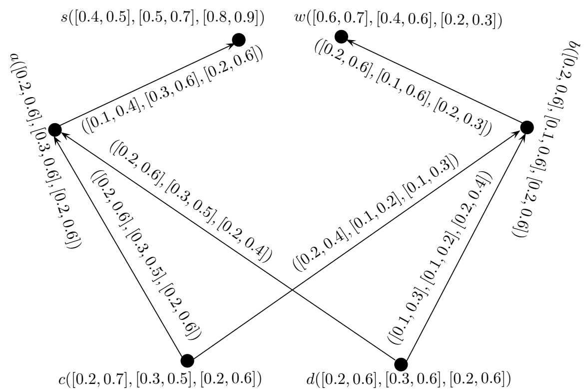

Figure 10. IVN-digraph

We calculate 2-step IVN-in-neighbourhoods as, $\mathbb{N}_2^-(s) = \{(c,$ [0.1,0.4], [0.3,0.5], [0.2,0.6]), $(d,$ [0.1,0.4], [0.3,0.5], [0.2,0.4])$\}$ and $\mathbb{N}_2^-(w) = \{(c,$ [0.2,0.4], [0.1,0.2], [0.1,0.3]), $(d,$ [0.1,0.3], [0.1,0.2], [0.2,0.3])$\}$.

Definition 2.33. Suppose $\overrightarrow{G} = (A,\ \overrightarrow{B})$ is an IVN-digraph. The m-step IVNC-graph of IVN-digraph $\overrightarrow{G}$ is denoted by $\mathbb{C}_m(\overrightarrow{G}) = (A,\ B)$ which has same IVN-set of vertices as in $\overrightarrow{G}$ and has an edge between two vertices $s,\ w \in X$ in $\mathbb{C}_m(\overrightarrow{G})$ if and only if $(\mathbb{N}_m^+(s) \cap \mathbb{N}_m^+(w))$ is a non-empty IVN-set in $\overrightarrow{G}$. The interval-valued truth-membership value of edge $(s,\ w)$ in $\mathbb{C}_m(\overrightarrow{G})$ is $t_B^l(s,\ w) = [t_A^l(s) \wedge t_A^l(w)]h_1^l(\mathbb{N}_m^+(s) \cap \mathbb{N}_m^+(w))$, and $t_B^u(s,\ w) = [t_A^u(s) \wedge t_A^u(w)]h_1^u(\mathbb{N}_m^+(s) \cap \mathbb{N}_m^+(w))$, the interval-valued indeterminacy-membership value of edge $(s,\ w)$ in $\mathbb{C}_m(\overrightarrow{G})$ is $i_B^l(s,\ w) = [i_A^l(s) \wedge i_A^l(w)]h_2^l(\mathbb{N}_m^+(s) \cap \mathbb{N}_m^+(w))$, and $i_B^u(s,\ w) = [i_A^u(s) \wedge i_A^u(w)]h_2^u(\mathbb{N}_m^+(s) \cap \mathbb{N}_m^+(w))$, the interval-valued falsity-membership value of edge $(s,\ w)$ in $\mathbb{C}_m(\overrightarrow{G})$ is $f_B^l(s,\ w) = [f_A^l(s) \wedge f_A^l(w)]h_3^l(\mathbb{N}_m^+(s) \cap \mathbb{N}_m^+(w))$, and $f_B^u(s,\ w) = [f_A^u(s) \wedge f_A^u(w)]h_3^u(\mathbb{N}_m^+(s) \cap \mathbb{N}_m^+(w))$.

The 2−step IVNC-graph is illustrated by the following example.

Example 2.34. Consider an IVN-digraph $G = (A, \overrightarrow{B})$ on $X = \{s, w, a, b, c, d\}$, such that $A = \{(s,\ [0.4, 0.5],\ [0.5, 0.7],\ [0.8, 0.9]),\ (w,\ [0.6, 0.7],\ [0.4, 0.6],\ [0.2, 0.3]),\ (a,$ $[0.2, 0.6],\ [0.3, 0.6],\ [0.2, 0.6]),\ (b,\ [0.2, 0.6],\ [0.1, 0.6],\ [0.2, 0.6]),\ (c,\ [0.2, 0.7],\ [0.3, 0.5],$ $[0.2, 0.6]),\ d([0.2, 0.6], [0.3, 0.6], [0.2, 0.6])\}$, and $B = \{(\overrightarrow{(s, a)},\ [0.1, 0.4],\ [0.3, 0.6],\ [0.2,$ $0.6]),\ (\overrightarrow{(a, c)},\ [0.2, 0.6],\ [0.3, 0.5],\ [0.2, 0.6]),\ (\overrightarrow{(a, d)},\ [0.2, 0.6],\ [0.3, 0.5],\ [0.2, 0.4]),$ $(\overrightarrow{(w, b)},\ [0.2, 0.6],\ [0.1, 0.6],\ [0.2, 0.3]),\ (\overrightarrow{(b, c)},\ [0.2, 0.4],\ [0.1, 0.2],\ [0.1, 0.3]),\ (\overrightarrow{(b, d)},$ $[0.1, 0.3],\ [0.1, 0.2],\ [0.2, 0.4])\}$, as shown in Fig. 11.

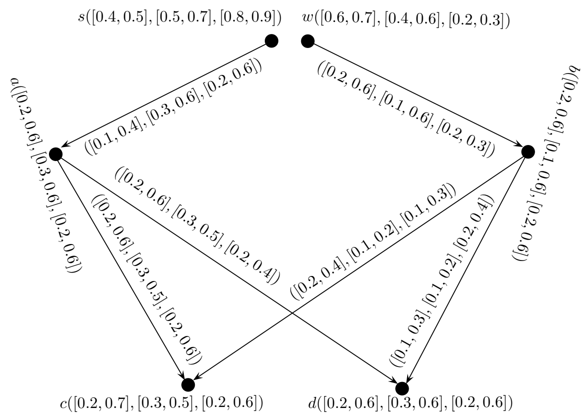

FIGURE 11. IVN-digraph

We calculate $\mathbb{N}_2^+(s) = \{(c,\ [0.1, 0.4],\ [0.3, 0.5],\ [0.2, 0.6]),\ (d,\ [0.1, 0.4],\ [0.3, 0.5],\ [0.2, 0.4])\}$ and $\mathbb{N}_2^+(w) = \{(c,\ [0.2, 0.4],\ [0.1, 0.2],\ [0.1, 0.3]),\ (d,\ [0.1, 0.3],\ [0.1, 0.2],\ [0.2, 0.3])\}$. Therefore, $\mathbb{N}_2^+(s) \cap \mathbb{N}_2^+(w) = \{(c,\ [0.1, 0.4],\ [0.1, 0.2],\ [0.2, 0.6]),\ (d,\ [0.1, 0.3],\ [0.1, 0.2],\ [0.2, 0.4])\}$. Thus, $t_B^l(s,\ w) = 0.04$, $t_B^u(s,\ w) = 0.20$, $i_B^l(s,\ w) = 0.04$, $i_B^u(s,\ w) = 0.12$, $f_B^l(s,\ w) = 0.04$ and $f_B^u(s,\ w) = 0.12$. This graph is depicted in Fig. 12.

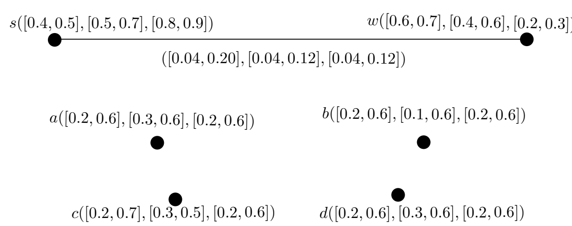

FIGURE 12. 2-Step IVNC-graph

If a predator s attacks one prey w, then the linkage is shown by an edge $\overrightarrow{(s, w)}$ in an IVN-digraph. But, if predator needs help of many other mediators $s_1, s_2, \ldots, s_{m-1}$, then linkage among them is shown by interval-valued neutrosophic directed path $\overrightarrow{P}^m_{s,w}$ in an IVN-digraph. So, m-step prey in an IVN-digraph is represented by a vertex which is the m-step out-neighbourhood of some vertices. Now, the strength of an IVNC-graphs is defined below.

Definition 2.35. Let $\overrightarrow{G} = (A,\ \overrightarrow{B})$ be an IVN-digraph. Let w be a common vertex of m-step out-neighbourhoods of vertices $s_1,\ s_2,\ \ldots,\ s_l$. Also, let $\overrightarrow{B_1^l}(u_1, v_1)$, $\overrightarrow{B_1^l}(u_2, v_2), \ldots, \overrightarrow{B_1^l}(u_r, v_r)$ and $\overrightarrow{B_1^u}(u_1, v_1)$, $\overrightarrow{B_1^u}(u_2, v_2), \ldots, \overrightarrow{B_1^u}(u_r, v_r)$ be the minimum interval-valued truth-membership values, $\overrightarrow{B_2^l}(u_1, v_1)$, $\overrightarrow{B_2^l}(u_2, v_2), \ldots, \overrightarrow{B_2^l}(u_r, v_r)$ and $\overrightarrow{B_2^u}(u_1, v_1)$, $\overrightarrow{B_2^u}(u_2, v_2), \ldots, \overrightarrow{B_2^u}(u_r, v_r)$ be the minimum indeterminacy-membership values, $\overrightarrow{B_3^l}(u_1,v_1)$, $\overrightarrow{B_3^l}(u_2,v_2)$, ..., $\overrightarrow{B_3^l}(u_r,v_r)$ and $\overrightarrow{B_3^u}(u_1,v_1)$, $\overrightarrow{B_3^u}(u_2,v_2)$, ..., $\overrightarrow{B_3^u}(u_r, v_r)$ be the maximum false-membership values, of edges of the paths $\overrightarrow{P}^m_{s_1,w}$, $\overrightarrow{P}^m_{s_2,w}$, ..., $\overrightarrow{P}^m_{s_r,w}$, respectively. The m-step prey $w \in X$ is strong prey if

$$\overrightarrow{B_1^l}(u_i,v_i) > 0.5, \quad \overrightarrow{B_2^l}(u_i,v_i) > 0.5, \quad \overrightarrow{B_3^l}(u_i,v_i) < 0.5,$$
$$\overrightarrow{B_1^u}(u_i,v_i) > 0.5, \quad \overrightarrow{B_2^u}(u_i,v_i) > 0.5, \quad \overrightarrow{B_3^u}(u_i,v_i) < 0.5, \text{ for all } i = 1, 2, \ldots, r.$$

The strength of the prey w can be measured by the mapping $S : X \to [0,1]$, such that:

$$S(w) = \frac{1}{r}\Bigg\{ \sum_{i=1}^{r}[\overrightarrow{B_1^l}(u_i,v_i)] + \sum_{i=1}^{r}[\overrightarrow{B_1^u}(u_i,v_i)] + \sum_{i=1}^{r}[\overrightarrow{B_2^l}(u_i,v_i)] + \sum_{i=1}^{r}[\overrightarrow{B_2^u}(u_i,v_i)] - \sum_{i=1}^{r}[\overrightarrow{B_3^l}(u_i,v_i)] - \sum_{i=1}^{r}[\overrightarrow{B_3^u}(u_i,v_i)] \Bigg\}.$$

Example 2.36. Consider an IVN-digraph $\overrightarrow{G} = (A, \overrightarrow{B})$ as shown in Fig. 11, the strength of the prey c is equal to

$$\frac{(0.2+0.2)+(0.6+0.4)+(0.1+0.1)+(0.6+0.2)-(0.2+0.1)-(0.3+0.3)}{2} = 1.5 > 0.5.$$

Hence, c is strong 2-step prey.

We state the following theorem without its proof.

Theorem 2.37. *If a prey w of $\overrightarrow{G} = (A, \overrightarrow{B})$ is strong, then the strength of w, $S(w) > 0.5$.*

Remark 2.38. The converse of the above theorem is not true, i.e. if $S(w) > 0.5$, then all preys may not be strong. This can be explained as:
Let $S(w) > 0.5$ for a prey w in $\overrightarrow{G}$. So,

$$S(w) = \frac{1}{r}\Bigg\{ \sum_{i=1}^{r}[\overrightarrow{B_1^l}(u_i,v_i)] + \sum_{i=1}^{r}[\overrightarrow{B_1^u}(u_i,v_i)] + \sum_{i=1}^{r}[\overrightarrow{B_2^l}(u_i,v_i)] + \sum_{i=1}^{r}[\overrightarrow{B_2^u}(u_i,v_i)] - \sum_{i=1}^{r}[\overrightarrow{B_3^l}(u_i,v_i)] - \sum_{i=1}^{r}[\overrightarrow{B_3^u}(u_i,v_i)] \Bigg\}.$$

Hence,

$$\Bigg\{ \sum_{i=1}^{r}[\overrightarrow{B_1^l}(u_i,v_i)] + \sum_{i=1}^{r}[\overrightarrow{B_1^u}(u_i,v_i)] + \sum_{i=1}^{r}[\overrightarrow{B_2^l}(u_i,v_i)] + \sum_{i=1}^{r}[\overrightarrow{B_2^u}(u_i,v_i)] - \sum_{i=1}^{r}[\overrightarrow{B_3^l}(u_i,v_i)] - \sum_{i=1}^{r}[\overrightarrow{B_3^u}(u_i,v_i)] \Bigg\} > \frac{r}{2}.$$

This result does not necessarily imply that

$$\overrightarrow{B_1^l}(u_i, v_i) > 0.5, \quad \overrightarrow{B_2^l}(u_i, v_i) > 0.5, \quad \overrightarrow{B_3^l}(u_i, v_i) < 0.5,$$
$$\overrightarrow{B_1^u}(u_i, v_i) > 0.5, \quad \overrightarrow{B_2^u}(u_i, v_i) > 0.5, \quad \overrightarrow{B_3^u}(u_i, v_i) < 0.5, \text{ for all } i = 1, 2, \ldots, r.$$

Since, all edges of the directed paths $\overrightarrow{P}^m_{s_1,w}$, $\overrightarrow{P}^m_{s_2,w}, \ldots, \overrightarrow{P}^m_{s_r,w}$, are not strong. So, the converse of the above statement is not true i.e., if $S(w) > 0.5$, the prey w of $\overrightarrow{G}$ may not be strong. Now, m-step interval-valued neutrosophic neighbouhood graphs are defines below.

Definition 2.39. The m-step IVN-out-neighbourhood of vertex s of an IVN-digraph $\overrightarrow{G} = (A,\ \overrightarrow{B})$ is IVN-set

$$\mathbb{N}_m(s) = (X_s,\ [t^l_s,\ t^u_s],\ [i^l_s,\ i^u_s],\ [f^l_s,\ f^u_s]), \quad \text{where}$$

$X_s = \{w|$ there exists a directed interval-valued neutrosophic path of length m from s to w, $\mathbb{P}^m_{s,w}\}$, $t^l_s : X_s \to [0,\ 1]$, $t^u_s : X_s \to [0,\ 1]$, $i^l_s : X_s \to [0,\ 1]$, $i^u_s : X_s \to [0,\ 1]$, $f^l_s : X_s \to [0,\ 1]$, $f^u_s : X_s \to [0,\ 1]$, are defined by $t^l_s = \min\{t^l(s_1,\ s_2)$, $(s_1,\ s_2)$ is an edge of $\mathbb{P}^m_{s,w}\}$, $t^u_s = \min\{t^u(s_1,\ s_2)$, $(s_1,\ s_2)$ is an edge of $\mathbb{P}^m_{s,w}\}$, $i^l_s = \min\{i^l(s_1,\ s_2)$, $(s_1,\ s_2)$ is an edge of $\mathbb{P}^m_{s,w}\}$, $i^u_s = \min\{i^u(s_1,\ s_2)$, $(s_1,\ s_2)$ is an edge of $\mathbb{P}^m_{s,w}\}$, $f^l_s = \min\{f^l(s_1,\ s_2)$, $(s_1,\ s_2)$ is an edge of $\mathbb{P}^m_{s,w}\}$, $f^u_s = \min\{f^u(s_1,\ s_2)$, $(s_1,\ s_2)$ is an edge of $\mathbb{P}^m_{s,w}\}$, respectively.

Definition 2.40. Suppose $G = (A,\ B)$ is an IVN-graph. Then m-step interval-valued neutrosophic neighbouhood graph $\mathbb{N}_m(G)$ is defined by $\mathbb{N}_m(G) = (A,\ \acute{B})$ where $A = ([A^l_1,\ A^u_1],\ [A^l_2,\ A^u_2],\ [A^l_3,\ A^u_3])$, $\acute{B} = ([\acute{B}^l_1,\ \acute{B}^u_1],\ [\acute{B}^l_2,\ \acute{B}^u_2],\ [\acute{B}^l_3,\ \acute{B}^u_3])$, $\acute{B}^l_1 : X \times X \to [0,\ 1]$, $\acute{B}^u_1 : X \times X \to [0,\ 1]$, $\acute{B}^l_2 : X \times X \to [0, 1]$, $\acute{B}^u_2 : X \times X \to [0,\ 1]$, $\acute{B}^l_3 : X \times X \to [0,\ 1]$, and $\acute{B}^u_3 : X \times X \to [0, -1]$ are such that:

$$\acute{B}^l_1(s, w) = A^l_1(s) \wedge A^l_1(w) h^l_1(\mathbb{N}_m(s) \cap \mathbb{N}_m(w)),$$
$$\acute{B}^l_2(s, w) = A^l_2(s) \wedge A^l_2(w) h^l_2(\mathbb{N}_m(s) \cap \mathbb{N}_m(w)),$$
$$\acute{B}^l_3(s, w) = A^l_3(s) \wedge A^l_3(w) h^l_3(\mathbb{N}_m(s) \cap \mathbb{N}_m(w)),$$
$$\acute{B}^u_1(s, w) = A^u_1(s) \wedge A^u_1(w) h^u_1(\mathbb{N}_m(s) \cap \mathbb{N}_m(w)),$$
$$\acute{B}^u_2(s, w) = A^u_2(s) \wedge A^u_2(w) h^u_2(\mathbb{N}_m(s) \cap \mathbb{N}_m(w)),$$
$$\acute{B}^u_3(s, w) = A^u_3(s) \wedge A^u_3(w) h^u_3(\mathbb{N}_m(s) \cap \mathbb{N}_m(w)), \text{respectively.}$$

We state the following theorems without thier proofs.

Theorem 2.41. *If all preys of $\overrightarrow{G} = (A,\ \overrightarrow{B})$ are strong, then all edges of $\mathbb{C}_m(\overrightarrow{G}) = (A,\ B)$ are strong.*

A relation is established between m-step IVNC-graph of an IVN-digraph and IVNC-graph of m-step IVN-digraph.

Theorem 2.42. *If $\overrightarrow{G}$ is an IVN-digraph and $\overrightarrow{G_m}$ is the m-step IVN-digraph of $\overrightarrow{G}$, then $\mathbb{C}(\overrightarrow{G}_m) = \mathbb{C}_m(\overrightarrow{G})$.*

Theorem 2.43. *Let $\overrightarrow{G} = (A, \overrightarrow{B})$ be an IVN-digraph. If $m > |X|$ then $\mathbb{C}_m(\overrightarrow{G}) = (A, B)$ has no edge.*

Theorem 2.44. *If all the edges of IVN-digraph* $\overrightarrow{G} = (A, \overrightarrow{B})$ *are independent strong, then all the edges of* $\mathbb{C}_m(\overrightarrow{G})$ *are independent strong.*

3. Conclusions

Graph theory is an enjoyable playground for the research of proof techniques in discrete mathematics. There are many applications of graph theory in different fields. We have introduced IVNC-graphs and k-competition IVN-graphs, p-competition IVN-graphs and m-step IVNC-graphs as the generalized structures of IVNC-graphs. We have described interval-valued neutrosophic open and closed-neighbourhood. Also we have established some results related to them. We aim to extend our research work to (1) Interval-valued fuzzy rough graphs; (2) Interval-valued fuzzy rough hypergraphs, (3) Interval-valued fuzzy rough neutrosophic graphs, and (4) Decision support systems based on IVN-graphs.

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